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Morita equivalence of groupoid C^*-algebras arising from dynamical systems

by

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Abstract. We show that the stable C^*-algebra and the related Ruelle algebra defined by I. Putnam from the irreducible Smale space associated with a topologically mixing expanding map of a compact metric space are strongly Morita equivalent to the groupoid C^*-algebras defined directly from the expanding map by C. Anantharaman-Delaroche and V. Deaconu. As an application, we calculate the K_*-group of the Ruelle algebra for a solenoid.

1. Introduction. In [4, 5], Ian F. Putnam constructed C^*-algebras from irreducible Smale spaces. These Smale spaces include subshifts of finite type, solenoids and Anosov diffeomorphims. The corresponding C^*-algebras are defined as the reduced groupoid C^*-algebras for various equivalence relations.

Let X be a compact metric space, and let σ be an expanding surjection on X. It is well known that the system (X, σ) can give rise to a Smale space (Ω, φ), as defined in [8]. For the Smale space (Ω, φ), one has the groupoid G_{s} and its semidirect product $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ given by the stable equivalence relation on Ω (see [4]). Note that the groupoids G_{s} and $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ are not r-discrete. In [1] and [2], C. Anantharaman-Delaroche and V. Deaconu considered two r-discrete groupoids R_∞ and $G(X, \sigma)$ defined directly from the dynamical system (X, σ). Ian F. Putnam [5] conjectured that there should be an explicit relation between the groupoids $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and $G(X, \sigma)$. In this paper, using the results on the correspondence of groupoids given by M. Stadler and M. O'uchi [9], we show that these two groupoids are equivalent in the sense

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of Muhly–Renault–Williams [3]. Consequently, the reduced groupoid C^*-algebras $C^*_{\mathrm{r}}(G_{\mathrm{s}}\times_\alpha\mathbb{Z})$ and $C^*_{\mathrm{r}}(G(X,\sigma))$ are strongly Morita equivalent. Since $G(X,\sigma)$ is given directly by the dynamical system (X,σ), without using the complicated process of inverse limit, and more importantly, it is also r-discrete, it is more tractable than $G_{\mathrm{s}}\times_\alpha\mathbb{Z}$. As an application, we can compute the K_*-group of $C^*_{\mathrm{r}}(G_{\mathrm{s}}\times_\alpha\mathbb{Z})$ for solenoids.

We recall some definitions and notions.

Let G be a second countable locally compact Hausdorff groupoid. We denote by s (resp. r) the source (resp. range) map of G. The unit space is denoted by $G^{(0)}$. Set $G_x=s^{-1}(x)$, $G^x=r^{-1}(x)$ for $x\in G^{(0)}$. We denote by $G^{(2)}$ the set of composable pairs, and by $C^*_{\mathrm{r}}(G)$ the reduced groupoid C^*-algebra of G. For general results on groupoids and groupoid C^*-algebras, we refer to [7].

Definition 1.1 (see [8]). Let X be a compact metric space, with metric d, and σ a continuous surjective map of X. We say that σ is *expanding* if there exist a positive number ε and $\lambda\in(0,1)$ such that the following property holds:

If $d(\sigma(x),y')<2\varepsilon$, then there exists a unique y such that $\sigma(y)=y'$ and $d(x,y)<2\varepsilon$. Furthermore, $d(x,y)\le\lambda d(\sigma(x),\sigma(y))$.

Remark. By the definition, $d(\sigma x,\sigma y)\ge\lambda^{-1}d(x,y)$ whenever $d(x,y)<2\lambda\varepsilon$; and σ is *expansive*, i.e., there is a constant $c>0$ such that $x\ne y$ implies $d(\sigma^n(x),\sigma^n(y))\ge c$ for some integer $n\ge 0$. In particular, σ is an open map and a local homeomorphism.

Let σ be an expanding map of a compact metric space X. We define

$$\Omega=\{(x_n)_{n\ge0}\mid x_n\in X,\ \sigma(x_{n+1})=x_n,\ n=0,1,\ldots\}.$$

Then Ω is a compact metric space with respect to the metric

$$d(x,y)=\sum_{n\ge0}\frac{d(x_n,y_n)}{2^n}\quad\text{ for } x=(x_n)_{n\ge0},y=(y_n)_{n\ge0}\in\Omega.$$

Let φ and φ^{-1} be two maps from Ω onto itself defined by

$$(\varphi(x))_n=\sigma x_n,\quad (\varphi^{-1}(x))_n=x_{n+1}\quad\text{ for } n\ge0 \text{ and } x=(x_k)_{k\ge0}\in\Omega.$$

Then φ is a homeomorphism of Ω with inverse φ^{-1}. Obviously, $\sigma\pi=\pi\varphi$, where π is the continuous open map from Ω onto X defined by

$$\pi(x)=x_0\quad\text{ for } x=(x_n)_{n\ge0}\in\Omega.$$

Remark. By [8], Ω with the homeomorphism φ is a Smale space, called the Smale space canonically associated with the expanding map σ; moreover σ is topologically transitive (respectively, mixing) if and only if φ is. For general results on Smale spaces, we refer to [4, 8].

Throughout this paper let (Ω, d, φ) be the Smale space associated with an expanding dynamics (X, σ). We assume that σ is topologically mixing, hence the Smale space (Ω, d, φ) is irreducible. By the definition of a Smale space, for every x in Ω, there are two subsets $V^{\mathrm{s}}(x, \varepsilon)$ and $V^{\mathrm{u}}(x, \varepsilon)$ of Ω such that $V^{\mathrm{u}}(x, \varepsilon) \times V^{\mathrm{s}}(x, \varepsilon)$ is homeomorphic to a neighborhood of x in Ω, where ε is some small positive parameter.

Consider the stable equivalence G_{s} on Ω and its semidirect product with $\mathbb{Z}$, as defined in [4, 6]. We recall from [4], [6] that

$$G_{\mathrm{s}} = \{(x, y) \in \Omega \times \Omega \mid d(\varphi^n(x), \varphi^n(y)) \to 0 \text{ as } n \to +\infty\},$$
$$G_{\mathrm{s}} \times_\alpha \mathbb{Z} = \{(x, n, y) \in \Omega \times \mathbb{Z} \times \Omega \mid n \in \mathbb{Z},\ (\varphi^n(x), y) \in G_{\mathrm{s}}\}.$$

Then G_{s} is a principal groupoid and $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ is a groupoid with the range map, source map, inverse map and multiplication given respectively by

$$r(x, n, y) = x, \qquad s(x, n, y) = y,$$
$$(x, n, y)^{-1} = (y, -n, x), \qquad (x, m, y)(y, n, z) = (x, n + m, z).$$

Under the maps $x \in \Omega \mapsto (x, x) \in G_{\mathrm{s}}$ and $x \in \Omega \mapsto (x, 0, x) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, we regard Ω as the unit space of G_{s} and $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, respectively.

Let

$$G_{\mathrm{s}}^0 = \{(x, y) \in \Omega \times \Omega : y \in V^{\mathrm{s}}(x, \varepsilon_0)\}, \qquad G_{\mathrm{s}}^n = (\varphi \times \varphi)^{-n}(G_{\mathrm{s}}^0)$$

for each $n = 1, 2, \ldots$, where $V^{\mathrm{s}}(x, \varepsilon_0)$ is as in [4]. Then

$$G_{\mathrm{s}} = \bigcup_{n=1}^{\infty} G_{\mathrm{s}}^n.$$

If each G_{s}^n is given the relative topology of $\Omega \times \Omega$ and G_{s} is given the inductive limit topology, then G_{s}, called the *stable equivalence*, is a second countable locally compact Hausdorff principal groupoid. To obtain a Haar system for G_{s}, we proceed as follows. Let μ be the Bowen measure on the Smale space (Ω, φ). Fix x in Ω, and restrict μ to the set $V^{\mathrm{u}}(x, \varepsilon) \times V^{\mathrm{s}}(x, \varepsilon)$ which can be identified with a neighborhood of x in Ω. Then the restriction of the measure μ to a neighborhood of x is a product measure $\mu_{\mathrm{u}}^x \times \mu_{\mathrm{s}}^x$. Here the measures μ_{u}^x and μ_{s}^x depend on x and satisfy some conditions. Fix $x \in \Omega$. Let δ_x denote the unit mass at x. We define a measure on G_{s}^0 by $\delta_x \times \mu_{\mathrm{s}}^x$ and then on G_{s}^n by

$$\lambda^{-n}(\delta_{\varphi^n(x)} \times \mu_{\mathrm{s}}^{\varphi^n(x)}) \circ (\varphi \times \varphi)^n,$$

where $\lambda > 1$ and $\log(\lambda)$ is the topological entropy of (Ω, φ). In this way, we obtain a measure $\widetilde{\mu_{\mathrm{s}}^x}$ on G_{s}. By [4, 5], $\{\widetilde{\mu_s^x} : x \in G_{\mathrm{s}}^{(0)}\}$ forms a Haar system for G_{s}.

The map η sending $((x, y), n)$ in $G_{\mathrm{s}} \times \mathbb{Z}$ to $(x, n, \varphi^n(y))$ in $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ is bijective, and we transfer the product topology from $G_{\mathrm{s}} \times \mathbb{Z}$ over $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ via this map. With this topology, $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ is a second countable locally compact Hausdorff groupoid with a Haar system (see [5]).

For a dynamical system (X,σ), C. Anantharaman-Delaroche [1] and V. Deaconu [2] considered the following groupoid $G(X,\sigma)$:

$$G(X,\sigma) = \{(x,n,y) \in X \times \mathbb{Z} \times X \mid \exists k,l \geq 0,\ n = k-l,\ \sigma^k(x) = \sigma^l(y)\}$$

with the range map, source map, inverse map and multiplication given respectively by

$$r(x,n,y) = x, \qquad s(x,n,y) = y,$$
$$(x,n,y)^{-1} = (y,-n,x), \qquad (x,m,y)(y,n,z) = (x,n+m,z).$$

Then the unit space X is embedded in $G(X,\sigma)$ by $x \mapsto (x,0,x)$. Moreover $G(X,\sigma)$ is given the topology having as a basis of open sets those of the form

$$\Lambda(U,V,k,l) = \{(x,k-l,\sigma_V^{-l}\sigma^k(x)) \mid x \in U\},$$

where U and V are open subsets of X, and $k,l \geq 0$ are such that $\sigma^k|_U$ and $\sigma^l|_V$ are homeomorphisms with the same range (and σ_V^{-l} is the inverse of σ_V^l). Then $G(X,\sigma)$ is a second countable locally compact Hausdorff r-discrete groupoid and the counting measure is a Haar system.

We denote by $C_{\mathrm{r}}^*(G_{\mathrm{s}})$, $C_{\mathrm{r}}^*(G_{\mathrm{s}} \times_\alpha \mathbb{Z})$ and $C_{\mathrm{r}}^*(G(X,\sigma))$ the reduced groupoid C^*-algebra of the groupoid G_{s}, $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and $G(X,\sigma)$, respectively. In [4], the C^*-algebra $C_{\mathrm{r}}^*(G_{\mathrm{s}} \times_\alpha \mathbb{Z})$ is called the *Ruelle algebra* associated with the stable equivalence.

In the rest of this section, we recall some results on the correspondence and equivalence of groupoids. For $i = 1,2$, let G_i be a second countable locally compact Hausdorff groupoid and let Z be a second countable locally compact Hausdorff space.

Definition 1.2 (see [3, 9]). A *left action* of G_1 on Z is a pair of maps consisting of a continuous surjection $\varrho : Z \to G_1^{(0)}$ and a continuous map $(g,z) \mapsto g \cdot z$ from the space $G_1 * Z = \{(g,z) \in G_1 \times Z \mid s(g) = \varrho(z)\}$ of composable pairs into Z, having the following properties:

1. $\varrho(g \cdot z) = r(g)$ for $(g,z) \in G_1 * Z$;
2. if $(g',g) \in G_1^{(2)}$, $(g,z) \in G_1 * Z$, then $g' \cdot (g \cdot z) = (g'g) \cdot z$;
3. $\varrho(z) \cdot z = z$ for $z \in Z$.

We say that the left G_1-action is *proper* if the map $(g,z) \in G_1 * Z \mapsto (g \cdot z, z) \in Z \times Z$ is proper, that is, the inverse images of compact sets are compact. We say that the left G_1-action is *free* if $g \cdot z = z$ only when g is a unit, i.e., $g \in G_1^{(0)}$.

Remark. A right action of G_2 on Z is defined by a pair of maps consisting of a continuous surjection $\tau : Z \to G_2^{(0)}$ and a continuous map

$(z, g) \mapsto z \cdot g$ from the space $Z * G_2 = \{(z, g) \in Z \times G_2 \mid \tau(z) = r(g)\}$ of composable pairs into Z, with properties similar to those above. Similarly, we can also define proper and free actions of G_2 on Z.

DEFINITION 1.3 (see [9]). Let G_1 and G_2 be two second countable locally compact Hausdorff groupoids and let Z be a second countable locally compact Hausdorff space. The space Z is called a *correspondence* from G_1 to G_2 if it has the following properties:

(i) there exists a proper left action of G_1 on Z such that ϱ is an open map;
(ii) there exists a proper right action of G_2 on Z;
(iii) the G_1 and G_2 actions commute;
(iv) the map ϱ induces a bijection of Z/G_2 onto $G_1^{(0)}$.

DEFINITION 1.4 (see [3]). Let G_1, G_2 and Z be as in Definition 1.3. We say that the space Z is a (G_1, G_2)-*equivalence* if it has the following properties:

(i) there exists a proper free left action of G_1 on Z such that ϱ is an open map;
(ii) there exists a proper free right action of G_2 on Z such that τ is an open map;
(iii) the G_1 and G_2 actions commute;
(iv) the map ϱ induces a bijection of Z/G_2 onto $G_1^{(0)}$;
(v) the map τ induces a bijection of $G_1 \setminus Z$ onto $G_2^{(0)}$.

Let G_1 and G_2 be two second countable locally compact Hausdorff groupoids. We say G_1 and G_2 are *equivalent in the sense of Muhly–Renault–Williams* if there exists a (G_1, G_2)-equivalence.

THEOREM 1.1 (see [9]). *Let G_1 and G_2 be two second countable locally compact Hausdorff groupoids, let f be a continuous homomorphism of G_1 onto G_2 and let H be the kernel of f. Suppose that the following conditions are satisfied*:

(C1) *the range map $r : G_1 \to G_1^{(0)}$ is an open map*;
(C2) *the quotient map $q_1 : G_1 \to G_1/H$ is an open map*;
(C3) $(r, s)_H : H \to H^{(0)} \times H^{(0)}$ *is a proper map*;
(C4) $f(G_{1,x}) = G_{2,f(x)}$ *for all $x \in G_1^{(0)}$*;
(C5) *f is an open map*;
(C6) *the restriction $f^{(0)}$ of f to the unit space $G_1^{(0)}$ is locally one-to-one, and maps onto $G_2^{(0)}$.*

Then G_1/H is a correspondence from G_1 to G_2.

2. Main results. Let X be a compact metric space, let σ be an expanding surjection on X, and let (Ω,φ) be the Smale space associated with the map σ. Let $G_{\mathrm{s}}\times_\alpha\mathbb{Z}$ and $G(X,\sigma)$ be as in the Introduction. In this section we state and prove the following main result of this paper.

Theorem 2.1. *The groupoids $G_{\mathrm{s}}\times_\alpha\mathbb{Z}$ and $G(X,\sigma)$ are equivalent in the sense of Muhly–Renault–Williams. Consequently, the reduced groupoid C^*-algebras $C^*_{\mathrm{r}}(G_{\mathrm{s}}\times_\alpha\mathbb{Z})$ and $C^*_{\mathrm{r}}(G(X,\sigma))$ are Morita equivalent.*

In order to prove Theorem 2.1, we need the following lemmas. Firstly, for the topology on the groupoid G_{s}, by the definitions of Ω and φ, we have the following reduction.

Lemma 2.1. *Let x and y be in Ω. Then $(x,y)\in G_{\mathrm{s}}$ if and only if there exists an integer $k\ge 0$ such that $\sigma^k(\pi x)=\sigma^k(\pi y)$. Set $G_{\mathrm{s},k}=\{(x,y)\in\Omega\times\Omega\mid\sigma^k(\pi x)=\sigma^k(\pi y)\}$ for $k=0,1,2,\ldots$ and endow each $G_{s,k}$ with the relative topology of the product topology on $\Omega\times\Omega$, and $G_{\mathrm{s}}=\bigcup_{k=1}^\infty G_{\mathrm{s},k}$ with the inductive limit topology. Then the inductive limit topology agrees with the one defined by Putnam in* [4], *hence in that topology, G_{s} is a second countable locally compact Hausdorff groupoid with a left Haar system λ.*

Proof. Note that, for $x=(x_n)_{n\ge0}$, $y=(y_n)_{n\ge0}$ and $l>0$,

$$d(\sigma^l(x_0),\sigma^l(y_0))\le d(\varphi^l(x),\varphi^l(y))=\sum_{n=0}^{l-1}\frac{d(\sigma^{l-n}(x_0),\sigma^{l-n}(y_0))}{2^n}+\frac{d(x,y)}{2^l}.$$

If $(x,y)\in G_{\mathrm{s}}$, then $\lim_{l\to\infty}d(\sigma^l(x_0),\sigma^l(y_0))=0$. Since σ is expansive, there is $k\ge0$ such that $\sigma^k(x_0)=\sigma^k(y_0)$.

If there is $k\ge0$ such that $\sigma^k(x_0)=\sigma^k(y_0)$, then for $l\ge k$,

$$d(\varphi^l(x),\varphi^l(y))=\frac{1}{2^l}(d(x,y)+\ldots+2^{k-1}d(\sigma^{k-1}(x_0),\sigma^{k-1}(y_0))).$$

So $\lim_{l\to\infty}d(\varphi^l(x),\varphi^l(y))=0$.

The rest of the proof is straightforward and we omit it.

We define

$$f:G_{\mathrm{s}}\times_\alpha\mathbb{Z}\to G(X,\sigma),\qquad f(x,n,y)=(\pi(x),n,\pi(y)),$$

for $(x,n,y)\in G_{\mathrm{s}}\times_\alpha\mathbb{Z}$, and prove that the statements (C1)–(C5) are true for $G_{\mathrm{s}}\times_\alpha\mathbb{Z}$, $G(X,\sigma)$ and f.

Remark. We show that f is well defined. Let $(x,n,y)\in G_{\mathrm{s}}\times_\alpha\mathbb{Z}$. Then $(\varphi^n(x),y)\in G_{\mathrm{s}}$. Choose $k_0\ge1$ such that $k_0+n\ge1$. Then $(\varphi^{k_0+n}(x),\varphi^{k_0}(y))\in G_{\mathrm{s}}$. It follows from Lemma 2.1 that there exists an integer $l_0\ge0$ such that

$$\sigma^{l_0}\pi\varphi^{k_0+n}(x)=\sigma^{l_0}\pi\varphi^{k_0}(y).$$

Consequently, $\sigma^{l_0+k_0+n}\pi(x)=\sigma^{l_0+k_0}\pi(y)$, hence $(\pi(x),n,\pi(y))\in G(X,\sigma)$.

LEMMA 2.2. *The map f is a continuous homomorphism of $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ onto $G(X, \sigma)$. So $H = \ker(f)$ is an open and closed subgroupoid of $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$.*

Proof. It is clear that, by the definition of the topology on $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, we only need to check that the map

$$F = f\eta : G_{\mathrm{s}} \times \mathbb{Z} \to G(X, \sigma), \qquad F((x, y), n) = (\pi(x), n, \pi(\varphi^n(y))),$$

is continuous.

Fix $((x, y), n)$ in $G_{\mathrm{s}} \times \mathbb{Z}$, and assume $n \geq 0$. Then there exists $k > n$ such that $(x, y) \in G_{\mathrm{s},k}$. If we let $\pi(x) = x_0$ and $\pi(y) = y_0$, then $F((x, y), n) = (x_0, n, \sigma^n(y_0))$ and $\sigma^k(x_0) = \sigma^k(y_0)$. Set $n = k - l$. For every open neighborhood Λ of $(x_0, n, \sigma^n(y_0))$ in $G(X, \sigma)$, noting that $\sigma^k(x_0) = \sigma^l\sigma^n(y_0)$, we take two open subsets U and V of X such that $\sigma^k|_U$ and $\sigma^l|_V$ are homeomorphisms with the same range and $\Lambda(U, V, k, l)$ is a base neighborhood, contained in Λ, of $(x_0, n, \sigma^n(y_0))$ in $G(X, \sigma)$, where

$$\Lambda(U, V, k, l) = \{(u, k - l, \sigma_V^{-l}\sigma^k(u)) \mid u \in U\}.$$

Obviously, $x_0 \in U$, $\sigma^n(y_0) = \sigma_V^{-l}\sigma^k(x_0) \in V$. Since the map $\Phi(p, q) = (\pi(p), \pi(\varphi^n(q)))$, from $\Omega \times \Omega$ onto $X \times X$, is continuous, it follows that $\Phi^{-1}(U \times V)$ is an open subset of $\Omega \times \Omega$. Noting that $(x, y) \in \Phi^{-1}(U \times V)$, one finds that $W = \Phi^{-1}(U \times V) \cap G_{\mathrm{s},k}$ is a neighborhood of (x, y) in G_{s}, and therefore, that $W \times \{n\}$ is a neighborhood of $((x, y), n)$ in $G_{\mathrm{s}} \times \mathbb{Z}$. In order to establish the claim, we only need to prove that $F(W \times \{n\}) \subseteq \Lambda(U, V, k, l)$.

In fact, let $((p, q), n) \in W \times \{n\}$ with $p = (p_n)_{n \geq 0}$ and $q = (q_n)_{n \geq 0}$. Then $(p_0, \sigma^n(q_0)) \in U \times V$ and $\sigma^k(p_0) = \sigma^k(q_0)$. Therefore $\sigma^k(p_0) = \sigma^k(q_0) = \sigma^{n+l}(q_0)$. Since $\sigma^n(q_0) \in V$ and $p_0 \in U$, we have $\sigma^n(q_0) = \sigma_V^{-l}(\sigma^k(p_0))$, which implies that $F((p, q), n) = (p_0, n, \sigma^n(q_0)) \in \Lambda(U, V, k, l)$.

For the case $n < 0$, noting that $\pi\varphi^n(y) = y_{-n}$ and $\sigma^{-n}(y_{-n}) = y_0$, we can prove that F is continuous at $((x, y), n)$ in a similar way.

Since $G(X, \sigma)$ is r-discrete, the unit space $G(X, \sigma)^{(0)}$ is an open and closed subset of $G(X, \sigma)$. Therefore H is an open and closed subgroupoid of $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ by the continuity of f.

LEMMA 2.3 (see [10]). *Let G be a locally compact groupoid with a left Haar system $\lambda = \{\lambda^u \mid u \in G^{(0)}\}$. Let G' be an open subgroupoid of G with an induced measure system $\lambda' = \{\lambda^u|_{G'} \mid u \in G'^{(0)}\}$, where $G'^{(0)} = G' \cap G^{(0)}$. Then $\lambda' = \{\lambda^u|_{G'} \mid u \in G'^{(0)}\}$ is a left Haar system of G'.*

LEMMA 2.4 (see [7]). *Let G be a locally compact groupoid with a left Haar system λ and H a closed subgroupoid of G containing $G^{(0)}$ and admitting a Haar system λ_H. Consider the relation on G defined by $x \sim y$ if and only if $r(x) = r(y)$ and $x^{-1}y \in H$. Then*

(1) *the range map $r : G \to G^{(0)}$ is an open map*;

(2) *the quotient space G/H defined by the above equivalence relation is locally compact and Hausdorff in the quotient topology, and the quotient map $G \to G/H$ is a continuous open map.*

From Lemmas 2.3 and 2.4, we have the following corollary.

COROLLARY 2.1. *Let $H = \ker(f)$. Then*

(1) *the quotient map $q_1 : G_{\mathrm{s}} \times_\alpha \mathbb{Z} \to (G_{\mathrm{s}} \times_\alpha \mathbb{Z})/H$ is continuous and open*;

(2) *the range map $r : G_{\mathrm{s}} \times_\alpha \mathbb{Z} \to (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(0)}$ is open.*

Proof. Since H is an open and closed subgroupoid of $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ by Lemma 2.2, and also since $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ is a locally compact groupoid with a left Haar system, one sees that H has a left Haar system by Lemma 2.3. Hence Lemma 2.4 implies (1) and (2).

LEMMA 2.5. *The map $(r,s)_H : H \to H^{(0)} \times H^{(0)}$ is a proper map.*

Proof. It is clear that $H = \{(x,0,y) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z} \mid \pi(x) = \pi(y)\}$. Then $\eta^{-1}(H) = \{((x,y),0) \in G_{\mathrm{s}} \times \mathbb{Z} \mid \pi(x) = \pi(y)\}$, which is homeomorphic to $\{(x,y) \in G_{\mathrm{s}} \mid \pi(x) = \pi(y)\} = G_{\mathrm{s},0}$. Noting that $G_{\mathrm{s},0}$ is a compact subset of G_{s} in the inductive limit topology defined in Lemma 2.1, we deduce that H is compact in $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, and hence $(r,s)_H$ is a proper map.

LEMMA 2.6. $f((G_{\mathrm{s}} \times_\alpha \mathbb{Z})_x) = G(X,\sigma)_{f(x)}$ *for all* $x \in \Omega \cong (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(0)}$.

Proof. Straightforward.

LEMMA 2.7. *The homomorphism f is an open map.*

Proof. It is clear that we need to prove that the map $F = f\eta$ from $G_{\mathrm{s}} \times \mathbb{Z}$ to $G(X,\sigma)$ defined by $F((x,y),n) = (\pi(x), n, \pi\varphi^n(y))$ is an open map.

By Lemma 2.1, G_{s} has the topology having as a basis of open sets the sets

$$\Lambda(U,V,k) = \{(x,y) \in U \times V \mid \pi\varphi^k(x) = \pi\varphi^k(y)\},$$

where U, V are open subsets of Ω, and k is a positive integer; hence we only need to show that $F(\Lambda(U,V,k) \times \{n\})$ is an open subset of $G(X,\sigma)$ for all $n \in \mathbb{Z}$. Without loss generality, we assume that $n \geq 0$.

Fix $(x_0, n, u_0) \in F(\Lambda(U,V,k) \times \{n\})$. Then there exists $(x,y) \in \Lambda(U,V,k)$ such that $F((x,y),n) = (x_0,n,u_0)$. Hence $x \in U$, $y \in V$, $\pi(x) = x_0$, $\pi\varphi^n(y) = u_0$ and $\pi\varphi^k(x) = \pi\varphi^k(y)$. If we set $\pi(y) = y_0$, then $\sigma^n(y_0) = \pi\varphi^n(y) = u_0$, $\sigma^k(x_0) = \sigma^k(y_0)$. Since π is an open map and σ is a local homeomorphism, we can choose neighborhoods V_{x_0} and V_{y_0} of x_0 and y_0 respectively in X such that the following conditions hold:

(i) $V_{x_0} \subseteq \pi(U)$, $V_{y_0} \subseteq \pi(V)$;
(ii) $\sigma^k|_{V_{x_0}}$ and $\sigma^k|_{V_{y_0}}$ are homeomorphisms with the same open range;
(iii) $\sigma^n|_{V_{y_0}}$ is a homeomorphism;

(iv) $\sigma^n|_{\sigma^k(V_{x_0})}$ $(= \sigma^n|_{\sigma^k(V_{y_0})})$ and $\sigma^k|_{\sigma^n(V_{y_0})}$ are homeomorphisms with the same open range.

By (iv), we have $(x_0, n, u_0) = (x_0, n, \sigma_{V_{u_0}}^{-k}(\sigma^{k+n}(x_0)))$, which belongs to $\Lambda(V_{x_0}, V_{u_0}, k+n, k)$, where $V_{u_0} = \sigma^n(V_{y_0})$, a neighborhood of u_0. So the claim will follow from $\Lambda(V_{x_0}, V_{u_0}, k+n, k) \subseteq F(\Lambda(U, V, k) \times \{n\})$.

Fix (s_0, n, t_0) in $\Lambda(V_{x_0}, V_{u_0}, k+n, k)$. Then $s_0 \in V_{x_0}$, $t_0 = \sigma_{V_{u_0}}^{-k}(\sigma^{k+n}(s_0)) \in V_{u_0}$, and hence $\sigma^{k+n}(s_0) = \sigma^k(t_0)$. Choose $s \in U$ and $r \in V$ such that $\pi(s) = s_0$, $\pi(r) = r_0 \in V_{y_0}$ and $t_0 = \sigma^n(r_0)$. Then we have $(s_0, n, t_0) = F((s, r), n)$. Since $\sigma^n(\sigma^k(s_0)) = \sigma^{k+n}(s_0) = \sigma^k(t_0) = \sigma^k(\sigma^n(r_0)) = \sigma^n(\sigma^k(r_0))$ and both $\sigma^k(r_0)$ and $\sigma^k(s_0)$ are in $\sigma^k(V_{x_0})$ from (ii), condition (iv) implies that $\sigma^k(s_0) = \sigma^k(r_0)$ and so $((s, r), n) \in \Lambda(U, V, k) \times \{n\}$. Consequently, $(s_0, n, t_0) = F((s, r), n) \in F(\Lambda(U, V, k) \times \{n\})$, which implies that $\Lambda(V_{x_0}, V_{u_0}, k+n, k) \subseteq F(\Lambda(U, V, k) \times \{n\})$.

Remark. From Corollary 2.1 and Lemmas 2.5–2.7, conditions (C1)–(C5) in Theorem 1.1 hold for $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, $G(X, \sigma)$ and f, but (C6) does not hold. However, by inspecting the proof of Theorem 1.1 in [9], we find that condition (C6) is only used to prove that the right action of G_2 on Z is continuous. In our case, without (C6), we can still prove the continuity of the right action.

We recall from [9] the left and right actions on $Z = (G_{\mathrm{s}} \times_\alpha \mathbb{Z})/H$. Let $\varrho : Z \to (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(0)}$ be defined by $\varrho(q_1(x, n, y)) = r(x, n, y) = x$, where q_1 is the quotient map of Corollary 2.1. Then we have $(G_{\mathrm{s}} \times_\alpha \mathbb{Z}) * Z = \{((x, n, y), q_1(y, m, z)) \in (G_{\mathrm{s}} \times_\alpha \mathbb{Z}) \times Z\}$ and the left action of $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ on Z is defined as follows:

$$(x, n, y)q_1(y, m, z) = q_1(x, n+m, z)$$

for $((x, n, y), q_1(y, m, z)) \in (G_{\mathrm{s}} \times_\alpha \mathbb{Z}) * Z$.

Let $\tau : Z \to G(X, \sigma)^{(0)}$ be defined by $\tau(q_1(x, n, y)) = sf(x, n, y) = \pi(y)$. Then we have the right action, denoted by α_2, of $G(X, \sigma)$ on Z:

$$\alpha_2(q_1(x, n, y), (\pi(y), m, p_0)) = q_1(x, n+m, p)$$

for $(q_1(x, n, y), (\pi(y), m, p_0)) \in Z * G(X, \sigma)$, where p is an arbitrary element in Ω such that $\pi(p) = p_0$.

Lemma 2.8. *The right action α_2 of $G(X, \sigma)$ on Z is continuous.*

Proof. We first observe that α_2 is well defined.

Let $(q_1(x, n, y), (\pi(y), m, p_0))$ be an element in $Z * G(X, \sigma)$ and let $p \in \Omega$ satisfy $\pi(p) = p_0$. Since $(x, n, y) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$, we have $(\varphi^n(x), y) \in G_{\mathrm{s}}$ and so $(\varphi^{n+m}(x), \varphi^m(y))$ is in G_{s}. Also since $(\pi(y), m, p_0) \in G(X, \sigma)$, there are integers $k, l \geq 0$ such that $m = k - l$ and $\sigma^k\pi(y) = \sigma^l\pi(p)$ $(= \sigma^l(p_0))$. Hence, $\pi\varphi^k(y) = \pi\varphi^l(p)$, and so $(\varphi^k(y), \varphi^l(p)) \in G_{\mathrm{s}}$, which implies $(\varphi^m(y), p) \in G_{\mathrm{s}}$. Consequently, $(\varphi^{m+n}(x), p) \in G_{\mathrm{s}}$, hence $(x, n+m, p) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$. On the other hand, if we have another element $p' \in \Omega$ such that $\pi(p') = \pi(p) = p_0$, then by the definition of Z, it is clear that $q_1(x, n+m, p) = q_1(x, n+m, p')$ in Z. This establishes the observation.

We denote by β_1 the multiplication from $(G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(2)}$, the set of composable pairs, into $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ defined by $\beta_1(\gamma, \gamma') = \gamma\gamma'$. Fix an element $(q_1(x, n, y), (\pi(y), m, p_0)) \in Z * G(X, \sigma)$. We show that α_2 is continuous at $(q_1(x, n, y), (\pi(y), m, p_0))$.

Let V be an open neighborhood of $q_1(x, n+m, p)$ in Z, where $q_1(x, n+m, p) = \alpha_2(q_1(x, n, y), (\pi(y), m, p_0))$, $p \in \Omega$ and $\pi(p) = p_0$. Corollary 2.1 implies that $q_1^{-1}(V)$ is an open neighborhood of $(x, n+m, p) = \beta_1((x, n, y), (y, m, p))$ (note that if $(\pi(y), m, \pi(p)) \in G(X, \sigma)$ then $(y, m, p) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$). Since β_1 is continuous, there exist an open neighborhood U of (x, n, y) and an open neighborhood W of (y, m, p) in $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ such that $\beta_1((U \times W) \cap (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(2)}) \subseteq q_1^{-1}(V)$. By the definition of the topology on $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and Lemma 2.1, we can suppose that

$$U = \eta(\widetilde{U} \times \{n\}), \qquad W = \eta(\widetilde{W} \times \{m\}),$$

where

$$\widetilde{U} = \Lambda(V_x, \varphi^{-n}(V_y), k) = \{(u, v) \in V_x \times \varphi^{-n}(V_y) \mid \sigma^k \pi(u) = \sigma^k \pi(v)\},$$
$$\widetilde{W} = \Lambda(V_y, \varphi^{-m}(V_p), k) = \{(u, v) \in V_y \times \varphi^{-m}(V_p) \mid \sigma^k \pi(u) = \sigma^k \pi(v)\},$$

V_x, V_y and V_p are neighborhoods of x, y and p respectively in Ω and $k \geq 0$. Let $U' = q_1(U)$ and $W' = f(W)$. From Corollary 2.1 and Lemma 2.7, one sees that U' and W' are open neighborhoods of $q_1(x, n, y)$ and $f(y, m, p) = (\pi(y), m, \pi(p)) = (\pi(y), m, p_0)$ in Z and $G(X, \sigma)$, respectively. We only need to show that $\alpha_2((U' \times W') \cap (Z * G(X, \sigma)))$ is contained in V.

For, let $(a, n, b) \in U, (b', m, q) \in W$ be such that $(q_1(a, n, b), f(b', m, q))$ belongs to $(U' \times W') \cap (Z * G(X, \sigma))$. Then

$$\pi(b) = \pi(b'), \qquad \alpha_2(q_1(a, n, b), f(b', m, q)) = q_1(a, n+m, q).$$

By the choice of U and W, we have $a \in V_x$, $b \in V_y$, $\sigma^k \pi(a) = \sigma^k \pi \varphi^{-n}(b)$ and also $b' \in V_y$, $q \in V_p$, $\sigma^k \pi(b') = \sigma^k \pi \varphi^{-m}(q)$. Therefore $\sigma^k \pi(b) = \sigma^k \pi(b') = \sigma^k \pi \varphi^{-m}(q)$, which implies that $(b, \varphi^{-m}(q)) \in \widetilde{W}$. Consequently, $(b, m, q) = \eta((b, \varphi^{-m}(q)), m)$, which belongs to W. Also since

$$\alpha_2(q_1(a, n, b), f(b', m, q)) = q_1(a, n+m, q) = q_1\beta_1((a, n, b), (b, m, q)),$$

we have $\alpha_2(q_1(a, n, b), f(b', m, q)) \in q_1\beta_1((U \times W) \cap (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(2)}) \subseteq V$. This completes the proof.

From the above arguments and Theorem 1.1, we have the following corollary.

Corollary 2.2. *The space $Z = (G_{\mathrm{s}} \times_\alpha \mathbb{Z})/H$ is a correspondence from $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ to $G(X, \sigma)$.*

LEMMA 2.9. *The map $\tau : G_{\mathrm{s}} \times_\alpha \mathbb{Z}/H \to G(X,\sigma)^{(0)}$ is an open map.*

Proof. By the definition of τ, it is clear that $\tau q_1 = sf$. Note that s and f are continuous open surjections. Then τq_1 is a continuous open surjection. Also since q_1 is a continuous open surjection, τ is an open map.

LEMMA 2.10. *The left and right actions are free.*

Proof. Let $(x,n,y) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and $q_1(y,m,q) \in Z$ be such that

$$(x,n,y)q_1(y,m,q) = q_1(y,m,q).$$

Then from the definition of the left action, $q_1(x,n+m,q) = q_1(y,m,q)$ in Z. It follows that $x = y$, $n+m = m$, and therefore $(x,n,y) = (x,0,x) \in (G_{\mathrm{s}} \times_\alpha \mathbb{Z})^{(0)}$. Consequently, the left action is free.

Let $q_1(x,n,y) \in Z$ and $(\pi(y),m,p_0) \in G(X,\sigma)$ be such that

$$q_1(x,n,y)(\pi(y),m,p_0) = q_1(x,n,y).$$

Then from the definition of the right action, one has $m = 0$ and $p_0 = \pi(y)$, which implies that $(\pi(y),m,p_0) = (\pi(y),0,\pi(y)) \in G(X,\sigma)^{(0)}$. So the right action is free.

LEMMA 2.11. *The map τ induces a bijection τ' from $G_{\mathrm{s}} \times_\alpha \mathbb{Z} \setminus Z$ onto $G(X,\sigma)^{(0)}$.*

Proof. We recall that τ' is defined as follows: $\tau'([q_1(x,n,y)]) = \pi(y)$, where $[q_1(x,n,y)]$ denotes the equivalence class of $q_1(x,n,y)$ in $G_{\mathrm{s}} \times_\alpha \mathbb{Z} \setminus Z$.

Obviously, τ' is surjective. We have to prove its injectivity. Suppose that $\tau'([q_1(x,n,y)]) = \tau'([q_1(p,m,q)])$, i.e., $\pi(y) = \pi(q)$. Since (x,n,y) and (p,m,q) belong to $G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and $\pi(y) = \pi(q)$, we have $g = (p,m-n,x) \in G_{\mathrm{s}} \times_\alpha \mathbb{Z}$ and $gq_1(x,n,y) = q_1(p,m,y)$. Also since $q_1(p,m,y) = q_1(p,m,q)$ in Z, we have $gq_1(x,n,y) = q_1(p,m,q)$, which implies that $[q_1(x,n,y)] = [q_1(p,m,q)]$ in $G_{\mathrm{s}} \times_\alpha \mathbb{Z} \setminus Z$.

Proof of Theorem 2.1. Definition 1.4, Corollary 2.2 and Lemmas 2.9–2.11 complete the proof of the theorem.

REMARK. In the same way, one can prove that G_{s} is equivalent to the r-discrete groupoid R_∞ defined as in [2]. Therefore, $C^*_{\mathrm{r}}(G_{\mathrm{s}})$ is Morita equivalent to $C^*_{\mathrm{r}}(R_\infty)$.

3. Example. As an application of Theorem 2.1, we can calculate the K_*-groups of the groupoid C^*-algebras in the case of solenoids.

Let $X = \mathbb{S}^1$, the unit circle in the complex plane, and let $\sigma(x) = x^p$, $p \geq 2$. Define $d(e^{i\alpha_1}, e^{i\alpha_2}) = |\alpha_1 - \alpha_2| \leq \pi$ as the metric on $\mathbb{S}^1$. Obviously, σ is an expanding map. The dynamical system $(\mathbb{S}^1,\sigma)$ gives rise to a Smale space (Ω, φ), which is called a solenoid. From [2], we know that

$$K_0(C^*_{\mathrm{r}}(R_\infty)) \cong \mathbb{Z}[1/p], \qquad K_1(C^*_{\mathrm{r}}(R_\infty)) \cong \mathbb{Z},$$
$$K_0(C^*_{\mathrm{r}}(G(\mathbb{S}^1, \sigma))) \cong \mathbb{Z} \oplus \mathbb{Z}_{p-1}, \quad K_1(C^*_{\mathrm{r}}(G(\mathbb{S}^1, \sigma))) \cong \mathbb{Z}.$$

Since C^*-algebras which are Morita equivalent have the same K_*-groups, we have

$$K_0(C^*_{\mathrm{r}}(G_{\mathrm{s}})) \cong \mathbb{Z}[1/p], \qquad K_1(C^*_{\mathrm{r}}(G_{\mathrm{s}})) \cong \mathbb{Z},$$
$$K_0(C^*_{\mathrm{r}}(G_{\mathrm{s}} \times_\alpha \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}_{p-1}, \quad K_1(C^*_{\mathrm{r}}(G_{\mathrm{s}} \times_\alpha \mathbb{Z})) \cong \mathbb{Z}.$$

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