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Squeezing the Sierpiński sponge

by

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Abstract. We give an example relating to the regularity properties of mappings with finite distortion. This example suggests conditions to be imposed on the distortion function in order to avoid "cavitation in measure".

1. Introduction. Recently a cohesive theory of mappings of finite distortion has begun to emerge as a generalization of the theory of quasiconformal mappings. The connections with non-linear elasticity as developed by Antman and Ball [1, 3] are a primary motivation for this extension of the classical theory. Mappings of finite distortion appear naturally as the minimizers of certain non-linear stored energy functionals. The regularity properties of these solutions are issues of fundamental importance in the theory. Here we provide an example which appears to us to lead naturally to a precise conjecture relating the integrability properties of a distortion functions and the associated geometric properties of the mapping. Let us first give a definition.

DEFINITION. A mapping $f : \Omega \to \mathbb{R}^n$ is said to have *finite distortion* if:

(i) $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$,

(ii) the Jacobian determinant of f is locally integrable and does not change sign in Ω,

(iii) there is a measurable *distortion* function $\mathcal{K} = \mathcal{K}(x) \geq 1$, finite almost everywhere, such that f satisfies the *distortion inequality*

$$|Df(x)|^n \leq \mathcal{K}(x)|J(x, f)| \quad \text{a.e. in } \Omega. \tag{1}$$

The conditions (i)–(iii) above are *not* enough to imply $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, unless of course the distortion $\mathcal{K}$ is a bounded function. If $\mathcal{K}(x) \leq K$, then f is referred to as a K-*quasiregular mapping*, or quasiconformal if f is additionally a homeomorphism (onto its image); see [15]. There is an extant

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literature concerning the cases when $\mathcal{K}$ is not bounded, but well controlled. For instance, openness and discreteness of such mappings were treated in [7, 11, 12]. Also, if $\mathcal{K}$ is bounded by a BMO function, then the interplay between the dual spaces BMO and $\mathcal{H}_1$, or a better space where the Jacobian might lie [14], can be exploited to develop an interesting theory. See [2, 8, 9, 16] where the pioneering work of David [4] is extended, particularly in the planar case.

The relevance of the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ in which one looks for the solutions of the distortion inequality (1) is clear. However, one of the major advances in the modern theory of quasiconformal mappings relies fundamentally on the properties of very weak solutions to (1). Very weak solutions are those which *a priori* belong to a weaker class than the natural space $W^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ on which the classical theory is based. The goal in dealing with very weak solutions is to show, using properties of the distortion function, that in fact they are classical solutions and perhaps in an even higher Sobolev class, e.g. $W^{1,n+\varepsilon}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. Such improved regularity largely relies on integrability properties of the Jacobian determinant. We mention here that estimates for very weak solutions, below the natural exponent n, are essential in studying the nature of the singularities of mappings of finite (or even bounded) distortion. These things are discussed in some detail in our monograph [10].

We further emphasise that the three conditions (i)–(iii) above do not *a priori* guarantee that the Jacobian does not vanish on a set of positive measure. That in certain circumstances, for instance $\mathcal{K}$ bounded, the Jacobian does not vanish on such sets is a deep analytic fact.

There are a few things to note concerning the definition of a mapping of finite distortion. Because the differential inequality at (1) is assumed to hold only almost everywhere, the values of $\mathcal{K}$ on a set of measure zero are immaterial. For this reason $\mathcal{K}$ is treated as a function which is only finite almost everywhere. Furthermore, Hadamard's inequality asserts that pointwise $J(x,f) \le |Df(x)|^n$, thus the assumption $\mathcal{K}(x) \ge 1$ is imposed on us.

Here we establish the following theorem.

THEOREM 1.1. *Let $\mathbf{Q} = [-1,1]^n$ denote the unit cube in $\mathbb{R}^n$. There is a homeomorphism $f : \mathbf{Q} \to \mathbb{R}^n$ such that*

- *f is a Lipschitz map and therefore*
- *f lies in every Sobolev class $W^{1,p}(\mathbf{Q},\mathbb{R}^n)$, $1 \le p \le \infty$,*
- *f is a mapping of finite distortion,*
- *$H(\cdot,f), K_I(\cdot,f) \in L\log^\alpha L(\mathbf{Q})$ for all $\alpha < -1$ but not for $\alpha = -1$,*
- *$K_O(\cdot,f) \in L^p(\mathbf{Q})$ for all $p < 1/(n-1)$ but not for $p = 1/(n-1)$,*
- *$J(x,f)$ vanishes on a set S of positive measure,*
- *$f(S)$ has zero measure.*

Moreover $f(\mathbf{Q}) = \mathbf{Q}$ *and the inverse map* $g : \mathbf{Q} \to \mathbf{Q}$ *has the following properties*:

- g *lies in every Sobolev class* $W^{1,p}(\mathbf{Q}, \mathbb{R}^n)$, $p < n$,
- g *is a mapping of finite distortion*,
- $K_I(\cdot, g) \in L^q$ *if and only if* $q < n/(n-1)$,
- $K_O(\cdot, g) \in L^p$ *if and only if* $p < n$ *and*
- g *maps some set of zero measure* (*and Hausdorff dimension* n) *onto a set of positive measure.*

Here we are referring to the following distortion functions which we recall for the reader:

- the *linear* distortion

$$H(x, f) = |Df(x)| \cdot |Df(x)^{-1}| = \frac{\max\{|Df(x)\zeta| : |\zeta| = 1\}}{\min\{|Df(x)\zeta| : |\zeta| = 1\}}, \tag{2}$$

- the *outer* distortion

$$K_O(x, f) = \frac{|Df(x)|^n}{J(x, f)}, \tag{3}$$

- the *inner* distortion

$$K_I(x, f) = K_O(f(x), f^{-1}) = \frac{|D^{\#} f(x)|^n}{J(x, f)^{n-1}}, \tag{4}$$

where $D^{\#} f(x)$ denotes the $n \times n$ matrix whose entries are the $(n-1) \times (n-1)$ minors of Df.

Here and in what follows, the distortion functions of a singular matrix are assumed to be equal to ∞, except for those of the zero matrix where we make the convention that they are all equal to 1.

The theorem we are yet to prove, and our belief in the extremality of the example we present, motivate the conjecture:

Conjecture 1.1. *Let* $f : \Omega \to \mathbb{R}^n$ *be a non-constant mapping of finite distortion such that*

$$K_I(x, f) \in L^P(\Omega) \tag{5}$$

where

$$\int_1^\infty P(s) \, \frac{ds}{s^2} = \infty.$$

Then

- $J(x, f) > 0$ *almost everywhere in* Ω,
- f *is open and discrete*,
- *if* $|E| = 0$, *then* $|f(E)| = 0$.

Here L^P denotes the Orlicz–Sobolev space with norm generated by the function P. ($P(t) = t^p$ gives the usual L^p spaces. See [10] for this and a discussion of Orlicz–Sobolev spaces with applications in this area.)

The following result from [8] is also pertinent to our discussion. It shows that in many natural situations the integrability of the Jacobian is automatic; this extends a result of Gehring and Lehto [6] to higher dimensions.

THEOREM 1.2. *Suppose that the function $f : \Omega \to \mathbb{R}^n$ belongs to the space $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and that for some positive integer N there is some set A of measure zero such that $f : \Omega \setminus A \to \mathbb{R}^n$ is at most N to 1. Then $J(x, f) \in L^1_{\mathrm{loc}}(\Omega)$. In particular, we have $J(x, f) \in L^1_{\mathrm{loc}}(\Omega)$ for each local homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.*

2. Squeezing the sponge; the mapping f. The construction of our mapping f is based on the classical "Sierpiński sponge", whence the title. The map we construct squeezes the sponge in an interesting way. The holes, initially forming a Cantor set of positive volume, get squeezed down to a set of measure zero. Yet the map is of finite distortion with its distortion function in a nice class. Let us get on with the construction before pointing out other nice features of this example.

We make use of the maximum norm in $\mathbb{R}^n$ which we denote by

$$|x| = \max\{|x_i| : i = 1, \dots, n\}. \tag{6}$$

As a starting point we consider the cube

$$\mathbf{Q} = \{x : |x| \le 1\}$$

and a sequence $\{r_\nu\}_{\nu=1}^\infty$ of positive numbers such that

$$1 = r_1 > 2r_2 > \ldots > 2^{\nu-1}\, r_\nu > \ldots \tag{7}$$

We divide $\mathbf{Q}$ up into 2^n congruent subcubes denoted by $\mathbf{Q}\big(a, \frac{1}{2}r_1\big)$, where the centers are given by $a = \big(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\big) = \big(\pm\frac{1}{2}r_1, \dots, \pm\frac{1}{2}r_1\big)$, corresponding to an arbitrary choice of signs. We remove from each $\mathbf{Q}\big(a, \frac{1}{2}r_1\big)$ the concentric cube $\mathbf{Q}(a, r_2)$ leaving a rectangular *frame*:

$$F_1(a) = \big\{x : r_2 < |x - a| \le \tfrac{1}{2}r_1\big\}.$$

Their union is denoted by

$$F_1 = \bigcup F_1(a)$$

where the union runs over all centers $a = \big(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\big)$. Next we repeat this construction within each cube $\mathbf{Q}(a, r_2)$. The dyadic subcubes are centered at the points

$$a + \tfrac{1}{2}(\pm r_2, \dots, \pm r_2) = \tfrac{1}{2}(\pm r_1 \pm r_2, \dots, \pm r_1 \pm r_2).$$

The cubes we remove from $\mathbf{Q}(a, r_2)$ are concentric and have radii r_3. We iterate this procedure generating the following:

The νth generation of centers A_ν consists of the $2^{n\nu}$ points of the form

$$a = \tfrac{1}{2}(\pm r_1 \pm \ldots \pm r_\nu, \ldots, \pm r_1 \pm \ldots \pm r_\nu)$$

where the signs are chosen arbitrarily. The condition at (7) ensures that different choices of sign yield different points of A_ν. Now at each stage ν and for every point $a \in A_\nu$ there is an associated rectangular frame

$$F_\nu(a) = \left\{x : r_{\nu+1} < |x - a| \le \tfrac{1}{2} r_\nu\right\} \tag{8}$$

and their union is

$$F_\nu = \bigcup\{F_\nu(a) : a \in A_\nu\}. \tag{9}$$

Sierpiński's sponge is simply the union of all the frames

$$F = \bigcup_{\nu=1}^{\infty} F_\nu. \tag{10}$$

The complement of the sponge is a Cantor subset of the cube $\mathbf{Q}$:

$$S = \mathbf{Q} \setminus F. \tag{11}$$

We shall view S as a singular set. It is easy to see, and important to note, that any sequence $\{a_\nu\}_{\nu=1}^{\infty}$ of centers in A_ν converges to a point in S. Conversely, every point of S is the limit of exactly one such sequence of centers. It is therefore convenient to parameterise the points of the Cantor set S by sequences of centers.

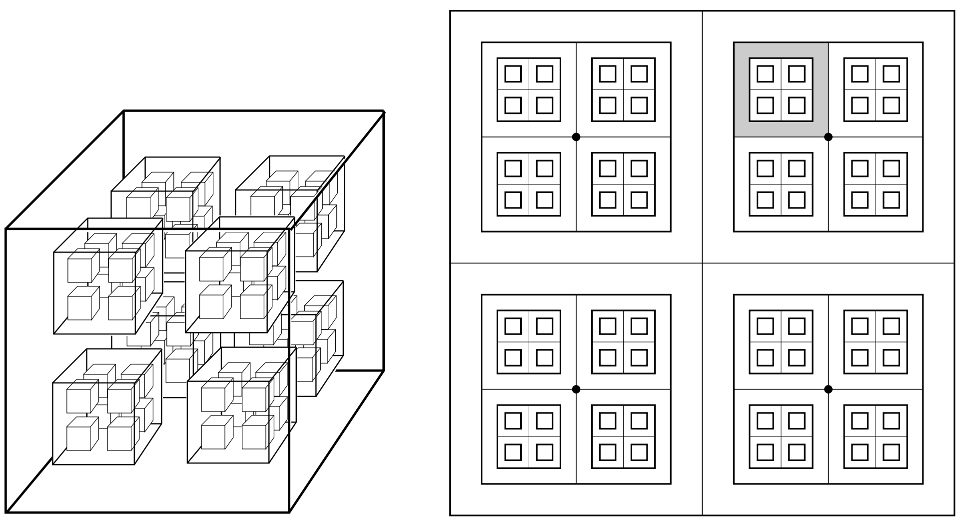

Fig. 1. Sierpiński sponge

The measure of the set S is easily found to be

$$|S| = \lim_{\nu\to\infty} \Big| \bigcup_{a\in A_\nu} \mathbf{Q}\big(a, \tfrac{1}{2}r_\nu\big)\Big| = \lim_{\nu\to\infty} 2^{n\nu} r_\nu^n. \tag{12}$$

The "squeezed" Sierpiński sponge will be another such sponge corresponding to a sequence of radii

$$1 = r_1' > 2r_2' > \ldots > 2^{\nu-1} r_\nu' > \ldots \tag{13}$$

Other terms of this new sponge will simply be tagged with the prime notation.

We are going to construct a homeomorphism (indeed a Lipschitz map) of finite distortion whose differential vanishes on the singular set S of positive measure and nowhere else.

Let

$$r_\nu' = \frac{2^{(\nu-1)(\nu-2)/2} r_\nu^\nu}{r_1 \ldots r_\nu} \tag{14}$$

and note that the condition at (13) is satisfied by the sequence $\{r_\nu'\}$ as

$$\frac{r_{\nu+1}'}{r_\nu'} = \left(\frac{r_{\nu+1}}{r_\nu}\right)^\nu 2^{\nu-1} < \frac{2^{\nu-1}}{2^\nu} = \frac{1}{2}.$$

We now define the map f on each frame $F_\nu(a)$ as a "radial stretching" in the max norm $|\cdot|$. That is,

$$f(x) = \alpha_\nu |x-a|^{\nu-1}(x-a) + a', \tag{15}$$

with the constant

$$\alpha_\nu = \frac{2^{\nu(\nu-1)/2}}{r_1 \ldots r_\nu}.$$

It follows from the choice of r_ν' that f maps each frame $F_\nu(a)$ homeomorphically onto the corresponding frame $F_\nu'(a')$. Indeed we note that if $|x-a| = \frac{1}{2}r_\nu$, then

$$|f(x) - a'| = \frac{2^{\nu(\nu-1)/2}\big(\frac{1}{2}r_\nu\big)^\nu}{r_1 \ldots r_\nu} = \frac{1}{2} r_\nu'.$$

And if $|x-a| = r_{\nu+1}$, then

$$|f(x) - a'| = \frac{2^{\nu(\nu-1)/2}(r_{\nu+1})^\nu}{r_1 \ldots r_\nu} = r_{\nu+1}'.$$

Similarly, we may verify that two different radial stretchings coincide on the common boundary of adjacent frames. These stretchings, when restricted to a common boundary, are similarity transformations, uniquely determined by the image of a face of the frame.

This piecewise definition of f, frame by frame, leaves us with a homeomorphism defined on their union F whose image is F'. As a final step we extend f uniquely to a homeomorphism $f : \mathbf{Q} \to \mathbf{Q}$ as follows. If $x \in S$, there is a unique sequence a_ν of nested centers with $x = \lim_{\nu\to\infty} a_\nu$. We put $f(x) = \lim_{\nu\to\infty} a'_\nu$, with the above mentioned correspondence between a_ν and a'_ν.

Next comes the computation of the differential and the distortion of f. Let us first examine a generic radial stretching,

$$h(x) = |x|^{\nu-1}x, \qquad \nu \ge 1. \tag{16}$$

Because of the various symmetries it suffices to make all the computations in the region where $|x| = x_1 \ge 0$. Thus h may be assumed to have the form $h(x) = x_1^{\nu-1}(x_1, \dots, x_n)$ defined on the region $|x_i| \le x_1$, $i = 1, \dots, n$. There we find

$$Dh(x) = x_1^{\nu-1} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + (\nu-1)x_1^{\nu-2} \begin{bmatrix} x_1 & 0 & \dots & 0 \\ x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \dots & 0 \end{bmatrix}$$

and

$$J(x,h) = \nu |x|^{n(\nu-1)}.$$

The differential of h has $n-2$ singular values equal to $|x|^{\nu-1}$ with the remaining two singular values, denoted by σ_+ and σ_-, satisfying

$$\sigma_+\sigma_- = \nu|x|^{2\nu-2},$$

$$C^{-1}|x|^{\nu-1} \le \sigma_- \le C|x|^{\nu-1}, \tag{17}$$

$$C^{-1}\nu|x|^{\nu-1} \le \sigma_+ \le C\nu|x|^{\nu-1}, \tag{18}$$

where $C \le 2n$ is a constant depending only on the dimension.

We now go about fixing the parameters. We choose the sequence of radii

$$r_\nu = \frac{\nu+1}{\nu\, 2^\nu}, \qquad \nu = 1, 2, \dots \tag{19}$$

The measure of $\mathbf{Q}$ is 2^n and the measure of the singular set S is

$$|S| = \lim_{\nu\to\infty} 2^{n\nu}\left(\frac{\nu+1}{\nu}\right)^n 2^{-n\nu} = 1. \tag{20}$$

The radii of the cubes in the image frames are

$$r'_\nu = \frac{1}{(\nu+1)2^{\nu-1}}\left(\frac{\nu+1}{\nu}\right)^\nu$$

and therefore the measure of the set S' is

$$|S'| = \lim_{\nu\to\infty} 2^{n\nu}\left(\frac{\nu+1}{\nu}\right)^n 2^{-n\nu-n}\frac{1}{(\nu+1)^n} = 0. \tag{21}$$

The mapping f restricted to the frame $F_\nu(a)$ is

$$f(x) = \frac{2^{\nu^2}}{\nu+1}|x-a|^{\nu-1}(x-a) + a'$$

where $n-2$ singular values of $Df(x)$ are equal to

$$(\nu+1)^{-1}2^{\nu^2}|x-a|^{\nu-1},$$

while the remaining two singular values satisfy the bounds

$$\frac{2^{\nu^2}}{2C(\nu+1)}|x-a|^{\nu-1} \le \sigma_- \le \frac{2C\,2^{\nu^2}}{\nu+1}|x-a|^{\nu-1},$$

$$\frac{2^{\nu^2}}{4C}|x-a|^{\nu-1} \le \sigma_+ \le 2C\,2^{\nu^2}|x-a|^{\nu-1}.$$

We also estimate the norm of the differential on the frame $F_\nu(a)$:

$$|Df(x)| = \sigma_+ \le 2C2^{\nu^2}|x-a|^{\nu-1} \le C2^{1+\nu^2}\left(\frac{r_\nu}{2}\right)^{\nu-1}$$

$$\le 4C\left(\frac{\nu+1}{\nu}\right)^{\nu-1} \le 24n.$$

Similarly

$$|Df(x)| \ge \frac{2^{\nu^2}}{4C}|x-a|^{\nu-1} \ge \frac{2^{\nu^2}}{4C}r_{\nu+1}^{\nu-1} \ge \frac{1}{4n}.$$

The bound on the differential seems at first surprising since piecewise we have mapped a rather thin frame to a rather thick frame. Thus the distortion is large (as we will see in a moment). However, this thick frame is at a much smaller scale, which gives us the control on the differential we want.

One important feature of Sierpiński's sponge, as we have constructed it, is that every pair of points $a, b \in F = \mathbf{Q}\setminus S$ can be connected by a piecewise linear curve in F whose segments are parallel to the axes and whose length is comparable to the euclidean distance between a and b. In fact one can show that there is such a path γ whose length does not exceed $\sqrt{n}\,|a-b|$. Such a curve is found by traversing the boundaries of suitable frames covering the line segment from a to b. It follows that

$$|f(a)-f(b)| \le \int_\gamma |Df| \le \sqrt{n}\,\|Df\|_\infty |a-b| \le 24n^{3/2}|a-b|.$$

Thus f is Lipschitz. This Lipschitz estimate remains valid in all of $\mathbf{Q}$ because f is continuous and F is dense. According to the Rademacher Theorem, f is therefore differentiable almost everywhere and its differential is bounded by $24n^{3/2}$. We have already seen that $|Df(x)| \ge 1/(4n)$ at each point of differentiability in F. We want to show that $Df(a) = 0$ at each point of differentiability $a \in S$. To this end we write $\{a\} = \bigcap_{\nu=1}^\infty \mathbf{Q}(a_\nu, r_{\nu+1})$ for an appropriate sequence of centers $a_\nu \in A_\nu$. On the boundary of each cube $\mathbf{Q}(a_\nu, r_{\nu+1})$ there is a point y_ν such that $|a - y_\nu| \ge r_{\nu+1}$. Since $|y_\nu - a_\nu| = r_{\nu+1}$, it follows that $|f(y_\nu) - a'_\nu| = r'_{\nu+1}$ as we have already seen. On the other hand $\{f(a)\} = \bigcap_{\nu=1}^{\infty} \mathbf{Q}(a'_\nu, r'_{\nu+1})$. This implies that $|f(a) - a'_\nu| \le r'_{\nu+1}$ and by the triangle inequality $|f(a) - f(y_\nu)| \le 2r'_{\nu+1}$. Putting this together shows

$$\frac{|f(a) - f(y_\nu)|}{|a - y_\nu|} \le \frac{2r'_{\nu+1}}{r_{\nu+1}} = \frac{4}{\nu+2}\left(\frac{\nu+2}{\nu+1}\right)^{\nu} \le \frac{12}{\nu}.$$

If we now let $\nu \to \infty$ we conclude $Df(a) = 0$.

Hence $Df(a) = 0$ on a subset of S of full measure.

The Jacobian determinant on the frame set $F_\nu(a)$ is given by

$$J(x,f) = \frac{\nu\, 2^{n\nu^2}}{(\nu+1)^n}|x - a|^{n(\nu-1)}$$

and hence does not vanish. Therefore we can write

$$|Df(x)|^n = K_O(x,f)J(x,f) \tag{22}$$

at *all* points of $\mathbf{Q}$, where, by convention, $K_O(a,f) = 1$ for $a \in S$. Elsewhere we see

$$K_O(x,f) = \frac{(\sigma_+)^n}{J(x,f)} \le \frac{[2C(\nu+1)]^\nu}{\nu} \le (8n)^n \nu^{n-1}$$

for $x \in F_\nu$. Thus f is a mapping of finite distortion. Indeed the linear distortion has the better bound

$$H(x,f) \approx \frac{\sigma_+}{\sigma_-} \approx 4C^2(\nu+1) \approx C(n)\nu \tag{23}$$

for $x \in F_\nu$ with large ν.

It is of interest to compute the inner distortion $K_I(x,f)$ of the mapping f as well. From the singular values above, we find that on each frame F_ν,

$$\frac{\nu}{C} \le K_I(x,f) \le C\nu \tag{24}$$

for a constant C which depends only on the dimension.

We now investigate the integrability properties of the distortion functions.

Let P be an Orlicz function. We shall estimate the P-norm of $\mathcal{K} = K_I(x,f)$. These will be controlled by the estimate at (24). If $\lambda > 0$, then by the integral test for infinite series we find that

$$\int_{\mathbf{Q}} P(\lambda\mathcal{K}) \approx \sum_{\nu=1}^{\infty} P(\lambda\nu)|F_\nu| + |S|P(\lambda).$$

Now as

$$|F_\nu| = 2^{n\nu}(r_\nu^n - 2^n r_{\nu+1}^n) = \left(\frac{\nu+1}{\nu}\right)^n - \left(\frac{\nu+2}{\nu+1}\right)^n$$

and hence

$$\frac{n}{\nu(\nu+1)} \le |F_\nu| \le \frac{n\,2^{n-1}}{\nu(\nu+1)}$$

we have the estimate

$$\int_{\mathbf{Q}} P(\lambda\mathcal{K}) \approx \sum_{\nu=1}^{\infty} \frac{P(\lambda\nu)}{\nu^2} \approx \int_1^\infty P(\lambda t)\,\frac{dt}{t^2} = \lambda \int_\lambda^\infty P(s)\,\frac{ds}{s^2}.$$

We find therefore that $\mathcal{K}(x) = K_I(x,f) \in L^P(\mathbf{Q})$ if and only if

$$\int_1^\infty P(s)\,\frac{ds}{s^2} < \infty$$

and in this case we have

$$\|\mathcal{K}\|_P \approx \frac{1}{P(1)} \int_1^\infty P(s)\,\frac{ds}{s^2}.$$

In particular $K_I(x,f) \in L\log^\alpha L$ for all $\alpha < -1$. However, $K_I(x,f) \notin L\log^{-1} L$. The estimate at (23) shows the same is true for the linear distortion function.

In summary, we have found a homeomorphism $f : \mathbf{Q} \to \mathbb{R}^n$ satisfying the first list of conditions in Theorem 1.1.

2.1. *Releasing the sponge*; *the mapping* g. Having squeezed the Sierpiński sponge and found it to be a very interesting example, it is of considerable interest to investigate the inverse mapping $g = f^{-1}$. On each frame $F'_\nu(a'_\nu)$ we have

$$g(y) = (\nu+1)^{1/\nu} 2^{-\nu} |y - a'_\nu|^{1/\nu - 1}(y - a'_\nu) + a_\nu.$$

Hence we find that

$$|Dg(y)| \le C\,2^{-\nu}|y - a'_\nu|^{1/\nu-1} \le C\,2^{-\nu}(r'_{\nu+1})^{1/\nu-1} \le C\nu.$$

The measure of F'_ν is computed as

$$\begin{aligned}|F'_\nu| &= 2^{n\nu}[(r'_\nu)^n - (2r'_{\nu+1})^n] \le n\,2^{n\nu}(r'_\nu - 2r'_{\nu+1})(r'_\nu)^{n-1} \\ &\le C(n)\nu^{1-n}\left[\frac{1}{\nu+1}\left(\frac{\nu+1}{\nu}\right)^\nu - \frac{1}{\nu+2}\left(\frac{\nu+2}{\nu+1}\right)^\nu\right] \\ &\le C(n)\frac{1}{\nu^{n+1}}.\end{aligned}$$

Now let P be an Orlicz function such that

$$\int_1^\infty P(t)\,\frac{dt}{t^{n+1}} < \infty. \tag{25}$$

Applying the integral test we find

$$\int_{\mathbf{Q}} P\left(\frac{|Dg(y)|}{C}\right) dy \le C(n) \int_1^\infty P(t)\,\frac{dt}{t^{n+1}}. \tag{26}$$

This implies, in particular, that $g \in W^{1,P}(\mathbf{Q}, \mathbb{R}^n)$ and therefore g too is a mapping of finite distortion. Also notice that the outer distortion function of g at the point $y = f(x)$ is $K_O(y, g) = K_I(x, f) \approx C(n)\nu$ for $y \in F'_\nu$. Reasoning as before we find $K_O(\cdot, g) \in L^P(\mathbf{Q})$, with P as at (25).

There is one final property of g that we want to discuss. The map g deforms the set F' of full measure into the cube $\mathbf{Q}$ which has the "hole" S of positive measure. This phenomenon of *cavitation in measure* is curious for mappings of finite distortion which lie in such a reasonable Sobolev-Orlicz space. A lesson from this example is that the Sobolev–Orlicz class $W^{1,P}(\Omega, \mathbb{R}^n)$ with P satisfying (25) is insufficient to guarantee strong regularity properties of mappings of finite distortion. Note that $P(t) = t^{n-\varepsilon}$, giving the usual Sobolev spaces $W^{1,n-\varepsilon}$, satisfies (25). Thus $g \in W^{1,p}(\mathbf{Q}, \mathbb{R}^n)$ for all $p < n$. This shows why we are interested in the Zygmund classes just below $W^{1,n}$.

In our earlier investigations of mappings of finite distortion [10] we have repeatedly met the condition $\int_1^\infty P(t)\,\frac{dt}{t^{n+1}} = \infty$ as being necessary for many regularity properties of these mappings.

The question arises just how big the set S' is. It has zero Lebesgue measure, but we shall see in a moment that its Hausdorff dimension is n. It is possible that this is optimal in the sense that the dimension of S' should be n.

To make a more subtle distinction we need to discuss the concept of weighted Hausdorff measure.

Let $\alpha \in C[0, \infty)$ be a continuous increasing function with $\alpha(0) = 0$ and $\lim_{t\to\infty} \alpha(t) = \infty$. Given a set X in $\mathbb{R}^n$, for each $\delta > 0$ we consider a countable covering of X by cubes $\mathbf{Q}_i$ (or balls) of diameter less than δ, $X \subset \bigcup_{i=1}^\infty \mathbf{Q}_i$, and compute the sum

$$\sum_{i=1}^\infty \alpha(\operatorname{diam} \mathbf{Q}_i).$$

The infimum of all such sums is denoted by $\mathcal{H}_\alpha^\delta(X)$. It is clear that for fixed X and α, the function $\mathcal{H}_\alpha^\delta(X)$ is non-decreasing in the parameter δ. The *α-Hausdorff content* (or measure) of X is defined to be the limit of this quantity as $\delta \to 0$,

$$\mathcal{H}_\alpha(X) = \lim_{\delta\to 0} \mathcal{H}_\alpha^\delta(X). \tag{27}$$

Of course the weight function $\alpha(t) = t^p$, $0 < p \le n$, is most often used and gives the usual Hausdorff p-measure, denoted by $\mathcal{H}_p(\cdot)$. There is a slight notational inconsistency here, but it should cause no problems.

For X fixed, $\mathcal{H}_p(X)$ is a non-increasing function of p. In this way we come to define the *Hausdorff dimension* of X as

$$\begin{aligned} \dim_{\mathcal{H}}(X) &= \inf\{p \geq 0 : \mathcal{H}_p(X) < \infty\} \\ &= \inf\{p \geq 0 : \mathcal{H}_p(X) = 0\} \\ &= \sup\{p \geq 0 : \mathcal{H}_p(X) > 0\} \\ &= \sup\{p \geq 0 : \mathcal{H}_p(X) = \infty\}. \end{aligned} \tag{28}$$

All these identities are readily verified. Of course any subset of $\mathbb{R}^n$ has Hausdorff dimension at most n.

We now turn back to our set S' of measure zero. We shall show that

$$0 < \mathcal{H}_\alpha(S') < \infty \tag{29}$$

where

$$\alpha(t) = \frac{t^n}{\log^n(e + 1/t)}. \tag{30}$$

It then follows that $\dim_{\mathcal{H}}(S') = n$. We recall that $S' = \bigcap_{i=1}^\infty F'_\nu$ where the set F'_ν consists of $2^{n\nu}$ cubes of radius $\frac{1}{2}r'_\nu$. Hence

$$\begin{aligned} \sum_{i=1}^{2^{n\nu}} \alpha(\operatorname{diam} \mathbf{Q}_i) = 2^{n\nu}\alpha(\sqrt{n}\, r'_\nu) &= 2^{n\nu} \frac{n^{n/2}(r'_\nu)^n}{\log(e + \sqrt{n}\, r'_\nu)} \\ &\approx \frac{C(n)2^{n\nu}\log^n(e + \nu\, 2^\nu)}{(\nu\, 2^\nu)^n} \approx C(n). \end{aligned}$$

That it suffices to consider only this particular cover of S' to estimate the measure relies on fairly elementary properties of Hausdorff measure and can be found in most texts on Geometric Measure Theory (see [5, 13]). The point is that S' is a "regular" set.

In summary, we have found a homeomorphism $g : \mathbf{Q} \to \mathbb{R}^n$ satisfying the second list of conditions in Theorem 1.1.

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