STUDIA MATHEMATICA 149 (2) (2002)

Generalized non-commutative tori

by

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Abstract. The generalized non-commutative torus T_ϱ^k of rank n is defined by the crossed product $A_{m/k} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of a rational rotation algebra $A_{m/k}$ are trivial, and $C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$ is a non-commutative torus A_ϱ. It is shown that T_ϱ^k is strongly Morita equivalent to A_ϱ, and that $T_\varrho^k \otimes M_{p^\infty}$ is isomorphic to $A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.

Introduction. Let G be a locally compact abelian group. A *multiplier* on G is a measurable function $\omega : G \times G \to \mathbb{T}^1$ which satisfies

$$\omega(xy, z)\omega(x, y) = \omega(x, yz)\omega(y, z), \quad x, y, z \in G,$$
$$\omega(x, e) = \omega(e, x) = 1, \quad x \in G,$$

where e is the identity in G. Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^n, \omega)$ is said to be a *non-commutative torus of rank* n and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and ω is called *totally skew* if the symmetry group S_ω is trivial; the torus A_ω is then called *completely irrational* (see [1]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra.

Boca [3] showed that almost all completely irrational non-commutative tori are isomorphic to inductive limits of circle algebras, where the "circle algebra" means a C^*-algebra which is a finite direct sum of C^*-algebras of the form $C(\mathbb{T}^1) \otimes M_q(\mathbb{C})$. We will assume that each completely irrational non-commutative torus appearing in this paper is an inductive limit of circle algebras.

2000 *Mathematics Subject Classification*: Primary 46L05, 46L87.

Key words and phrases: non-commutative torus, equivalence bimodule, circle algebra, real rank 0, crossed product, UHF-algebra, Cuntz algebra.

Research supported by grant No. 1999-2-102-001-3 from the interdisciplinary Research program year of the KOSEF. The author would also like to thank the referee for a number of valuable suggestions to a previous version of this paper.

In [5], it was shown that two separable C^*-algebras A and B are stably isomorphic if and only if they are strongly Morita equivalent, i.e., there exists an A-B-equivalence bimodule defined in [14]. In [4], M. Brabanter constructed an $A_{m/k}$-$C(\mathbb{T}^2)$-equivalence bimodule. Modifying his construction, we are going to construct a T_ϱ^k-A_ϱ-equivalence bimodule.

It was shown in [2, Theorem 1.5] that each completely irrational non-commutative torus has real rank 0, where the "real rank 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. Combining Theorem 1.2 given in the first section and [6, Corollary 3.3] yields that if A_ϱ is simple then T_ϱ^k has real rank 0, since the non-commutative torus A_ϱ has real rank 0. And Lin and Rørdam's results [12, Propositions 2 and 3] say that if A_ϱ simple then T_ϱ^k is an inductive limit of circle algebras, since $T_\varrho^k \otimes \mathcal{K}(\mathcal{H}) \cong A_\varrho \otimes \mathcal{K}(\mathcal{H})$ is an inductive limit of circle algebras.

Combining Elliott's classification theorem [10, Theorem 7.1] and Ji and Xia's result [11, Theorem 1.3] yields that the completely irrational non-commutative tori A_ω of rank n and the simple generalized non-commutative tori T_ϱ^k of rank n are classified by the ranges of the traces, and so one can completely classify them up to isomorphism or up to strong Morita equivalence. Hence some completely irrational non-commutative tori A_ω of rank n are isomorphic to some simple generalized non-commutative tori T_ϱ^k of rank n, and this result can be applied to understand the (bundle) structure of C^*-algebras of sections of locally trivial continuous C^*-algebra bundles over CW-complexes with fibres completely irrational non-commutative tori.

It is moreover shown that $T_\varrho^k \otimes M_{p^\infty}$ is isomorphic to $A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p, that $\mathcal{O}_{2u} \otimes T_\varrho^k$ is isomorphic to $\mathcal{O}_{2u} \otimes A_\varrho \otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime, and that $\mathcal{O}_\infty \otimes T_\varrho^k$ is not isomorphic to $\mathcal{O}_\infty \otimes A_\varrho \otimes M_k(\mathbb{C})$ if $k > 1$, where $\mathcal{O}_u$ and $\mathcal{O}_\infty$ denote the Cuntz algebra and the generalized Cuntz algebra, respectively.

1. Generalized non-commutative tori. It was shown in [4, Proposition 1] that $A_{m/k}$ is the C^*-algebra of matrices $(f_{ij})_{i,j=1}^k$ of functions f_{ij} with

$$\begin{aligned} &f_{ij} \in C^*(k\mathbb{Z} \times k\mathbb{Z}) && \text{if } i,j \in \{1,\dots,k-1\} \text{ or } (i,j) = (k,k),\\ &f_{ik} \in \Omega \ \&\ f_{ki} \in \Omega^* && \text{if } i \in \{1,\dots,k-1\}, \end{aligned}$$

where Ω and Ω^* are the $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules defined as

$$\begin{aligned} \Omega &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f(z,1) = z^s f(z,0),\ \forall z \in \widehat{k\mathbb{Z}}\},\\ \Omega^* &= \{f \in C(\widehat{k\mathbb{Z}} \times [0,1]) \mid f^* \in \Omega\} \end{aligned}$$

for an integer s such that $sm = 1 \pmod{k}$.

The non-commutative torus A_ω of rank n is obtained by an iteration of $n-1$ crossed products by actions of $\mathbb{Z}$, the first action being on $C(\mathbb{T}^1)$. When A_ω has a primitive ideal space $\widehat{S}_\omega \cong \widehat{k\mathbb{Z}}$, A_ω is realized as the C^*-algebra of sections of a locally trivial continuous C^*-algebra bundle over $\widehat{k\mathbb{Z}}$ with fibres $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ for some totally skew multiplier ω_1, where $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong A_\varphi \otimes M_k(\mathbb{C})$ for A_φ a completely irrational non-commutative torus of rank $n-1$. By a change of basis, one can assume that $A_\omega \cong A_{m/k} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ are trivial, since the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ is factored out of the fibre $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ of A_ω (see [1, 9, 13]). This assures us of the existence of such actions α_i as in the definition of T_ϱ^k in the abstract.

1.1. Definition. The *generalized non-commutative torus* T_ϱ^k of rank n is defined to be the crossed product $A_{m/k} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, where the actions α_i of $\mathbb{Z}$ on the fibre $M_k(\mathbb{C})$ of a rational rotation algebra $A_{m/k}$ are trivial, and $C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$ is a non-commutative torus A_ϱ of rank n.

So the generalized non-commutative torus T_ϱ^k has a matrix representation induced from the matrix representation of the rational rotation subalgebra $A_{m/k}$.

1.2. Proposition. *The generalized non-commutative torus T_ϱ^k is isomorphic to the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ with*

$$g_{ij} \in A_\varrho \qquad \text{if } i,j \in \{1,\ldots,k-1\} \text{ or } (i,j)=(k,k),$$
$$g_{ik} \in \widetilde{\Omega} \ \&\ g_{ki} \in \widetilde{\Omega}^* \qquad \text{if } i \in \{1,\ldots,k-1\},$$

where $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are the A_ϱ-modules defined as

$$\widetilde{\Omega} = A_\varrho \cdot \Omega, \qquad \widetilde{\Omega}^* = A_\varrho \cdot \Omega^*.$$

Here Ω and Ω^* are the $C^*(k\mathbb{Z} \times k\mathbb{Z})$-modules defined above.

Proof. One sees from the definition of T_ϱ^k that the isomorphism between $A_{m/k}$ and the C^*-algebra of matrices $(f_{ij})_{i,j=1}^k$ satisfying the condition given above gives an isomorphism between T_ϱ^k and the C^*-algebra of matrices $(g_{ij})_{i,j=1}^k$ satisfying the condition given in the statement. Note that $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are the A_ϱ-modules defined by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ in $\Omega = C^*(k\mathbb{Z} \times k\mathbb{Z}) \cdot \Omega$ with $A_\varrho \cong C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, since the entries in the matrix representation of $A_{m/k}$ have a $C^*(k\mathbb{Z} \times k\mathbb{Z})$-module structure, and T_ϱ^k may be obtained by canonically replacing $C^*(k\mathbb{Z} \times k\mathbb{Z})$ with $A_\varrho \cong C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$. ∎

We are going to construct a T_ϱ^k-A_ϱ-equivalence bimodule.

1.3. THEOREM. *T_ϱ^k is strongly Morita equivalent to A_ϱ.*

Proof. Let X be the complex vector space $(\bigoplus_1^{k-1} \widetilde{\Omega}) \oplus A_\varrho$. We will consider the elements of X as $(k,1)$ matrices where the first $k-1$ entries are in $\widetilde{\Omega}$ and the last entry is in A_ϱ. If $x \in X$, denote by x^* the $(1,k)$ matrix resulting from x by transposition and involution so that $x^* \in (\bigoplus_1^{k-1} \widetilde{\Omega}^*) \oplus A_\varrho$. The space X is a left T_ϱ^k-module if module multiplication is defined by matrix multiplication $F \cdot x$, where $F = (g_{ij})_{i,j=1}^k \in T_\varrho^k$ and $x \in X$. If $g \in A_\varrho$ and $x \in X$, then $x \cdot [g]$ defines a right A_ϱ-module structure on X. Now we define a T_ϱ^k-valued and an A_ϱ-valued inner products $\langle \cdot, \cdot \rangle_{T_\varrho^k}$ and $\langle \cdot, \cdot \rangle_{A_\varrho}$ on X by

$$\langle x, y \rangle_{T_\varrho^k} = x \cdot y^*, \quad \langle x, y \rangle_{A_\varrho} = x^* \cdot y$$

for $x, y \in X$, with matrix multiplication on the right.

It is obvious that for $x, y \in X$, $x \cdot y^* \in T_\varrho^k$ and $x^* \cdot y \in A_\varrho$. Let $A_{m/k} = T_\varrho^k$. By [4, Theorem 3], $\{x \cdot y^* \mid x, y \in X\}$ is dense in $A_{m/k}$. Let us replace $C^*(k\mathbb{Z} \times k\mathbb{Z})$ in the vector space X for $A_{m/k}$ with $A_\varrho \cong C^*(k\mathbb{Z} \times k\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$. From the definitions of $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ and the structure of the generalized non-commutative torus T_ϱ^k of rank n, given in the proof of Proposition 1.2, one finds that $\{x \cdot y^* \mid x, y \in X\}$ is dense in T_ϱ^k. On the other hand, for any $a \in A_\varrho$, let $x = (0, 0, \ldots, 0, 1)$, $y = (0, 0, \ldots, 0, a) \in X$. Then $x^* \cdot y = a$. Hence $\{x^* \cdot y \mid x, y \in X\}$ is dense in A_ϱ. So X becomes a T_ϱ^k-A_ϱ-equivalence bimodule, as desired. ■

The generalized non-commutative torus T_ϱ^k of rank n is strongly Morita equivalent to the non-commutative torus A_ϱ of rank n, so $K_i(T_\varrho^k) \cong K_i(A_\varrho) \cong \mathbb{Z}^{2^{n-1}}$ (see [9, Theorem 2.2]). The non-commutative torus A_ϱ of rank n is the universal object for unitary ϱ-representations of $\mathbb{Z}^n$, so A_ϱ is realized as $C^*(u_1, \ldots, u_n \mid u_i u_j = e^{2\pi i \theta_{ji}} u_j u_i)$, where u_i are unitaries and θ_{ji} are real numbers for $1 \le i, j \le n$.

1.4. THEOREM. (1) $\mathrm{tr}(K_0(T_\varrho^k)) = k^{-1} \cdot \mathrm{tr}(K_0(A_\varrho))$ *if A_ϱ is completely irrational.*

(2) $[1_{T_\varrho^k}] \in K_0(T_\varrho^k)$ *is primitive.*

Proof. (1) T_ϱ^k has a matrix representation induced from the matrix representation of the rational rotation subalgebra $A_{m/k}$. The diagonal entries of the matrix representation are in A_ϱ, and so the range of the trace of $K_0(T_\varrho^k)$ is

$$\mathbb{Z} + \frac{1}{k}(\mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \ldots + \mathbb{Z}\gamma),$$

where $\mathrm{tr}(K_0(A_\varrho)) = \mathbb{Z} + \mathbb{Z}k + \mathbb{Z}\alpha + \mathbb{Z}\beta + \ldots + \mathbb{Z}\gamma$. Hence $\mathrm{tr}(K_0(T_\varrho^k)) = k^{-1} \cdot \mathrm{tr}(K_0(A_\varrho))$.

(2) We argue by induction on n. For $n = 2$, it is the Elliott result [9, Theorem 2.2]. Assume that the result is true for all T^k_ϱ with $n = i - 1$. Since T^k_ϱ is realized as $A_{m/k} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \ldots \times_{\alpha_n} \mathbb{Z}$, write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(A_{m/k}, u_3, \ldots, u_i)$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product of $\mathbb{S}_{i-1}$ by an action α_i of $\mathbb{Z}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(u_1^k, u_2^k, u_3, \ldots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i\theta_{ji}} u_j$, $j \neq 1, 2$, and sending u_j^k to $u_i u_j^k u_i^{-1} = e^{2\pi ik\theta_{ji}} u_j^k$, $j = 1, 2$), and which acts trivially on $M_k(\mathbb{C})$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner–Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \xrightarrow{1-(\alpha_i)_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $(\alpha_i)_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces to a split short exact sequence

$$\{0\} \to K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \to \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{i-2} = 2^{i-1}$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner–Voiculescu exact sequence is a split short exact sequence of torsion-free groups. ■

1.5. Corollary. *T^k_ϱ is not isomorphic to $A \otimes M_d(\mathbb{C})$ for a C^*-algebra A if $d > 1$.*

Proof. Assume T^k_ϱ is isomorphic to $A \otimes M_d(\mathbb{C})$. Then the unit $1_{T^k_\varrho}$ maps to $1_A \otimes I_d$. This implies that there is a projection e in T^k_ϱ such that $[1_{T^k_\varrho}] = d[e]$ in $K_0(T^k_\varrho)$, which contradicts Theorem 1.4 if $d > 1$. Thus no non-trivial matrix algebra can be factored out of T^k_ϱ. ■

2. Tensor products of generalized non-commutative tori with UHF-algebras and Cuntz algebras. Using the fact that $[1_{T^k_\varrho}] \in K_0(T^k_\varrho)$ is primitive, we investigate the structure of $T^k_\varrho \otimes M_{p^\infty}$ for M_{p^∞} a UHF-algebra of type p^∞.

2.1. Theorem. *$T^k_\varrho \otimes M_{p^\infty}$ is isomorphic to $A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of k is a subset of the set of prime factors of p. To show that $T^k_\varrho \otimes M_{p^\infty}$ is isomorphic to $A_\varrho \otimes M_k(\mathbb{C}) \otimes$

M_{p^∞}, it is enough to show that $T_\varrho^k \otimes M_{k^\infty} \cong A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{k^\infty}$. But there exist the C^*-algebra homomorphisms which are the canonical inclusions $T_\varrho^k \otimes M_{k^g}(\mathbb{C}) \hookrightarrow A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{k^g}(\mathbb{C})$ and the A_ϱ-module maps $A_\varrho \otimes M_{k^g}(\mathbb{C}) \hookrightarrow T_\varrho^k \otimes M_{k^g}(\mathbb{C})$:

$$T_\varrho^k \hookrightarrow A_\varrho \otimes M_k(\mathbb{C}) \hookrightarrow T_\varrho^k \otimes M_k(\mathbb{C}) \hookrightarrow A_\varrho \otimes M_{k^2}(\mathbb{C}) \hookrightarrow \ldots$$

The inductive limit of the odd terms

$$\ldots \to T_\varrho^k \otimes M_{k^g}(\mathbb{C}) \to T_\varrho^k \otimes M_{k^{g+1}}(\mathbb{C}) \to \ldots$$

is $T_\varrho^k \otimes M_{k^\infty}$, and the inductive limit of the even terms

$$\ldots \to A_\varrho \otimes M_{k^g}(\mathbb{C}) \to A_\varrho \otimes M_{k^{g+1}}(\mathbb{C}) \to \ldots$$

is $A_\varrho \otimes M_{k^\infty}$. Thus by the Elliott theorem [10, Theorem 2.1], $T_\varrho^k \otimes M_{k^\infty}$ is isomorphic to $A_\varrho \otimes M_{k^\infty}$.

Conversely, assume that $T_\varrho^k \otimes M_{p^\infty} \cong A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{T_\varrho^k} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{A_\varrho} \otimes 1_{M_{p^\infty}} \otimes I_k$. So

$$[1_{T_\varrho^k} \otimes 1_{M_{p^\infty}}] = [1_{A_\varrho} \otimes 1_{M_{p^\infty}} \otimes I_k],$$
$$[1_{T_\varrho^k} \otimes 1_{M_{p^\infty}}] = [1_{T_\varrho^k}] \otimes [1_{M_{p^\infty}}],$$
$$[1_{A_\varrho} \otimes 1_{M_{p^\infty}} \otimes I_k] = k([1_{A_\varrho}] \otimes [1_{M_{p^\infty}}]).$$

Under the assumption that the unit $1_{T_\varrho^k} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{A_\varrho} \otimes 1_{M_{p^\infty}} \otimes I_k$, if there is a prime factor q of k such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞}. So there is a projection $e \in T_\varrho^k$ such that $[1_{T_\varrho^k}] = q[e]$. This contradicts Theorem 1.4. Thus the set of prime factors of k is a subset of the set of prime factors of p.

Therefore, $T_\varrho^k \otimes M_{p^\infty}$ is isomorphic to $A_\varrho \otimes M_k(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of k is a subset of the set of prime factors of p. ■

Let us study the structure of the tensor products of generalized noncommutative tori with (even) Cuntz algebras.

The *Cuntz algebra* $\mathcal{O}_u$, $2 \leq u < \infty$, is the universal C^*-algebra generated by u isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \ldots + s_u s_u^* = 1$. Cuntz [7, 8] proved that $\mathcal{O}_u$ is simple and the K-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

2.2. Proposition. *Let u be a positive integer such that k and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes T_\varrho^k$ is not isomorphic to $\mathcal{O}_u \otimes A_\varrho \otimes M_k(\mathbb{C})$.*

Proof. Let p be a prime such that $p \mid k$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes T_\varrho^k$ is isomorphic to $\mathcal{O}_u \otimes A_\varrho \otimes M_k(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u \otimes T_\varrho^k}$ maps to the unit $1_{\mathcal{O}_u \otimes A_\varrho} \otimes I_k$. So $[1_{\mathcal{O}_u \otimes T_\varrho^k}] = [1_{\mathcal{O}_u \otimes A_\varrho} \otimes I_k] = k[1_{\mathcal{O}_u \otimes A_\varrho}]$. Hence there is a projection e in $\mathcal{O}_u \otimes T_\varrho^k$ such that $[1_{\mathcal{O}_u \otimes T_\varrho^k}] = k[e]$. But $[1_{\mathcal{O}_u \otimes T_\varrho^k}] =$

$[1_{\mathcal{O}_u}]\otimes[1_{T^k_\varrho}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u)\cong\mathbb{Z}/(u-1)\mathbb{Z}$ (see [8]). But $p\,|\,u-1$. We have $[1_{\mathcal{O}_u}]\neq p[e_*]$ for e_* a projection in $\mathcal{O}_u$. So $[1_{T^k_\varrho}]=p[e']$ for e' a projection in T^k_ϱ. This contradicts Theorem 1.4. Hence k and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u\otimes T^k_\varrho$ is not isomorphic to $\mathcal{O}_u\otimes A_\varrho\otimes M_k(\mathbb{C})$ if k and $u-1$ are not relatively prime. ∎

The following result is useful to understand the structure of $\mathcal{O}_u\otimes T^k_\varrho$.

2.3. Proposition [15, Theorem 7.2]. *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha: K_0(A)\to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A])=[1_B]$, then there is an isomorphism $\phi: A\to B$ which induces α.*

2.4. Corollary. (1) *Let p be an odd integer such that p and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{(2u-1)p+1}\otimes M_{p^\infty}$. That is, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{p^\infty}$.*

(2) *$\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{(2u)^\infty}$.*

2.5. Theorem. *$\mathcal{O}_{2u}\otimes T^k_\varrho$ is isomorphic to $\mathcal{O}_{2u}\otimes A_\varrho\otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime.*

Proof. Assume that k and $2u-1$ are relatively prime. Let $k=p2^v$ for some odd integer p. Then p and $2u-1$ are relatively prime. Then by Corollary 2.4, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{p^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{(2u)^\infty}\cong\mathcal{O}_{2u}\otimes M_{(2u)^\infty}\otimes M_{(2^v)^\infty}\cong\mathcal{O}_{2u}\otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{p^\infty}\otimes M_{(2^v)^\infty}\cong\mathcal{O}_{2u}\otimes M_{k^\infty}$. Thus by Theorem 2.1, $\mathcal{O}_{2u}\otimes T^k_\varrho$ is isomorphic to $\mathcal{O}_{2u}\otimes M_{k^\infty}\otimes T^k_\varrho$, which in turn is isomorphic to $\mathcal{O}_{2u}\otimes M_{k^\infty}\otimes A_\varrho\otimes M_k(\mathbb{C})$. Hence $\mathcal{O}_{2u}\otimes T^k_\varrho$ is isomorphic to $\mathcal{O}_{2u}\otimes A_\varrho\otimes M_k(\mathbb{C})$.

The converse was proved in Proposition 2.2.

Therefore, $\mathcal{O}_{2u}\otimes T^k_\varrho$ is isomorphic to $\mathcal{O}_{2u}\otimes A_\varrho\otimes M_k(\mathbb{C})$ if and only if k and $2u-1$ are relatively prime. ∎

Cuntz [8] computed the K-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty)=\mathbb{Z}$ and $K_1(\mathcal{O}_\infty)=0$. He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

2.6. Proposition. *$\mathcal{O}_\infty\otimes T^k_\varrho$ is not isomorphic to $\mathcal{O}_\infty\otimes A_\varrho\otimes M_k(\mathbb{C})$ if $k>1$.*

Proof. Suppose $\mathcal{O}_\infty\otimes T^k_\varrho$ is isomorphic to $\mathcal{O}_\infty\otimes A_\varrho\otimes M_k(\mathbb{C})$. The unit $1_{\mathcal{O}_\infty\otimes T^k_\varrho}$ maps to the unit $1_{\mathcal{O}_\infty\otimes A_\varrho}\otimes I_k$. By the same trick as in the proof of Proposition 2.2, one can show that $[1_{\mathcal{O}_\infty\otimes T^k_\varrho}]=k[e]$ for a projection $e\in\mathcal{O}_\infty\otimes T^k_\varrho$. We have $[1_{\mathcal{O}_\infty\otimes T^k_\varrho}]=[1_{\mathcal{O}_\infty}]\otimes[1_{T^k_\varrho}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [8]). So $[1_{T_\varrho^k}] = k[e']$ for a projection $e' \in T_\varrho^k$. This contradicts Theorem 1.4 if $k > 1$.

Therefore, $\mathcal{O}_\infty \otimes T_\varrho^k$ is not isomorphic to $\mathcal{O}_\infty \otimes A_\varrho \otimes M_k(\mathbb{C})$. ∎

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Received March 13, 2000
Revised version August 20, 2001 (4493)