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Every separable L_1-predual is complemented in a C^*-algebra

by

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Abstract. We show that every separable complex L_1-predual space X is contractively complemented in the CAR-algebra. As an application we deduce that the open unit ball of X is a bounded homogeneous symmetric domain.

1. The main results. This paper is concerned with L_1-predual spaces (over $\mathbb{C}$) and their connection to C^*-algebras. Let X be a separable Banach space (over $\mathbb{C}$) such that X^* is isometrically isomorphic to an L_1-space. For example, if K is a compact Hausdorff space then the Banach space $C(K)$ of all complex-valued continuous functions on K has a dual which is isometrically isomorphic to an L_1-space. However there are many examples where X is not isomorphic to any complemented subspace of any $C(K)$-space ([2]). On the other hand, a separable L_1-predual is always isomorphic to a quotient of a $C(K)$-space ([4]).

$C(K)$ is a commutative C^*-algebra. H. P. Rosenthal conjectured that the non-commutative situation might be different, i.e. that X might always be complemented in a (non-commutative) C^*-algebra. Furthermore there might even be a universal C^*-algebra containing all separable L_1-preduals as complemented subspaces. The CAR-algebra $\mathcal{A}$ might be a candidate for this.

The aim of the paper is to confirm Rosenthal's conjecture.

Fix a sequence of integers $0 \leq m_1 < m_2 < \ldots$ Then we define the C^*-algebra $\mathcal{A}_{(m_n)}$ as follows: For a Hilbert space H let $\mathcal{L}(H)$ be the space of all linear and bounded operators on H. Moreover let $\mathcal{M}_n = \mathcal{L}(l_2^n)$ be the space of all $n \times n$-matrices (over $\mathbb{C}$). Identify $B \in \mathcal{M}_{2^{m_n}}$ with

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$$\begin{pmatrix} B & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & B \end{pmatrix} \in \mathcal{M}_{2^{m_{n+1}}}.$$

Via this identification $\mathcal{M}_{2^{m_n}}$ becomes a $*$-subalgebra of $\mathcal{M}_{2^{m_{n+1}}}$. Now, put

$$\mathcal{A}_{(m_n)} = \overline{\bigcup_n \mathcal{M}_{2^{m_n}}}.$$

If $m_n = n$ for all n then $\mathcal{A}$ is called the C(*anonical*) A(*nti-commutation*) R(*elations*) *algebra* ([5]). It is easily seen that, for arbitrary (m_n), the C^*-algebra $\mathcal{A}_{(m_n)}$ is algebraically isometric to a unital contractively complemented subalgebra of the CAR-algebra. We call $\mathcal{A}_{(m_n)}$ a *natural* subalgebra of the CAR-algebra.

THEOREM. *Let $\mathcal{A}$ be the CAR-algebra. Then every separable L_1-predual space X (over $\mathbb{C}$) is isometrically isomorphic to a contractively complemented subspace of $\mathcal{A}$.*

Before we prove the theorem in Section 3, we discuss the following consequence. Let U be an open connected subset of a complex Banach space X. Recall that a map $\varphi : U \to X$ is called *holomorphic* if, for each $z_0 \in U$, there is a sequence of homogeneous polynomials $p_n : X \to X$ of degree n such that

$$\varphi(z) = \sum_{k=0}^{\infty} p_n(z - z_0) \quad \text{for all } z \in U.$$

(Here $p_n(z) = f_n(z, \dots, z)$ for some continuous, symmetric, n-linear map $f_n : X^n \to X$.)

COROLLARY. *Let X be a separable complex L_1-predual space and U its open unit ball. Then U is a bounded homogeneous symmetric domain. That is, for each $z \in U$ there exists a bijective holomorphic map $\varphi_z : U \to U$ such that φ_z^{-1} is holomorphic and we have $\varphi_z(0) = z$. Moreover, there is a bijective holomorphic map $\sigma_z : U \to U$ such that $\sigma_z(z) = z$, $\sigma_z^2 = \mathrm{id}_U$ and $\sigma_z'(z) = -\mathrm{id}_X$ where $\sigma_z'(z)$ is the Fréchet derivative of σ_z at z.*

Proof. Since X is contractively complemented in a C^*-algebra, it is a JB^*-triple ([3], [7], [13], for definitions see also [1], [14]). It follows that U satisfies the assertion of the Corollary according to [6], [15] and [14]. ■

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2. L_1-predual spaces. First we recall some basic facts concerning separable L_1-preduals X.

It is well known ([9], [8], [11]) that there are l_∞^n-spaces $\mathcal{E}_n$ such that

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \quad \text{and} \quad X = \overline{\bigcup \mathcal{E}_n}.$$

Let $\{e_{i,n}\}_{i=1}^n$ be the unit vector basis of $\mathcal{E}_n$. Then there are numbers $a_{i,n}$ with $\sum_{i=1}^n |a_{i,n}| \le 1$ such that, for a suitable order of the indices i of the $e_{i,n+1}$, we have

$$e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}, \qquad i = 1, \ldots, n,\ n = 1, 2, \ldots \ ([10]).$$

Moreover, let $\Phi_j \in X^*$ be the functional with

$$\Phi_j(e_{i,n}) = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases} \qquad n = j, j+1, \ldots$$

Then $\|\Phi_j\| = 1$ and $\Phi_{n+1}(e_{i,n}) = a_{i,n}$, $i = 1, \ldots, n$.

It is well known that an L_1-predual X is a simplex space, i.e. the space of all continuous affine functions on a Choquet simplex, if and only if the unit ball of X has an extreme point e ([8], [12]). This is equivalent to the fact that X has a representation of the form $X = \overline{\bigcup \mathcal{E}_n}$ as above where $e_{1,1} = e$. This implies that here the corresponding numbers $a_{i,n}$ satisfy $a_{i,n} \ge 0$ and $\sum_{i=1}^n a_{i,n} = 1$.

The following lemma is due to Lazar and Lindenstrauss in the real case ([8]). To keep the paper self-contained we include a proof which also covers the complex case.

LEMMA. *For every separable L_1-predual X there is a separable simplex space $Y \supset X$ and a contractive projection $P : Y \to X$.*

Proof. Let

$$X = \overline{\bigcup_n \mathcal{E}_n}, \qquad \mathcal{E}_1 \subset \ldots \subset l_\infty^n \cong \mathcal{E}_n \subset \mathcal{E}_{n+1} \subset \ldots$$

as before. Using the preceding remarks we find $\Phi_1, \ldots, \Phi_n \in X^*$ such that $\Phi_1|_{\mathcal{E}_n}, \ldots, \Phi_n|_{\mathcal{E}_n}$ are extreme points of the unit ball of $\mathcal{E}_n^*$ and, by evaluation, $\mathcal{E}_n$ can be isometrically embedded into $C(K_n)$ where $K_n = \{\theta\Phi_j : j = 1, \ldots, n,\ \theta \in \mathbb{C},\ |\theta| = 1\}$. For $f \in C(K_n)$ put

$$(P_n f)(\theta\Phi_j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\varphi} f(\theta e^{i\varphi}\Phi_j)\, d\varphi, \qquad j = 1, \ldots, n,\ |\theta| = 1.$$

Then $(P_n f)(\theta\Phi_j) = \theta(P_n f)(\Phi_j)$ and we see that P_n is a contractive projection from $C(K_n)$ onto $\mathcal{E}_n$.

Let $i_n : \mathcal{E}_n \to \mathcal{E}_{n+1}$ be the canonical injection. We extend i_n to an isometry from $C(K_n)$ into $C(K_{n+1})$ as follows. Let

$$\Phi_{n+1}|_{\mathcal{E}_n} = \sum_{i=1}^n \alpha_i \theta_i \Phi_i|_{\mathcal{E}_n} + \sum_{i=1}^n \alpha_{n+i} \theta_{n+i} (-\Phi_i)|_{\mathcal{E}_n}$$

for some $\theta_i \in \mathbb{C}$ with $|\theta_i| = 1$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{2n} \alpha_i = 1$. Let $|\theta| = 1$ and define, for $f \in C(K_n)$,

$$(i_n f)(\theta \Phi_{n+1}) = \sum_{i=1}^{n} \alpha_i f(\theta\theta_i \Phi_i) + \sum_{i=1}^{n} \alpha_{n+i} f(-\theta\theta_{n+i}\Phi_i)$$

and $(i_n f)(\theta\Phi_j) = f(\theta\Phi_j)$ if $j \leq n$. This definition extends i_n to an isometry from $C(K_n)$ into $C(K_{n+1})$ with $i_n 1_{K_n} = 1_{K_{n+1}}$. Moreover, we have $i_n \circ P_n = P_{n+1} \circ i_n$. Thus, if we identify $f \in C(K_n)$ with $i_n f \in C(K_{n+1})$ then we can define $Y = \overline{\bigcup_n C(K_n)}$. Then Y is an L_1-predual (see e.g. [9]) whose unit ball has an extreme point, namely $1_{K_1} = 1_{K_2} = \ldots$, and Y contains $X = \overline{\bigcup_n \mathcal{E}_n}$. The P_n yield a contractive projection $P : Y \to X$. ■

3. Proof of the main result. In the following we consider a Hilbert space H and an involutive isometry $S : H \to H$. Take $T \in \mathcal{L}(H)$. Then we define

$$E_S(T) = \tfrac{1}{2}(T + STS).$$

Of course, we have $E_S E_S(T) = E_S(T)$. Moreover, $E_S(T) = 0$ if and only if $T = 2^{-1}(T - STS)$ and $E_S(T) = T$ if and only if $ST = TS$.

We use the notion of isomorphism strictly in the category of Banach spaces (i.e. as linear map). If we deal with invertible continuous multiplicative linear maps then we speak of algebra isomorphisms.

Proof of the Theorem. We construct a Hilbert space H, a $*$-subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ and an involutive isometry $S : H \to H$ such that X is isometrically embedded in $E_S(\mathcal{A})$ and complemented in $\mathcal{A} + S\mathcal{A}S$. Moreover, $\mathcal{A}$ and S are such that $E_S|_{\mathcal{A}}$ is an isometry. Hence X is isometrically isomorphic to a contractively complemented subspace of $\mathcal{A}$. It turns out that $\mathcal{A}$ is a natural subalgebra of the CAR-algebra and hence $\mathcal{A}$ is complemented in the CAR-algebra. This proves the Theorem.

In view of the Lemma in Section 2 it suffices to assume that X is a simplex space. So, let $a_{i,n}$ be such that

$$a_{i,n} \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} a_{i,n} = 1$$

and such that these numbers, as indicated in the preliminaries, define the isometric embeddings

$$\tau_n : \mathcal{E}_n \cong l_\infty^n \to l_\infty^{n+1} \cong \mathcal{E}_{n+1} \tag{1}$$

where $X = \overline{\bigcup \mathcal{E}_n}$. The $\mathcal{E}_n$ will be recovered as certain subspaces of $\mathcal{L}(H)$ for a suitable Hilbert space H.

First, we use induction to define finite-dimensional Hilbert spaces H_n, and isometric embeddings $\iota_n : H_n \to H_{n+1}$, $\pi_n : \mathcal{L}(H_n) \to \mathcal{L}(H_{n+1})$

such that π_n is an isometric $*$-algebra isomorphism onto a $*$-subalgebra of $\mathcal{L}(H_{n+1})$. Moreover we find isometric copies of $\mathcal{E}_n$ (called $\mathcal{E}_n$ again) in $\mathcal{L}(H_n)$ and contractive projections $P_n : \mathcal{L}(H_n) \to \mathcal{E}_n$ such that the following relations hold:

$$P_{n+1} \circ \pi_n = \tau_n \circ P_n, \tag{2}$$

$$\iota_n \circ T = \pi_n(T) \circ \iota_n \quad \text{for all } T \in \mathcal{L}(H_n), \tag{3}$$

$$P_{n+1}(\pi_n(T))(\iota_n h) = \iota_n P_n(T) h \quad \text{for all } T \in \mathcal{L}(H_n) \text{ and } h \in H_n. \tag{4}$$

In particular the following diagrams commute:

$$\begin{array}{ccc} \mathcal{L}(H_n) & \xrightarrow{\pi_n} & \mathcal{L}(H_{n+1}) \\ {\scriptstyle P_n}\downarrow & & \downarrow{\scriptstyle P_{n+1}} \\ \mathcal{E}_n & \xrightarrow{\tau_n} & \mathcal{E}_{n+1} \end{array} \qquad \begin{array}{ccc} H_n & \xrightarrow{\iota_n} & H_{n+1} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \pi_n(T)} \\ H_n & \xrightarrow{\iota_n} & H_{n+1} \end{array}$$

It would be tempting to go over to the direct limits of the H_n and the $\mathcal{L}(H_n)$ and then, using the P_n, to build up a common projection P. Unfortunately we do not have $\pi_n|_{\mathcal{E}_n} = \tau_n$ in general. Hence we cannot find an isometric copy of X as a subspace of the direct limit of the $\mathcal{L}(H_n)$. This is the reason why we bring E_S into the play with respect to some isometric involution S. To this end we construct, in addition, involutive isometries $S_n : H_n \to H_n$ such that

$$\iota_n \circ S_n = S_{n+1} \circ \iota_n, \tag{5}$$

$$P_n(T) \circ S_n = S_n \circ P_n(T) \quad \text{for all } T \in \mathcal{L}(H_n), \tag{6}$$

$$P_n(T) = P_n(S_n T S_n) \quad \text{for all } T \in \mathcal{L}(H_n), \tag{7}$$

$$\|E_{S_{n+1}}(\pi_n(T))\| = \|T\| \quad \text{for all } T \in \mathcal{L}(H_n). \tag{8}$$

In particular, the diagram

$$\begin{array}{ccc} H_n & \xrightarrow{\iota_n} & H_{n+1} \\ {\scriptstyle S_n}\downarrow & & \downarrow{\scriptstyle S_{n+1}} \\ H_n & \xrightarrow{\iota_n} & H_{n+1} \end{array}$$

commutes and we obtain, by (6) with $S_n^2 = \mathrm{id}$,

$$\mathcal{E}_n \subset E_{S_n}(\mathcal{L}(H_n)) \quad \text{for each } n.$$

On the other hand, we do not have $\pi_n(S_n) = S_{n+1}$ in general.

(a) First we want to show how we can derive the essential part of the Theorem from the preceding assumptions.

Claim. *Assume that* (2)–(8) *are satisfied. Then there is a Hilbert space H, an involutive isometry $S \in \mathcal{L}(H)$ and a $*$-subalgebra $\mathcal{A} \subset \mathcal{L}(H)$ such that $E_S|_{\mathcal{A}}$ is an isometry. Moreover, there is a contractive projection $P : E_S\mathcal{A} \to E_S\mathcal{A}$ such that $P(E_S\mathcal{A})$ is an isometric copy of X.*

Proof. At first put

$$\widetilde{H} = \overline{\operatorname{span}}\{(\underbrace{0,\ldots,0}_{n-1}, h, \iota_n(h), \iota_{n+1}\iota_n(h),\ldots) : h \in H_n,\ n = 1,2,\ldots\}$$
$$\subset (H_1 \oplus H_2 \oplus \ldots)_{(\infty)}$$

(endowed with the norm $\|(h_k)\| = \sup_k \|h_k\|$). Moreover, define

$$N = \{(h_1, h_2, \ldots) \in \widetilde{H} : \lim_{n\to\infty} \|h_n\| = 0\}.$$

Put $H = \widetilde{H}/N$. Then H is a Hilbert space with scalar product

$$\langle (h_k) + N, (g_k) + N\rangle = \lim_{k\to\infty} \langle h_k, g_k\rangle$$

(recall that $\langle h_k, g_k\rangle = \langle \iota_k h_k, \iota_k g_k\rangle$ since the ι_k are isometries). Identify $h \in H_n$ with

$$(\underbrace{0,\ldots,0}_{n-1}, h, \iota_n h, \iota_{n+1}\iota_n h, \ldots) + N \in H.$$

Then $H_1 \subset H_2 \subset \ldots$ and $H = \overline{\bigcup H_n}$.

Define, for $T \in \mathcal{L}(H_n)$,

$$\text{(9)}\quad \widetilde{T}((\underbrace{0,\ldots,0}_{m-1}, h, \iota_m h, \iota_{m+1}\iota_m h, \ldots) + N)$$
$$= (\underbrace{0,\ldots,0}_{m-1}, (\pi_{m-1}\circ\ldots\circ\pi_n)(T)h, \iota_m(\pi_{m-1}\circ\ldots\circ\pi_n)(T)h, \ldots) + N$$
$$= (\underbrace{0,\ldots,0}_{m-1}, (\pi_{m-1}\circ\ldots\circ\pi_n)(T)h, (\pi_m\circ\pi_{m-1}\circ\ldots\circ\pi_n)(T)\iota_m h,$$
$$(\pi_{m+1}\circ\ldots\circ\pi_n)(T)\iota_{m+1}\iota_m h, \ldots) + N$$

if $h \in H_m$ and $m > n$ (see (3)). Put

$$\underline{\mathcal{L}}(H_n) = \{\widetilde{T} : T \in \mathcal{L}(H_n)\}.$$

Then $\underline{\mathcal{L}}(H_1) \subset \underline{\mathcal{L}}(H_2) \subset \ldots$ Define

$$\text{(10)}\qquad \mathcal{A} = \overline{\bigcup \underline{\mathcal{L}}(H_n)} \subset \mathcal{L}(H),$$

which is a $*$-subalgebra of $\mathcal{L}(H)$ since all π_m are $*$-algebra isomorphisms.

Moreover, put

$$S((\underbrace{0,\dots,0}_{n-1},h,\iota_n h,\dots)+N)$$
$$=(\underbrace{0,\dots,0}_{n-1},S_n h,S_{n+1}\iota_n h,S_{n+2}\iota_{n+1}\iota_n h,\dots)+N$$
$$=(\underbrace{0,\dots,0}_{n-1},S_n h,\iota_n S_n h,\iota_{n+1}\iota_n S_n h,\dots)+N.$$

This makes sense in view of (5). Then S is an involutive isometry on H. Unfortunately, S is not an element of $\mathcal{A}$ (since $S_{n+1}\neq\pi_n(S_n)$). For $T\in\mathcal{L}(H_n)$, $h\in H_n$, we have

$$S\widetilde{T}S((\underbrace{0,\dots,0}_{n-1},h,\iota_n h,\iota_{n+1}\iota_n h,\dots)+N)$$
$$=(\underbrace{0,\dots,0}_{n-1},S_nTS_n h,S_{n+1}\pi_n(T)S_{n+1}\iota_n h,$$
$$S_{n+2}(\pi_{n+1}\circ\pi_n)(T)S_{n+2}\iota_{n+1}\iota_n h,\dots)+N.$$

This implies

$$E_S(\widetilde{T})((\underbrace{0,\dots,0}_{n-1},h,\iota_n h,\dots)+N)$$
$$=(\underbrace{0,\dots,0}_{n-1},E_{S_n}(T)h,E_{S_{n+1}}(\pi_n(T))\iota_n h,E_{S_{n+2}}(\pi_{n+1}\circ\pi_n(T))\iota_{n+1}\iota_n h,\dots)+N.$$

Hence, by (8), $E_S:\mathcal{A}\to\mathcal{L}(H)$ is an isometry.

For $T\in\mathcal{L}(H_n)$ and $h\in H_n$ define

$$P(\widetilde{T})((\underbrace{0,\dots,0}_{n-1},h,\iota_n h,\iota_{n+1}\iota_n h,\dots)+N)$$
$$=(\underbrace{0,\dots,0}_{n-1},P_n(T)h,P_{n+1}(\pi_n(T))\iota_n h,P_{n+2}(\pi_{n+1}\circ\pi_n(T))\iota_{n+1}\iota_n h,\dots)+N$$
$$=(\underbrace{0,\dots,0}_{n-1},P_n(T)h,\iota_n P_n(T)h,\iota_{n+1}\iota_n P_n(T)h,\dots)+N$$

(see (4)). Hence P is well defined on $\mathcal{A}$ and, in view of (7), even on $\mathcal{A}+S\mathcal{A}S$. So P can be regarded as a contractive operator on $\mathcal{A}+S\mathcal{A}S$. Condition (6) implies that

$$P\mathcal{A}=P(\mathcal{A}+S\mathcal{A}S)\subset E_S\mathcal{A}\subset\mathcal{A}+S\mathcal{A}S.$$

Hence $P\mathcal{A}$ is contractively complemented in $E_S\mathcal{A}$ and $E_S\mathcal{A}$ is isometrically isomorphic to $\mathcal{A}$.

Finally, by the definition of P, in view of (1), (3) and (2), we have, if $T \in \mathcal{L}(H_n)$ and $h \in H_n$,

$$\begin{aligned}
&P(\widetilde{T})((\underbrace{0,\ldots,0}_{n-1}, h, \iota_n h, \iota_{n+1}\iota_n h, \ldots) + N) \\
&\qquad = (\underbrace{0,\ldots,0}_{n-1}, P_n(T)h, \iota_n P_n(T)h, \iota_{n+1}\iota_n P_n(T)h, \ldots) + N \\
&\qquad = (\underbrace{0,\ldots,0}_{n-1}, P_n(T)h, P_{n+1}(\pi_n(T))\iota_n h, \iota_{n+1}P_{n+1}(\pi_n(T))\iota_n h, \\
&\qquad\qquad\qquad \iota_{n+2}\iota_{n+1}P_{n+1}(\pi_n(T))\iota_n h, \ldots) + N \\
&\qquad = (\underbrace{0,\ldots,0}_{n-1}, P_n(T)h, \tau_n P_n(T)\iota_n h, \iota_{n+1}\tau_n P_n(T)\iota_n h, \ldots) + N \\
&\qquad = (\underbrace{0,\ldots,0}_{n}, \tau_n P_n(T)\iota_n h, \iota_{n+1}\tau_n P_n(T)\iota_n h, \ldots) + N.
\end{aligned}$$

The last equality follows from the definition of N.

This means that $\mathcal{E}_n \cong P\mathcal{L}(H_n)$ and $\mathcal{E}_n$ is identified with the subspace $\tau_n\mathcal{E}_n$ of $\mathcal{E}_{n+1} \cong P\mathcal{L}(H_{n+1})$. Hence $P\mathcal{A} = X$. This completes the proof of Claim (a). ■

(b) Now we show that we can realize (2)–(8). Consider the isometries $\tau_n : l_\infty^n \cong \mathcal{E}_n \to \mathcal{E}_{n+1} \cong l_\infty^{n+1}$ of (1).

Claim. *There are H_n, ι_n, S_n, π_n and P_n satisfying* (2)–(8).

Proof. We construct H_n, ι_n, S_n, π_n, P_n by induction. Let H_1 be a one-dimensional Hilbert space, $S_1 = \mathrm{id}_{H_1}$ and $P_1 =$ identity on $\mathcal{L}(H_1)$.

Assume next that we already have finite-dimensional Hilbert spaces $H_1, \ldots, H_n$, involutive isometries $S_1, \ldots, S_n$, isometric embeddings $\iota_1, \ldots, \iota_{n-1}$, isometric $*$-algebra isomorphisms $\pi_1, \ldots, \pi_{n-1}$ and projections $P_1, \ldots, P_n$ satisfying the relations corresponding to (2)–(8) for the indices $1, \ldots, n$. Moreover, we assume that $T_{i,n} \in \mathcal{L}(H_n)$, $i = 1, \ldots, n$, are the elements of the unit vector basis of $\mathcal{E}_n \cong l_\infty^n$ and that there are $h_{j,k} \in H_n$, $j = 1, \ldots, n$, $k = 1, \ldots, m_j$, for some m_j, which form an ON-system in H_n and satisfy $S_n h_{j,k} = h_{j,k}$ for all j and k and

$$T_{l,n}h_{j,k} = \begin{cases} h_{j,k} & \text{if } l = j, \\ 0 & \text{if } l \neq j, \end{cases} \qquad k = 1, \ldots, m_j.$$

Finally suppose that there are $\beta_{j,k} \geq 0$ such that

$$\Phi_j := \sum_{k=1}^{m_j} \beta_{j,k} h_{j,k} \otimes h_{j,k} \tag{11}$$

regarded as a linear functional on $\mathcal{L}(H_n)$ satisfies

$$\|\Phi_j\| = 1 \quad \text{and} \quad \Phi_j(T_{l,n}) = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \neq l. \end{cases} \tag{12}$$

By (11) we mean the functional with

$$\Phi_j(T) = \sum_{k=1}^{m_j} \beta_{j,k} \langle T h_{j,k}, h_{j,k} \rangle \quad \text{for all } T \in \mathcal{L}(H_n).$$

The $h_{i,k}$ may not span H_n. They are only needed to define the functionals Φ_j. (The values $T_{l,n}(h)$ are irrelevant if h is not in the span of the elements $h_{i,k}$ as long as we know that $\|T_{l,n}\| = 1$.)

Our hypothesis includes further that P_n is defined by

$$P_n(T) = \sum_{j=1}^{n} \Phi_j(T) T_{j,n}, \qquad T \in \mathcal{L}(H_n). \tag{13}$$

For the next step of the induction put

$$m_{n+1} = \sum_{j=1}^{n} m_j \quad \text{and} \quad M = 2^{m_{n+1}+1}. \tag{14}$$

Hence $M - 2 \geq m_{n+1}$. Let

$$H_{n+1} = (\underbrace{H_n \oplus \ldots \oplus H_n}_{M \text{ times}})_{(2)}$$

be endowed with the norm

$$\|(h_1, \ldots, h_M)\| = \sqrt{\textstyle\sum_{k=1}^{M} \|h_k\|^2}.$$

Define, for $h \in H_n$,

$$\iota_n h = (h, 0, \ldots, 0) \in H_{n+1}.$$

Moreover, for $T \in \mathcal{L}(H_n)$ put

$$\pi_n(T) = \begin{pmatrix} T & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & T \end{pmatrix} \in \mathcal{L}(H_{n+1}). \tag{15}$$

Then π_n is an isometric $*$-algebra isomorphism onto a $*$-subalgebra of $\mathcal{L}(H_{n+1})$. Clearly, $\iota_n \circ T = \pi_n(T) \circ \iota_n$ for all $T \in \mathcal{L}(H_n)$, which proves (3).

Now, with the given numbers $a_{j,n} \geq 0$ describing τ_n (see (1)) we define the following elements in H_{n+1}:

$$h_{n+1,l} = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \sqrt{a_{j,n}\beta_{j,k}} \exp\Big(i\,\frac{2\pi}{m_{n+1}}\, l\Big(\sum_{q=1}^{j-1} m_q + k\Big)\Big)(\underbrace{0,\dots,0}_{l}, h_{j,k}, 0, \dots), \tag{16}$$

$l = 1, \dots, m_{n+1}$. (Recall that $m_{n+1} \leq M - 2$.)

Since $\sum_{k=1}^{m_j} \beta_{j,k} = \Phi_j(T_{j,n}) = 1$ and $\sum_{j=1}^{n} a_{j,n} = 1$ we deduce that

$$\{\iota_n h_{j,k}\}_{j=1,\dots,n,\, k=1,\dots,m_j} \cup \{h_{n+1,l}\}_{l=1}^{m_{n+1}}$$

is an ON-system in H_{n+1}. We have

$$\begin{aligned}\langle \pi_n(T_{j,n}) h_{n+1,l}, h_{n+1,l'}\rangle &= \begin{cases} a_{j,n} \sum_{k=1}^{m_j} \beta_{j,k} \langle T_{j,n} h_{j,k}, h_{j,k}\rangle & \text{if } l = l',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} a_{j,n}\Phi_j(T_{j,n}) & \text{if } l = l',\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} a_{j,n} & \text{if } l = l',\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{17}$$

From now on we regard Φ_j as a functional on $\mathcal{L}(H_{n+1})$ by putting

$$\Phi_j \begin{pmatrix} U_{1,1} & \dots & U_{1,M} \\ \vdots & & \vdots \\ U_{M,1} & \dots & U_{M,M} \end{pmatrix} = \Phi_j(U_{1,1}), \qquad U_{i,k} \in \mathcal{L}(H_n). \tag{18}$$

We define

$$\Phi_{n+1} = \frac{1}{m_{n+1}} \sum_{l=1}^{m_{n+1}} h_{n+1,l} \otimes h_{n+1,l}. \tag{19}$$

Then $\Phi_{n+1}|_{\pi_n \mathcal{L}(H_n)} = \sum_{j=1}^{n} a_{j,n} \Phi_j|_{\pi_n\mathcal{L}(H_n)}$. Indeed, for $T \in \mathcal{L}(H_n)$ we have, in view of (19),

$$\begin{aligned}\Phi_{n+1}(\pi_n(T)) &= \frac{1}{m_{n+1}} \sum_{l=1}^{m_{n+1}} \sum_{j=1}^{n} \sum_{j'=1}^{n} \sum_{k=1}^{m_j} \sum_{k'=1}^{m_j} \sqrt{a_{j,n}a_{j',n}\beta_{j,k}\beta_{j',k'}}\\ &\quad\times \exp\Big(i\,\frac{2\pi}{m_{n+1}}\, l\Big(\Big(\sum_{q=1}^{j-1} m_q + k\Big) - \Big(\sum_{q=1}^{j'-1} m_q + k'\Big)\Big)\Big)\langle T h_{j,k}, h_{j',k'}\rangle\\ &= \sum_{j=1}^{n}\sum_{k=1}^{m_j} \beta_{j,k} a_{j,n} \langle T h_{j,k}, h_{j,k}\rangle = \sum_{j=1}^{n} a_{j,n}\Phi_j(T).\end{aligned} \tag{20}$$

Since $S_n h_{j,k} = h_{j,k}$ we also have

$$\Phi_{n+1}(\pi_n(S_n T S_n)) = \sum_{j=1}^{n} a_{j,n}\Phi_j(T).$$

Put

$$G = (\underbrace{H_n \oplus \ldots \oplus H_n}_{M-2\,\text{times}})_{(2)}$$

and regard G as a subspace of H_{n+1}, i.e. identify $g \in G$ with $(0, g, 0)$ in H_{n+1}. Let $Q : G \to \operatorname{span}\{h_{n+1,l}\}_{l=1}^{m_{n+1}}$ be the orthogonal projection. Moreover, let $S : G \to G$ be the involutive isometry with $S|_{QG} = \mathrm{id}$ and $S|_{(\mathrm{id}-Q)G} = -\mathrm{id}$. Hence, if $U \in \mathcal{L}(G)$ then

$$SUS = QUQ + (\mathrm{id}-Q)U(\mathrm{id}-Q) - QU(\mathrm{id}-Q) - (\mathrm{id}-Q)UQ. \tag{21}$$

Define S_{n+1} on $H_{n+1} = H_n \oplus G \oplus H_n$ by

$$S_{n+1}(h_1, g, h_2) = (S_n h_1, Sg, h_2).$$

Then $\iota_n \circ S_n = S_{n+1} \circ \iota_n$, which proves (5).

We obtain

$$E_{S_{n+1}}(\pi_n(T))(h_1, g, h_2) = (E_{S_n}(T)h_1, E_S(\pi_n(T)|_G)g, Th_2).$$

Hence (8) is satisfied. Moreover we have

$$S_{n+1}\iota_n h_{j,k} = \iota_n S_n h_{j,k} = \iota_n h_{j,k},$$

if $j = 1, \ldots, n$, and $S_{n+1}h_{n+1,l} = h_{n+1,l}$ by the definition of S.

For $T \in \mathcal{L}(H_n)$, (21) implies

$$E_S(\pi_n(T)|_G) = Q\pi_n(T)|_G Q + (\mathrm{id}-Q)\pi_n(T)|_G(\mathrm{id}-Q). \tag{22}$$

Put

$$T_{j,n+1} = \begin{pmatrix} T_{j,n} & 0 & 0 \\ 0 & E_S(\pi_n(T_{j,n})|_G) & 0 \\ 0 & 0 & T_{j,n} \end{pmatrix} - a_{j,n}\begin{pmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad j = 1, \ldots, n,$$

and

$$T_{n+1,n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that $\{T_{j,n+1}\}_{j=1}^{n+1}$ is the unit vector basis of l_∞^{n+1}. Indeed, by definition we have

$$T_{j,n+1}\iota_n h_{l,k} = \begin{cases} \iota_n h_{j,k} & \text{if } l = j, \\ 0 & \text{otherwise,} \end{cases} \qquad j = 1, \ldots, n,$$

and $T_{n+1,n+1}\iota_n h_{l,k} = 0$. Moreover, by (17) and (22) we obtain

$$E_S(\pi_n(T_{j,n}))h_{n+1,l} = a_{j,n}h_{n+1,l},$$

which yields

$$T_{j,n+1}h_{n+1,k} = \begin{cases} h_{n+1,k} & \text{if } j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (18), (19), (20) and (22) imply

$$\Phi_j(T_{l,n+1}) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{otherwise,} \end{cases} \qquad j = 1, \ldots, n+1. \tag{23}$$

On the other hand, let $(h_1, g, h_2) \in H_{n+1}$ be of norm one and θ_i be complex numbers with $|\theta_i| = 1$, $i = 1, \ldots, n+1$. Then we obtain (by (22))

$$\begin{aligned} &\Big\| \sum_{j=1}^{n+1} \theta_j T_{j,n+1}(h_1, g, h_2) \Big\|^2 \\ &\quad = \Big\| \sum_{j=1}^{n} \theta_j T_{j,n} h_1 \Big\|^2 + \|Qg\|^2 \\ &\qquad + \Big\| (\mathrm{id} - Q) \sum_{j=1}^{n} \theta_j \pi_n(T_{j,n}) \Big|_G (\mathrm{id} - Q) g \Big\|^2 + \Big\| \sum_{j=1}^{n} \theta_j T_{j,n} h_2 \Big\|^2 \\ &\quad \le \|h_1\|^2 + \|g\|^2 + \|h_2\|^2. \end{aligned}$$

Here we used the fact that $\|\sum_{j=1}^{n} \theta_j T_{j,n}\| \le 1$ by the hypothesis and that $T_{j,n+1}h_{n+1,k} = 0$ if $j < n+1$. Hence $T_{j,n+1}Qg = 0$. This proves that $\|\sum_{j=1}^{n+1} \theta_j T_{j,n+1}\| \le 1$. In connection with (23) we deduce that $\{T_{j,n+1}\}_{j=1}^{n+1}$ is the unit vector basis of l_∞^{n+1}.

If we put

$$\tau_n T_{j,n} = \begin{pmatrix} T_{j,n} & 0 & 0 \\ 0 & E_S(\pi_n(T_{j,n})|_G) & 0 \\ 0 & 0 & T_{j,n} \end{pmatrix}$$

then τ_n is an isometry from $\mathcal{E}_n = \mathrm{span}\{T_{j,n}\}_{j=1}^{n}$ into $\mathcal{E}_{n+1} = \mathrm{span}\{T_{j,n+1}\}_{j=1}^{n+1}$ with $\Phi_{n+1}(\iota_n T_{j,n}) = a_{j,n}$ by (20). We have already seen that

$$\begin{aligned} T_{j,n+1}\iota_n h_{l,k} &= \iota_n T_{j,n} h_{l,k} \\ &= \begin{cases} \iota_n h_{j,k} & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } j = 1, \ldots, n+1,\ l \le n, \end{aligned}$$

and

$$T_{j,n+1}h_{n+1,l} = \begin{cases} h_{n+1,l} & \text{if } j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we introduce

$$P_{n+1}(T) = \sum_{j=1}^{n+1} \Phi_j(T) T_{j,n+1} \qquad \text{if } T \in \mathcal{L}(H_{n+1}).$$

This is certainly a contractive projection. Since $\Phi_j(S_{n+1}TS_{n+1}) = \Phi_j(T)$, by (11), (18), (19), the definition of S_{n+1} and the fact that $S_n h_{j,k} = h_{j,k}$ we obtain $P_{n+1}(T) = P_{n+1}(S_{n+1}TS_{n+1})$ for all $T \in \mathcal{L}(H_{n+1})$. This proves (7).

Moreover, by hypothesis and the definitions of $T_{j,n+1}$ and S_{n+1} we obtain $T_{j,n+1}S_{n+1} = S_{n+1}T_{j,n+1}$, $j = 1, \ldots, n+1$. (Recall that $QS = SQ$.) This proves (6). Furthermore, if $T \in \mathcal{L}(H_n)$ and $h \in H_n$ then, by (18), (20) and the definition of $T_{j,n+1}$, we obtain

$$\begin{aligned} P_{n+1}(\pi_n(T))\iota_n h &= \sum_{j=1}^{n} \Phi_j(T) T_{j,n+1}\iota_n h + \sum_{j=1}^{n} a_{j,n}\Phi_j(T)T_{n+1,n+1}\iota_n h \\ &= \sum_{j=1}^{n} \Phi_j(T)\iota_n T_{j,n} h, \end{aligned}$$

i.e. $P_{n+1}(\pi_n(T)) \circ \iota_n = \iota_n \circ P_n(T)$, which proves (4).

Finally, for $T \in \mathcal{L}(H_n)$, we have

$$\begin{aligned} P_{n+1}(\pi_n(T)) &= \sum_{j=1}^{n} \Phi_j(T) T_{j,n+1} + \sum_{j=1}^{n} a_{j,n}\Phi_j(T)T_{n+1,n+1} \\ &= \sum_{j=1}^{n} \Phi_j(T) \begin{pmatrix} T_{j,n} & 0 & 0 \\ 0 & E_S(\pi_n(T_{j,n})|_G) & 0 \\ 0 & 0 & T_{j,n} \end{pmatrix} \\ &= \tau_n P_n(T), \end{aligned}$$

which proves (2).

This concludes the proof of Claim (b). ■

Now, (9), (10), (14) and (15) show that the C^*-algebra constructed in the proof of Claim (a) is of the form $\mathcal{A} = \mathcal{A}_{(m_n+1)}$. Hence $\mathcal{A}$ is contractively complemented in the CAR-algebra. This finishes the proof of the Theorem. ■

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