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Dual spaces generated by the interior of the set of norm attaining functionals

by

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Abstract. We characterize some isomorphic properties of Banach spaces in terms of the set of norm attaining functionals. The main result states that a Banach space is reflexive as soon as it does not contain ℓ_1 and the dual unit ball is the w^*-closure of the convex hull of elements contained in the "uniform" interior of the set of norm attaining functionals. By assuming a very weak isometric condition (lack of roughness) instead of not containing ℓ_1, we also obtain a similar result. As a consequence of the first result, a convex-transitive Banach space not containing ℓ_1 and such that the set of norm attaining functionals has nonempty interior is in fact superreflexive.

Introduction. James' Theorem states that a Banach space X is reflexive as soon as the set $\mathrm{NA}(X)$ of norm attaining functionals coincides with the dual space X^* [13]. A result due to Bourgain and Stegall states that $\mathrm{NA}(X)$ is of first Baire category if X is separable and the unit ball is not dentable (see [5, Problem 3.5.6]). Kenderov, Moors and Sciffer showed the same result for $C(K)$ (K infinite and compact) [15, Theorem 4].

There are also characterizations of reflexivity by assuming non-empty interior of the set of norm attaining functionals for topologies weaker than the norm topology. For instance, Debs, Godefroy and Saint-Raymond proved that a separable space X such that the w^*-interior of $\mathrm{NA}(X)$ relative to the dual unit sphere S_{X^*} is not empty, is reflexive [6, Lemma 11]. Jiménez Sevilla and Moreno showed the same result for any Banach space X [14, Proposition 3.2]. A result by Petunin and Plichko is along the same lines: a separable Banach space X is isometric to a dual space whenever there

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is a closed and weak* dense subspace of X^* contained in the set of norm attaining functionals [16].

Assuming that $\mathrm{NA}(X)$ has nontrivial (norm) interior, no isomorphic assumption can imply reflexivity of X. As a matter of fact, every Banach space can be equivalently renormed so as to have nonempty interior of the set of norm attaining functionals [1]. However, it was proven by Acosta and Ruiz Galán that a separable Banach space X that is very smooth or satisfies the Mazur intersection property has to be reflexive if $\mathrm{NA}(X)$ has nonempty interior [1]. Jiménez and Moreno showed the same result for spaces with the Mazur intersection property without the assumption of separability [14]. Acosta and Ruiz Galán showed that, for a space with Hahn–Banach smooth norm, either X is reflexive or $\mathrm{NA}(X)$ has empty interior [2].

In this note, we get results along the same lines, but assuming an isomorphic hypothesis on the space and a condition stating that the dual unit ball is generated by a subset of the interior of $\mathrm{NA}(X)$ (Theorem 1). We show that any Banach space X can be equivalently renormed so that the dual unit ball is the w^*-closure of the elements of the unit sphere in the interior of $\mathrm{NA}(X)$.

Afterwards, we prove that an isometric condition (nonroughness of the space) and the generation by the w^*-closure of the convex hull of the points in the dual unit ball contained in the "uniform" norm interior of $\mathrm{NA}(X)$ also imply reflexivity (Proposition 4). As a consequence, the previous result is valid for any Asplund space. Also we exhibit an example of a space X, isomorphic to c_0 and satisfying a certain condition of differentiability of the norm (weaker than Fréchet), such that $\mathrm{NA}(X)$ has nonempty interior. This example answers in the negative an open question posed in [2].

Results. We will assume that the spaces considered are **real**, but the results also hold in the complex case and the adaptation of the proofs is immediate.

In the following, X will be a Banach space, B_X and S_X the closed unit ball and the unit sphere of X, respectively. X^* will be the topological dual of X and $\mathrm{NA}(X) \subseteq X^*$ the subset of norm attaining functionals. For $r > 0$ we will write

$$\mathrm{NA}_r(X) := \{x^* \in X^* : x^* + rB_{X^*} \subset \mathrm{NA}(X)\}.$$

As mentioned in the introduction, no isomorphic condition can imply reflexivity, even if we assume that the set of norm attaining functionals has nonempty interior. Up to now, all the relevant results (except James' Theorem) use an additional isometric assumption. Here we will use an isomorphic condition and a stronger assumption on the set of norm attaining functionals to get a new characterization of reflexivity.

THEOREM 1. *Let X be a Banach space not containing an isomorphic copy of ℓ_1 and assume that for some $r > 0$,*

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}\{x^* \in S_{X^*} : x^* + rB_{X^*} \subset \mathrm{NA}(X)\},$$

where $\overline{\mathrm{co}}^{w^*}$ denotes the w^*-closure of the convex hull. Then X is reflexive.

Proof. We argue by contradiction. Assume that X is not reflexive. Since X does not contain ℓ_1, X does not have the Grothendieck property (see [19, Theorem 1] or [12, Proposition 1]). Hence, there exists a w^*-null sequence $\{x_n^*\}$ in S_{X^*} which is not weakly null. If $\Phi \in B_{X^{***}} \setminus \{0\}$ is a $\sigma(X^{***}, X^{**})$-cluster point of $\{x_n^*\}$, it is clear that $\Phi(X) = \{0\}$. If $x^* \in S_{X^*} \cap \mathrm{NA}_r(X)$ and $x^{**} \in S_{X^{**}}$, we can assume, by passing to a subsequence if necessary, that

$$x^{**}(x_n^*) \to \Phi(x^{**}).$$

Now it follows from Simons' inequality ([7, Lemma I.3.7]) for the functions $f_n := x^* + rx_n^*$ and the sets $B_X \subseteq B_{X^{**}}$ that

$$x^{**}(x^*) + r\Phi(x^{**}) \le 1, \qquad \forall x^{**} \in S_{X^{**}},$$

that is,

$$x^{**}(x^*) + r\Phi(x^{**}) \le \|x^{**}\|, \qquad \forall x^{**} \in X^{**}. \tag{1}$$

Now, fix $x_0^{**} \in S_{X^{**}}$ and $\varepsilon > 0$. By using again the fact that X does not contain ℓ_1 and [18, Theorem 10] (or [10, Theorem 1]) there are $x_0 \in S_X$ and $\alpha > 0$ so that

$$\operatorname{Osc} x_0^{**}(S(B_{X^*}, x_0, \alpha)) < \varepsilon, \tag{2}$$

where $S(B_{X^*}, x_0, \alpha) = \{x^* \in B_{X^*} : x^*(x_0) > 1 - \alpha\}$ and Osc denotes oscillation (i.e. $\sup - \inf$).

Since $\|x_0\| = 1$, by assumption there is a sequence $\{x_n^*\}$ in X^* such that

$$x_n^*(x_0) \to 1 \ \text{ and } \ x_n^* \in \mathrm{NA}_r(X) \cap S_{X^*}, \quad \forall n. \tag{3}$$

Choose a w^*-cluster point $x^{***} \in X^{***}$ of $\{x_n^*\}$. By (3) we can apply inequality (1) to each x_n^* and the element $x_0 + tx_0^{**}$ for any $t > 0$, so

$$x_n^*(x_0) + tx_0^{**}(x_n^*) + rt\Phi(x_0^{**}) \le \|x_0 + tx_0^{**}\|.$$

Since x^{***} is a w^*-cluster point of $\{x_n^*\}$, it follows from (3) that

$$1 + tx^{***}(x_0^{**}) + rt\Phi(x_0^{**}) \le \|x_0 + tx_0^{**}\|,$$

that is,

$$x^{***}(x_0^{**}) + r\Phi(x_0^{**}) \le \frac{\|x_0 + tx_0^{**}\| - 1}{t}, \qquad \forall t > 0. \tag{4}$$

We have $\lim_{t \to 0^+} (\|x_0 + tx_0^{**}\| - 1)/t = \max V(x_0, x_0^{**})$ ([8, Theorem V.9.5]), where

$$V(x_0, x_0^{**}) = \{y^{***}(x_0^{**}) : y^{***} \in S_{X^{***}},\ y^{***}(x_0) = 1\}.$$

By Goldstine's Theorem the slice $S(B_{X^*}, x_0, \alpha)$ is w^*-dense in the slice $S(B_{X^{***}}, x_0, \alpha)$, so

$$\operatorname{Osc} x_0^{**}(S(B_{X^{***}}, x_0, \alpha)) = \operatorname{Osc} x_0^{**}(S(B_{X^*}, x_0, \alpha)).$$

Since x^{***} is a w^*-cluster point of $\{x_n^*\}$, (3) gives us $x^{***}(x_0) = 1$. By (2), (4) and the previous observation we get

$$x^{***}(x_0^{**}) + r\Phi(x_0^{**}) \le \max V(x_0, x_0^{**}) \le x^{***}(x_0^{**}) + \varepsilon.$$

The inequality $r\Phi(x_0^{**}) \le \varepsilon$, valid for any $\varepsilon > 0$ and $x_0^{**} \in S_{X^{**}}$, gives $\Phi = 0$, a contradiction. ∎

It has been known before that the condition $\mathrm{NA}_r(X) \neq \emptyset$ does not suffice to get reflexivity of the space. To show that the first condition imposed ($X \not\supset \ell_1$) is needed, let us consider the next basic example:

Remark 2. *For $X = \ell_1$,*

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}(\mathrm{NA}_{1/2}(X) \cap S_{X^*}).$$

Proof. Clearly, the convex hull of the points in S_{X^*} with finite support is w^*-dense in B_{X^*}. Fix one such point $z_0 \in S_{X^*}$. If $z \in S_{X^*}$ and $\|z - z_0\| \le 1/2$, then $|z(n)| \le 1/2$ for every $n \notin \operatorname{supp} z_0$, while $|z(k)| \ge 1/2$ for some $k \in \operatorname{supp} z_0$ and therefore $z \in \mathrm{NA}(X)$. ∎

Also, the second assumption imposed in Theorem 1 is needed:

Proposition 3. *For any Banach space Z, there is a Banach space X isomorphic to Z so that*

$$B_{X^*} = \overline{\mathrm{co}}\Big(\bigcup_{r>0} [\mathrm{NA}_r(X) \cap S_{X^*}]\Big).$$

Proof. Of course, we can assume that $\dim Z \ge 2$. Let M be a closed linear subspace of Z and $0 \neq z_0 \in Z$ so that $Z = \mathbb{R}z_0 \oplus M$ and consider

$$X = \mathbb{R}z_0 \oplus_1 M,$$

that is, $B_X = \mathrm{co}\{\pm z_0 \cup B_M\}$ and so

$$|x^*| = \max\{|x^*(z_0)|, \|x^*_{|M}\|\},$$

where $\|\ \|$ is the original norm on Z and $|\ |$ denotes the new norm.

Define the functional z_0^* by

$$z_0^*(z_0) = 1, \qquad z_0^*(m) = 0, \qquad \forall m \in M.$$

Clearly, $|z_0^*| = 1$, and for any $x^* \in X^*$ with $|x^*| \le 1$ we have

$$x^* = \frac{1 + x^*(z_0)}{2}(z_0^* + x^*P_M) + \frac{1 - x^*(z_0)}{2}(-z_0^* + x^*P_M),$$

where P_M is the natural projection from X to M. Since

$$|z_0^* + x^*P_M| = |-z_0^* + x^*P_M| = \max\{|z_0^*(z_0)|, \|x^*_{|M}\|\} = 1,$$

we have proved that $B_{X^*} \subseteq \mathrm{co}\{x^* \in B_{X^*} : |x^*(z_0)| = 1\}$.

It is enough to approximate each element x_0^* in B_{X^*} satisfying $x_0^*(z_0) = 1$ by functionals in the interior of $\mathrm{NA}(X)$. For such an x_0^*, if we take

$$x_n^* = z_0^* + \left(1 - \frac{1}{n}\right) x_0^* P_M \qquad (n \in \mathbb{N}),$$

we get

$$x_0^* = \lim_n x_n^*.$$

Since $|x_n^*| = \max\{|x_n^*(z_0)|, (1-1/n)\|x^*_{n|M}\|\}$ and

$$(1-1/n)\|x^*_{n|M}\| \le 1 - 1/n < 1 = |x_n^*(z_0)|,$$

the same inequality holds in some open set containing x_n^*. That is, there is an open set of norm attaining functionals containing x_n^*. ■

By looking carefully at the proof of Theorem 1, one may think that the existence in the dual space of slices with small diameter (a stronger assumption than condition (2) appearing in the proof of Theorem 1) could make a similar result hold. We will get a new result in this direction by assuming that the space is nonrough (instead of not containing ℓ_1). First of all, recall that the norm $\|\cdot\|$ on a Banach space is *rough* if for some $\varepsilon > 0$,

$$\limsup_{h \to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} \ge \varepsilon, \qquad \forall x \in X.$$

Note that Asplund spaces admit no equivalent rough norm.

By assuming a condition on the size of $\mathrm{NA}(X)$—satisfied by ℓ_1—and a geometric assumption on X, we will get reflexivity.

PROPOSITION 4. *A Banach space X is reflexive if, and only if, its norm is not rough and for some $r > 0$,*

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}(\mathrm{NA}_r(X) \cap S_{X^*}).$$

Proof. A reflexive space is Asplund, so the norm is not rough and clearly satisfies the second condition since $\mathrm{NA}(X) = X^*$. Assume now that the norm of X is not rough and the dual unit ball can be generated by $\mathrm{NA}_r(X)$ as above. [7, Proposition I.1.11] yields $x \in S_X$ and $\alpha > 0$ satisfying

$$\mathrm{diam}\, S(B_{X^*}, x, \alpha) < r,$$

where $S(B_{X^*}, x, \alpha)$ is the w^*-slice given by

$$S(B_{X^*}, x, \alpha) = \{x^* \in B_{X^*} : x^*(x) > 1 - \alpha\}.$$

By using the second assumption and the definition of $S(B_{X^*}, x, \alpha)$, it follows that $S(B_{X^*}, x, \alpha)$ contains some element $x^* \in S_{X^*} \cap \mathrm{NA}_r(X)$. Therefore,

$$S(B_{X^*}, x, \alpha) \subset x^* + rB_{X^*} \subset \mathrm{NA}(X)$$

and by [14, Lemma 3.1], X is reflexive. ∎

COROLLARY 5. *X is reflexive if, and only if, the following conditions are satisfied*:

(i) *X is Asplund or at least its norm is Fréchet differentiable at some point.*

(ii) *There is $r > 0$ such that*

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}(\mathrm{NA}_r(X) \cap S_{X^*}).$$

It is known that Banach spaces X with the Mazur intersection property such that $\mathrm{NA}(X)$ has nonempty interior, are reflexive [14, Proposition 3.3]. In Proposition 4 we assumed a weaker condition on the space and a stronger assumption on $\mathrm{NA}(X)$.

Note that there are nonreflexive Asplund spaces X with $\mathrm{NA}(X)$ having nonempty interior. In fact, we will give an example of a space X isomorphic to c_0 such that the interior of $\mathrm{NA}(X)$ is not empty and the norm of X satisfies a certain differentiability condition. Recall that the norm $\|\cdot\|$ on a Banach space X is *strongly subdifferentiable* (see [9, 11]) at a point $u \in S_X$ if

$$\lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} = \tau(u, x) \quad \text{uniformly for } x \in B_X.$$

A smooth norm satisfying the previous condition at any point u of the unit sphere is Fréchet differentiable indeed.

The following example also answers in the negative a question posed in [2, Open Question 1].

EXAMPLE 6. *There is a Banach space X isomorphic to c_0 so that the norm of X is strongly subdifferentiable* (*at any point of S_X*) *and* $\mathrm{NA}(X)$ *has nonempty interior. Therefore, X is not reflexive but Asplund.*

Proof. It is enough to take as X the space c_0 endowed with the equivalent norm $\|\cdot\|$ given by

$$\|x\| := \tfrac{1}{2}|x(1)| + \max_{n \ge 2} |x(n)| \quad (x \in c_0).$$

This norm is strongly subdifferentiable as both the summands are.

If $\{e_n\}$ is the usual basis of c_0, we consider the open set in X^* given by

$$O = \Big\{x^* \in X^* : 2|x^*(e_1)| > \sum_{n=2}^{\infty} |x^*(e_n)|\Big\}.$$

For $x^* \in O$ and $x \in X$ we have

$$\begin{aligned}|x^*(x)| &= \Big|\sum_{n=1}^{\infty} x^*(e_n)x(n)\Big| \\ &\le |x^*(e_1)x(1)| + \sum_{n=2}^{\infty} |x^*(e_n)| \max_{n\ge 2} |x(n)| \\ &\le 2|x^*(e_1)|\big(\tfrac{1}{2}|x(1)| + \max_{n\ge 2}|x(n)|\big) \\ &\le 2|x^*(e_1)|\,\|x\|,\end{aligned}$$

hence $\|x^*\| \le 2|x^*(e_1)|$. Since $\|2e_1\| = 1$, we have $\|x^*\| = 2|x^*(e_1)|$ and x^* attains its norm, therefore $\mathrm{NA}(X)$ has nonempty interior. ■

The presence of points with special properties in the interior of $\mathrm{NA}(X)$ and the nonroughness of X force reflexivity. In order to be more precise, we recall some definitions.

For a Banach space X, let $\mathcal{G}$ be the group of all surjective linear isometries on X. The space X is said to be *convex-transitive* if $B_X = \overline{\mathrm{co}}\,\mathcal{G}(u)$ for any $u \in S_X$. The spaces $L_1[0,1]$, $L_\infty[0,1]$ and the Calkin algebra are convex-transitive (see [17, Theorem 9.6.4] and [4, Corollary 4.6]).

Our aim now is to exhibit a class of Banach spaces X which are superreflexive in the case where the interior of $\mathrm{NA}(X)$ is nonempty.

PROPOSITION 7. *Let X be a separable Banach space, which is convex-transitive and has* $\mathrm{NA}(X)$ *of second Baire category* (*in the norm topology*). *Then X is superreflexive.*

Proof. Since X is separable and $\mathrm{NA}(X)$ is of second Baire category, the unit ball of X is dentable (see [5, Theorem 3.5.5 and Problem 3.5.6]). By [7, Proposition I.1.11], this implies that the norm of X^* is not rough, and by using the convex-transitivity of X, X is superreflexive in view of [3, Theorem 3.2]. ■

PROPOSITION 8. *Let X be a convex-transitive Banach space not containing ℓ_1 such that* $\mathrm{NA}(X)$ *has nonempty interior. Then X is superreflexive.*

Proof. Since $\mathrm{NA}(X)$ is a cone with nonempty interior, there is $x^* \in S_{X^*}$ so that $x^* + rB_{X^*} \subset \mathrm{NA}(X)$ for some $r > 0$. That is, $x^* \in \mathrm{NA}_r(X) \cap S_{X^*}$. As X is convex-transitive, we have

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}\Big(\bigcup_{T\in\mathcal{G}} T^*x^*\Big). \tag{1}$$

Now, $x^* + ry^* \in \mathrm{NA}(X)$ for any $y^* \in B_{X^*}$, so there is $y \in S_X$ satisfying

$$|(x^* + ry^*)(y)| = \|x^* + ry^*\|.$$

Then, for any $T \in \mathcal{G}$,

$$T^*(x^* + ry^*)(T^{-1}(y)) = (x^* + ry^*)(y) = \|x^* + ry^*\| = \|T^*(x^* + ry^*)\|$$

and $\|T^{-1}(y)\| = \|y\| = 1$, that is, $T^*(x^* + ry^*) \in \mathrm{NA}(X)$. Since the above condition holds for any $y^* \in B_{X^*}$, and T^* is a surjective isometry on X^*, we have $T^*x^* \in S_{X^*} \cap \mathrm{NA}_r(X)$. By using (1) it follows that

$$B_{X^*} = \overline{\mathrm{co}}^{w^*}(\mathrm{NA}_r(X) \cap S_{X^*}).$$

Since we are assuming that X does not contain ℓ_1, in view of Theorem 1, X is reflexive. Also, by [3, Theorem 3.2], since X is convex-transitive, it is, in fact, superreflexive. ■

The previous positive results suggest the following question:

Open problem 9. *Let X be a convex-transitive Banach space such that* $\mathrm{NA}(X)$ *has nonempty interior. Is X superreflexive?*

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