STUDIA MATHEMATICA 160 (2) (2004)

Strichartz inequalities for the Schrödinger equation with the full Laplacian on the Heisenberg group

by

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Abstract. We prove Strichartz inequalities for the solution of the Schrödinger equation related to the full Laplacian on the Heisenberg group. A key point consists in estimating the decay in time of the L^∞ norm of the free solution; this requires a careful analysis due also to the non-homogeneous nature of the full Laplacian.

1. Introduction. In a recent paper ([BGX]), H. Bahouri, P. Gérard and C.-J. Xu studied the following Cauchy problem for the wave equation on the Heisenberg group $\mathbf{H}_n$ of topological dimension $2n+1$ and homogeneous dimension $N = 2n+2$:

$$(1)\qquad \begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = f(x,t) \in L^1((0,T), L^2(\mathbf{H}_n)), & t > 0, \\ u(x,0) = u_0(x) \in \dot{H}^1(\mathbf{H}_n), \\ \partial_t u(x,0) = u_1(x) \in L^2(\mathbf{H}_n), \end{cases}$$

where Δ is the sub-Laplacian on $\mathbf{H}_n$ (to be defined in Section 2) and $\dot{H}^1(\mathbf{H}_n)$ is the Sobolev space in $\mathbf{H}_n$ which is related to the sub-Laplacian operator. Their main aim was to establish a sort of Strichartz inequalities for the solution $(u, \partial_t u) \in C([0,T], \dot{H}^1(\mathbf{H}_n) \times L^2(\mathbf{H}_n))$ of (1); the results they obtained are the following (the Besov spaces $\dot{B}_r^{\varrho,q}(\mathbf{H}_n)$ will be defined in Section 4):

THEOREM 1 ([BGX], Théorème 1.2). *Let $u_0 \in \dot{B}_1^{N-1/2,1}(\mathbf{H}_n)$, $u_1 \in \dot{B}_1^{N-1/2-1,1}(\mathbf{H}_n)$ (for example $u_0, u_1 \in \mathcal{S}(\mathbf{H}_n)$), $f = 0$ in (1) and let u be a solution of (1). Then, for every $t > 0$,*

$$\|u(t)\|_{L^\infty(\mathbf{H}_n)} \le Ct^{-1/2}(\|u_0\|_{\dot{B}_1^{N-1/2,1}(\mathbf{H}_n)} + \|u_1\|_{\dot{B}_1^{N-1/2-1,1}(\mathbf{H}_n)}).$$

Moreover, there exist $u_0, u_1 \in \mathcal{S}(\mathbf{H}_n)$ such that the corresponding solution of (1) with $f = 0$ satisfies

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2000 *Mathematics Subject Classification*: 22E25, 35B65, 58J35.

Key words and phrases: Strichartz inequalities, Schrödinger equation, full Laplacian, Heisenberg group.

$$\|u(t)\|_{L^\infty(\mathbf{H}_n)} \geq C' t^{-1/2} \quad \textit{for } t \geq 1$$

with $C' > 0$.

THEOREM 2 ([BGX], Théorème 4.1). *Let* $r_1, r_2 \in [2, \infty]$; *suppose moreover that* $\varrho_1, \varrho_2 \in \mathbb{R}$, $p_1, p_2 \in [1, \infty]$ *satisfy*:

(i) $2/p_i + 1/r_i \leq 1/2$ *for* $i = 1, 2$;
(ii) $1/p_1 + N/r_1 - \varrho_1 = N/2 - 1$;
(iii) $1/p_2 + N/r_2 - \varrho_2 = N/2$.

Let r_i', p_i' *be such that* $1/r_i' + 1/r_i = 1$ *and* $1/p_i' + 1/p_i = 1$ *for* $i = 1, 2$. *Finally let* $u = v + w$ *be the solution of the Cauchy problem* (1) *where* v *is the solution corresponding to* $f = 0$ *and* w *is the solution corresponding to* $u_0 = u_1 = 0$. *Then*, *for every real interval* I *containing* 0, *the following estimates hold*:

$$\|v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}^{\varrho_1,2}_{r_1}(\mathbf{H}_n))} + \|\partial_t v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}^{\varrho_1-1,2}_{r_1}(\mathbf{H}_n))} \leq C(\|u_0\|_{\dot{H}^1(\mathbf{H}_n)} + \|u_1\|_{L^2(\mathbf{H}_n)}),$$

$$\|w\|_{L^{p_1}_{I}(\dot{B}^{\varrho_1,2}_{r_1}(\mathbf{H}_n))} + \|\partial_t w\|_{L^{p_1}_{I}(\dot{B}^{\varrho_1-1,2}_{r_1}(\mathbf{H}_n))} \leq C\|f\|_{L^{p_2'}_{I}(\dot{B}^{-\varrho_2,2}_{r_2'}(\mathbf{H}_n))},$$

the constant C *being independent of the interval* I.

Strichartz inequalities in the real setting have been proved for many dispersive equations (for the Schrödinger and wave equations, see for example [Str], [GV1], [GV2] and the more recent [KT], [Tao], [Vil]); the key point for proving them is to estimate the L^∞ norm of the solution of the free equation (that is, with $f = 0$) at a fixed time t. In particular, Strichartz inequalities are quite easy to prove for the Schrödinger equation in $\mathbb{R}^n$ (here L is the Laplacian on $\mathbb{R}^n$):

$$\text{(2)} \qquad \begin{cases} \partial_t u(x,t) - iLu(x,t) = f(x,t) \in L^1((0,T), L^2(\mathbb{R}^n)), & t > 0, \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^n). \end{cases}$$

In fact, the kernel of the Schrödinger operator can be written explicitly and the solution of (2) for $f = 0$ is

$$u(x,t) = (4\pi i t)^{-n/2}\, e^{i|\cdot|^2/(4t)} * u_0(x), \qquad x \in \mathbb{R}^n,\ t > 0.$$

Thus, it is natural to wonder whether such a generalization for Strichartz inequalities, obtained for the wave equation on $\mathbf{H}_n$ (with the sub-Laplacian), remains true also for the corresponding Schrödinger equation:

$$\text{(3)} \qquad \begin{cases} \partial_t u(x,t) - i\Delta u(x,t) = f(x,t) \in L^1((0,T), L^2(\mathbf{H}_n)), & t > 0, \\ u(x,0) = u_0(x) \in L^2(\mathbf{H}_n). \end{cases}$$

The answer is, surprisingly, no and an interesting counterexample was given by H. Bahouri, P. Gérard and C.-J. Xu in [BGX] (see Section 3 below).

Moreover, it has recently been proved by A. Sikora and J. Zienkiewicz ([SZ]) that the convolution kernel of the Schrödinger operator related to the sub-Laplacian is not even in $L^\infty(\mathbf{H}_n)$, because at every fixed time t it has a Calderón–Zygmund type singularity localized at a particular point in $\mathbf{H}_n$ (depending on t). In the same paper, the authors have found an explicit formulation also for the convolution kernel related to the full Laplacian on $\mathbf{H}_n$ and they have showed that the latter is indeed smooth at every time t even though it is not bounded. Another interesting difference in the behavior of the sub-Laplacian and the full Laplacian may be found in a paper by J. Zienkiewicz ([Zie1]) dealing with the Carleson problem about pointwise convergence to its initial data of the solution of the free Schrödinger equation. This result has recently been improved in [Zie2].

One of the reasons why the sub-Laplacian is considered a good generalization to $\mathbf{H}_n$ for the Laplacian in $\mathbb{R}^n$, more than the full Laplacian, is its homogeneity property as a differential operator, namely:

$$\Delta(f(rz, r^2 s)) = r^2 \Delta f(rz, r^2 s), \qquad r > 0,\ (z, s) \in \mathbf{H}_n.$$

This is not true any more for the full Laplacian and this lack of homogeneity involves some technical difficulties.

Nevertheless, in this paper we will prove that, for the Schrödinger equation related to the full Laplacian $\mathcal{L}$, which reads:

$$\begin{cases} \partial_t u(x,t) - i\mathcal{L}u(x,t) = f(x,t) \in L^1((0,T), L^2(\mathbf{H}_n)), & t > 0, \\ u(x,0) = u_0(x) \in L^2(\mathbf{H}_n), \end{cases} \tag{4}$$

a decay in time for the solution of the free equation is still true, even if the convolution kernel is not bounded (Proposition 9 and Corollary 10); we will prove moreover that such a decay is sharp (Proposition 9 and Corollary 10) and, as a consequence of the decay in time, we will establish new Strichartz inequalities for the solution of the equation with external term (Theorem 11).

We would like to thank Giancarlo Mauceri for his precious help and encouragement.

2. Notation and preliminaries. In this paper $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}^*$ the set of non-zero real numbers.

In this section we recall some basic facts about harmonic analysis on the Heisenberg group. For the proofs and further information, see e.g. [BJRW], [Far], [Gel], [Nac].

Fix $n \in \mathbb{Z}_+$; the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}_n$ is the nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, with the following non-commutative multiplication law:

$$(z,s)(z',s') = (z+z', s+s'+2\,\mathrm{Im}\langle z,z'\rangle)$$

where $\langle z,z'\rangle = \sum_{j=1}^n z_j \overline{z}'_j$. The Lie algebra of $\mathbf{H}_n$ is generated by the left-invariant vector fields $Z_1,\dots,Z_n,\overline{Z}_1,\dots,\overline{Z}_n,S$, where

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j\frac{\partial}{\partial s}, \qquad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - iz_j\frac{\partial}{\partial s}, \qquad S = \frac{\partial}{\partial s}.$$

$\mathbf{H}_n$ is a stratified group endowed with a family of dilations $\{\delta_r : r>0\}$ defined by

$$\delta_r(z,s) = (rz, r^2 s).$$

The homogeneous dimension of $\mathbf{H}_n$ is therefore $N = 2n+2$. The sub-Laplacian on $\mathbf{H}_n$ is

$$\varDelta = 2\sum_{j=1}^n (Z_j\overline{Z}_j + \overline{Z}_j Z_j).$$

The full Laplacian is

$$\mathcal{L} = \varDelta + S^2.$$

The bi-invariant Haar measure on $\mathbf{H}_n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. The convolution $f*g$ of two functions $f,g\in L^1(\mathbf{H}_n)$ is defined by

$$(f*g)(x) = \int_{\mathbf{H}_n} f(xy^{-1})g(y)\,dy, \qquad x\in\mathbf{H}_n.$$

As usual, we denote by $\mathcal{S}(\mathbf{H}_n)$ the Schwartz space of rapidly decreasing smooth functions on $\mathbf{H}_n$ and by $\mathcal{S}'(\mathbf{H}_n)$ the dual space of $\mathcal{S}(\mathbf{H}_n)$, i.e. the space of tempered distributions on $\mathbf{H}_n$. A function f on $\mathbf{H}_n$ is said to be *radial* if the value of $f(z,s)$ depends only on $|z|$ and s, where $|z| = (\sum_{j=1}^n |z_j|^2)^{1/2}$. We denote by $\mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n)$ and by $L^p_{\mathrm{rad}}(\mathbf{H}_n)$, $1\le p\le\infty$, the spaces of radial functions in $\mathcal{S}(\mathbf{H}_n)$ and in $L^p(\mathbf{H}_n)$, respectively. The space $L^1_{\mathrm{rad}}(\mathbf{H}_n)$ is a commutative, closed $*$-subalgebra of $L^1(\mathbf{H}_n)$. The Gelfand spectrum $\varSigma$ of $L^1_{\mathrm{rad}}(\mathbf{H}_n)$ can be identified, as a measure space, with the space $\mathbb{N}\times\mathbb{R}^*$ equipped with the Godement–Plancherel measure μ defined by

$$\int_{\varSigma} F(\psi)\,d\mu(\psi) = \frac{2^{n-1}}{\pi^{n+1}}\sum_{m=0}^{\infty}\binom{m+n-1}{m}\int_{\mathbb{R}^*} F(m,\lambda)|\lambda|^n\,d\lambda.$$

Let $L_m^{(\alpha)}$ be the Laguerre polynomial of type $\alpha\in\mathbb{N}$ and degree $m\in\mathbb{N}$, given by

$$L_m^{(\alpha)}(\tau) = \sum_{k=0}^m \frac{(-1)^k}{k!}\binom{m+\alpha}{k+\alpha}\tau^k, \qquad \tau\in\mathbb{R}.$$

The Gelfand transform of a function $f \in L^1_{\mathrm{rad}}(\mathbf{H}_n)$ is given by

$$\widehat{f}(m,\lambda) = \int_{\mathbf{H}_n} f(x)\omega_{m,\lambda}(x)\,dx, \qquad (m,\lambda) \in \mathbb{N} \times \mathbb{R}^*,$$

where $\omega_{m,\lambda}$ is the spherical function on $\mathbf{H}_n$ defined by

$$\omega_{m,\lambda}(z,s) = \binom{m+n-1}{m}^{-1} e^{i\lambda s}\, e^{-|\lambda||z|^2} L_m^{(n-1)}(2|\lambda|\,|z|^2). \tag{5}$$

By the classical properties of spherical functions

$$\|\omega_{m,\lambda}\|_{L^\infty(\mathbf{H}_n)} = \omega_{m,\lambda}(0,0) = 1.$$

Moreover, if $f \in \mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n)$, then Δf and $\mathcal{L} f$ are in $\mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n)$ and

$$\widehat{\Delta f}(m,\lambda) = -4(2m+n)|\lambda|\widehat{f}(m,\lambda) \tag{6}$$

and

$$\widehat{\mathcal{L} f}(m,\lambda) = -(4(2m+n)|\lambda| + \lambda^2)\widehat{f}(m,\lambda). \tag{7}$$

By the Godement–Plancherel theory, the Gelfand transform extends uniquely to a unitary operator $\mathcal{G} : L^2_{\mathrm{rad}}(\mathbf{H}_n) \to L^2(\Sigma)$. We will write again $\widehat{f}$ instead of $\mathcal{G}f$. If $f \in L^2_{\mathrm{rad}}(\mathbf{H}_n)$ and $\widehat{f} \in L^1(\Sigma)$, then the following inversion formula holds:

$$f(x) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} \binom{m+n-1}{m} \int_{\mathbb{R}^*} \widehat{f}(m,\lambda)\omega_{m,-\lambda}(x)|\lambda|^n\,d\lambda, \qquad x \in \mathbf{H}_n. \tag{8}$$

By the spectral theorem the multiplier operators $e^{it\Delta}$ and $e^{it\mathcal{L}}$ are bounded on $L^2(\mathbf{H}_n)$ for every $t \in \mathbb{R}$. Moreover, if $f \in L^2_{\mathrm{rad}}(\mathbf{H}_n)$, by (6) and (7) we have

$$\widehat{e^{it\Delta} f}(m,\lambda) = e^{-4it(2m+n)|\lambda|}\widehat{f}(m,\lambda) \tag{9}$$

and

$$\widehat{e^{it\mathcal{L}} f}(m,\lambda) = e^{-it(4(2m+n)|\lambda|+\lambda^2)}\widehat{f}(m,\lambda). \tag{10}$$

The operators $e^{it\Delta}$ and $e^{it\mathcal{L}}$ commute with left translations, so by Schwartz's kernel theorem for homogeneous groups ([KVW, Theorem 3.2]) they are given by right convolution with tempered distributions, which are called the *kernels* of $e^{it\Delta}$ and $e^{it\mathcal{L}}$, respectively.

The space $\mathcal{G}(\mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n))$ has been completely characterized in [BJR]. For our purposes, it is sufficient to remark that if $f \in \mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n)$ then $\widehat{f} \in L^1(\Sigma)$, so f can be recovered by the inversion formula (8). Moreover, for any $R \in \mathcal{S}(\mathbb{R})$ the function

$$F(m,\lambda) = R((2m+n)|\lambda|)$$

is in $\mathcal{G}(\mathcal{S}_{\mathrm{rad}}(\mathbf{H}_n))$ (see [Hul], [Mau]).

3. The counterexample in the sub-Laplacian case. In this section, we would like to comment on the counterexample given in [BGX]. Let us fix a function $Q \in C_{\rm c}^{\infty}((1,2))$ and consider the Schrödinger equation (3) with $f = 0$ and

$$u_0(z,s) = \frac{2^{n-1}}{\pi^{n+1}} \int_1^2 e^{-i\lambda s} Q(\lambda) e^{-\lambda |z|^2} \lambda^n \, d\lambda. \tag{11}$$

Thus, the initial data $u_0 \in \mathcal{S}_{\rm rad}(\mathbf{H}_n)$ satisfy

$$\widehat{u}_0(m,\lambda) = \begin{cases} 0 & \text{if } m \neq 0,\ \lambda \in \mathbb{R}^*, \\ Q(\lambda) & \text{if } m = 0,\ \lambda \in \mathbb{R}^*, \end{cases}$$

as is possible to check by comparing (11) with the inversion formula (8). The solution of the Cauchy problem can therefore be written explicitly thanks to (8) and (9):

$$\begin{aligned} u(z,s,t) = (e^{it\Delta} u_0)(z,s) &= \frac{2^{n-1}}{\pi^{n+1}} \int_1^2 e^{-i\lambda(s+4nt)} Q(\lambda) e^{-\lambda|z|^2} \lambda^n \, d\lambda \\ &= u_0(z, s+4nt). \end{aligned}$$

Thus, for all $p \in [1,\infty]$ and $t \in \mathbb{R}$, we have $\|u(t)\|_{L^p(\mathbf{H}_n)} = \|u_0\|_{L^p(\mathbf{H}_n)}$. In particular, we have no decay in time for $\|u(t)\|_{L^\infty(\mathbf{H}_n)}$. The key point in this counterexample is that the Gelfand transform of the sub-Laplacian is indeed linear in λ. On the contrary, the Gelfand transform of the full Laplacian is not and this will be crucial in what follows.

4. Littlewood–Paley decomposition and Besov spaces. Let R be a non-negative, even function in $C_{\rm c}^{\infty}((-4,-1/2) \cup (1/2,4))$ such that

$$\sum_{j\in\mathbb{Z}} R(2^{-2j}\tau) = 1, \qquad \tau \neq 0.$$

As in [BGX], for any $j \in \mathbb{Z}$ we define on $\mathbf{H}_n$ the function $\varphi_j \in \mathcal{S}_{\rm rad}(\mathbf{H}_n)$ such that

$$\widehat{\varphi}_j(m,\lambda) = R((2m+n)2^{-2j}\lambda). \tag{12}$$

The fact that $\varphi_j \in \mathcal{S}_{\rm rad}(\mathbf{H}_n)$ is guaranteed by the remarks at the end of Section 2. Moreover, a direct application of the inversion formula (8) gives the following homogeneity property:

$$\varphi_j(z,s) = 2^{Nj} \varphi_0(2^j z, 2^{2j} s). \tag{13}$$

Since $\widehat{\varphi_j * \varphi_{j'}} = \widehat{\varphi}_j \widehat{\varphi}_{j'}$, the support properties of the functions $\widehat{\varphi}_j$ also imply

$$\varphi_j * \varphi_{j'} = 0 \quad \text{if } |j - j'| \geq 2. \tag{14}$$

We will write

$$\Delta_j f = f * \varphi_j.$$

In [BGX] it is proved that for any $f \in L^2(\mathbf{H}_n)$ we have

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \quad \text{in } L^2(\mathbf{H}_n). \tag{15}$$

This decomposition is called a *homogeneous Littlewood–Paley decomposition* of f (more generally, the setting where this decomposition holds is actually the space $\mathcal{S}'(\mathbf{H}_n)$ modulo polynomials, but we slide over this detail). Moreover, the Littlewood–Paley theory applies to this decomposition and we have the following characterization of the $L^p(\mathbf{H}_n)$ spaces for $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in $\mathcal{S}'(\mathbf{H}_n)$:

$$f \in L^p(\mathbf{H}_n) \;\Leftrightarrow\; \Big(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2\Big)^{1/2} \in L^p(\mathbf{H}_n), \qquad 1 < p < \infty,$$

with equivalence between the two L^p norms. We stress the inequality

$$\|f\|_{L^\infty(\mathbf{H}_n)} \le \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^\infty(\mathbf{H}_n)}, \qquad f \in L^2(\mathbf{H}_n), \tag{16}$$

which is an immediate consequence of (15). Both sides of (16) are allowed to be infinite.

As in [BGX], we will also consider the functions

$$\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \qquad j \in \mathbb{Z}, \tag{17}$$

which satisfy the identity

$$\varphi_j = \varphi_j * \widetilde{\varphi}_j, \qquad j \in \mathbb{Z}. \tag{18}$$

We now recall the definition of a homogeneous Besov space on $\mathbf{H}_n$.

Definition 3. Let $\varrho \in \mathbb{R}$ and $q, r \in [1,\infty]$. The *homogeneous Besov space* on $\mathbf{H}_n$, denoted by $\dot{B}^{\varrho,q}_r(\mathbf{H}_n)$, is defined as follows:

$$\dot{B}^{\varrho,q}_r(\mathbf{H}_n) = \{f \in \mathcal{S}'(\mathbf{H}_n)/\mathcal{P} : \{2^{j\varrho}\|\Delta_j f\|_{L^r(\mathbf{H}_n)}\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})\}$$

where $\mathcal{P}$ is the space of polynomials on $\mathbf{H}_n$.

A careful introduction to these spaces and their inhomogeneous version may be found in [BGX] and [BG]. For our purposes, it will be enough to recall the properties contained in the following proposition.

Proposition 4 ([BGX], [Bou], [Tri]). (i) $\dot{B}^{\varrho,q}_r(\mathbf{H}_n)$, *endowed with the norm*

$$\|f\|_{\dot{B}^{\varrho,q}_r(\mathbf{H}_n)} = \big\|\{2^{j\varrho}\|\Delta_j f\|_{L^r(\mathbf{H}_n)}\}_{j\in\mathbb{Z}}\big\|_{l^q(\mathbb{Z})},$$

is a Banach space of equivalence classes of tempered distributions; *moreover*, *for $\varrho < N/r$ it is possible to realize this space as a space of tempered distributions by choosing the element $\sum_{j\in\mathbb{Z}} \Delta_j f$ in each class $[f]$*;

(ii) *the definition of* $\dot{B}^{\varrho,q}_r(\mathbf{H}_n)$ *does not depend on the choice of the function* R *in the Littlewood–Paley decomposition*;

(iii) *for all* $\varrho \in \mathbb{R}$ *and* $q, r \in (1,\infty]$, $\dot{B}^{\varrho,q}_r(\mathbf{H}_n)$ *is the dual space of* $\dot{B}^{-\varrho,q'}_{r'}(\mathbf{H}_n)$ *where* $1/q' + 1/q = 1$ *and* $1/r' + 1/r = 1$;

(iv) *for all* $q \in [1,\infty]$ *we have*

$$\dot{B}^{\varrho_1,q}_{r_1}(\mathbf{H}_n) \subset \dot{B}^{\varrho_2,q}_{r_2}(\mathbf{H}_n), \qquad \frac{1}{r_1} - \frac{\varrho_1}{N} = \frac{1}{r_2} - \frac{\varrho_2}{N},\ \varrho_1 \geq \varrho_2;$$

(v) $\dot{B}^{0,2}_2(\mathbf{H}_n) = L^2(\mathbf{H}_n)$;

(vi) $\mathcal{S}(\mathbf{H}_n) \subset \dot{B}^{\varrho,q}_r(\mathbf{H}_n)$ *if* $\varrho > -N/r'$;

(vii) *for all* $\theta, \varrho_1, \varrho_2, q_1, q_2, r_1, r_2$ *satisfying* $\theta \in (0,1)$, $\varrho_i \in \mathbb{R}$, $q_i, r_i \in (1,\infty)$, *we have*

$$[\dot{B}^{\varrho_1,q_1}_{r_1}(\mathbf{H}_n), \dot{B}^{\varrho_2,q_2}_{r_2}(\mathbf{H}_n)]_\theta = \dot{B}^{\varrho,q}_r(\mathbf{H}_n)$$

with

$$\varrho = (1-\theta)\varrho_1 + \theta\varrho_2, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}.$$

5. A dispersive inequality. Our main aim, which we will achieve in Section 6, is to establish Strichartz inequalities for the solution of the Cauchy problem (4). From now on, we will denote by S_t the operator $e^{it\mathcal{L}}$, $t \in \mathbb{R}$. As for the classical Schrödinger equation (2) in $\mathbb{R}^n$, also in this case there exists a unique solution $u \in C([0,T], L^2(\mathbf{H}_n))$ of (4), which is given by the sum $u = v + w$ where

$$v(t) = S_t u_0 \tag{19}$$

is the solution of (4) with $f = 0$ and

$$w(t) = \int_0^t S_{t-\sigma} f(\sigma)\, d\sigma \tag{20}$$

is the solution of (4) with $u_0 = 0$.

In this section we will prove a dispersive inequality for v, i.e. an estimate for the decay in time of $\|v(t)\|_{L^\infty(\mathbf{H}_n)}$, which we will use in Section 6 to prove our Strichartz inequalities. Moreover, we will prove that such an estimate is sharp.

Let us begin by introducing the tools of our method; first of all, we recall the stationary phase lemma (see e.g. [Ste, pages 332–334]) that will be the central argument:

LEMMA 5. *Suppose* $g \in C^\infty([a,b])$ *and* $h \in C^\infty_c((a,b))$, *with* g *real-valued. Suppose also* $|g^{(k)}(x)| \geq \delta$ *for any* $x \in [a,b]$, *with* $k \in \mathbb{Z}_+$ *and* $\delta > 0$. *If* $k = 1$, *we also require that* g' *is monotonic in* $[a,b]$. *Then there exists a* *constant $C > 0$, which depends only on k but not on a, b, g, h, δ, such that for every $\sigma \in \mathbb{R}^*$,*

$$\Big|\int_a^b e^{-i\sigma g(x)} h(x)\,dx\Big| \le C|\sigma|^{-1/k}\delta^{-1/k}\int_a^b |h'(x)|\,dx.$$

In order to show the sharpness of our estimate, we will also need the following

Lemma 6. *Suppose $0 < a < \alpha < \beta < b$ and $n \in \mathbb{Z}_+$. Suppose also $h \in C_c^\infty(\mathbb{R})$ such that $h(x) = 1$ for any $x \in [a, b]$. Then there exist two positive constants C and L such that for any $\sigma \ge L$ and $\gamma \in [\alpha, \beta]$ we have*

$$\Big|\int_{\mathbb{R}} e^{-i\sigma(x^2-2\gamma x)} h(x)x^n\,dx\Big| \ge \frac{C}{\sqrt{\sigma}}.$$

Proof. Observe that

$$\int_{\mathbb{R}} e^{-i\sigma(x^2-2\gamma x)} h(x)x^n\,dx = e^{i\sigma\gamma^2}(J_{\sigma,\gamma} + K_{\sigma,\gamma}) \tag{21}$$

where

$$J_{\sigma,\gamma} = \gamma^n \int_{\mathbb{R}} e^{-(1+i\sigma)t^2}\,dt = \gamma^n \sqrt{\frac{\pi}{|1+i\sigma|}}\, e^{-\frac{i}{2}\arctan\sigma} \tag{22}$$

(see e.g. [Ste, page 335]) and

$$K_{\sigma,\gamma} = \int_{\mathbb{R}} e^{-i\sigma(x-\gamma)^2}(h(x)x^n - \gamma^n e^{-(x-\gamma)^2})\,dx. \tag{23}$$

By (22) there exist two positive constants C_0 and L_0 such that for any $\sigma \ge L_0$ and $\gamma \in [\alpha, \beta]$ we have

$$|J_{\sigma,\gamma}| \ge \frac{C_0}{\sqrt{\sigma}}. \tag{24}$$

On the other hand, for $\sigma > 0$, an integration by parts in (23) gives

$$K_{\sigma,\gamma} = \frac{1}{2i\sigma}\int_{\mathbb{R}} e^{-i\sigma(x-\gamma)^2} F_\gamma'(x)\,dx \tag{25}$$

where

$$F_\gamma(x) = \begin{cases} \dfrac{h(x)x^n - \gamma^n e^{-(x-\gamma)^2}}{x-\gamma} & \text{if } x \ne \gamma, \\ n\gamma^{n-1} & \text{if } x = \gamma. \end{cases}$$

It is straightforward to verify that $F_\gamma \in C^1(\mathbb{R})$. Moreover, one can prove that there exists $\delta > 0$ such that for any $\gamma \in [\alpha, \beta]$ we have $F_\gamma' \ge 0$ in $[\gamma - \delta, \gamma + \delta]$ and so

$$\int_{|x-\gamma|\le\delta} |F'_\gamma(x)|\,dx = F_\gamma(\gamma+\delta) - F_\gamma(\gamma-\delta) \le C_1 \tag{26}$$

where C_1 is independent of γ. On the other hand, for $\gamma \in [\alpha,\beta]$ it is easy to see that

$$\int_{|x-\gamma|>\delta} |F'_\gamma(x)|\,dx \le C_2 \tag{27}$$

where C_2 is independent of γ. By (25)–(27) there exists $C_3 > 0$ such that for any $\sigma > 0$ and $\gamma \in [\alpha,\beta]$ we have

$$|K_{\sigma,\gamma}| \le \frac{C_3}{\sigma}. \tag{28}$$

The conclusion follows from (21), (24) and (28). ∎

Moreover, we will use the following properties of the Laguerre polynomials (see [BGX], [EMOT]):

LEMMA 7. *For any $\alpha \in \mathbb{N}$ there exists $C_\alpha > 0$ such that for every $\tau \ge 0$ and $m \in \mathbb{N}$ the following inequalities hold*:

$$|L_m^{(\alpha)}(\tau)e^{-\tau/2}| \le C_\alpha(m+1)^\alpha, \qquad \left|\tau\frac{d}{d\tau}(L_m^{(\alpha)}(\tau)e^{-\tau/2})\right| \le C_\alpha(m+1)^\alpha.$$

Finally, we will exploit the following estimates, which can be easily proved by comparing the sums with the corresponding integrals:

LEMMA 8. *Fix $n \in \mathbb{Z}_+$. There exists a constant $C > 0$ such that for $0 < a < b$ we have*:

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\ge a}} (2m+n)^{-2} \le \frac{C}{a}, \tag{29}$$

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\le b}} (2m+n)^{-1/2} \le Cb^{1/2}, \tag{30}$$

$$\sum_{\substack{m\in\mathbb{N}\\ a\le 2m+n\le b}} (2m+n)^{-1} \le \log(Cb/a). \tag{31}$$

Now we can state our main proposition which concerns the Littlewood–Paley functions φ_j defined in Section 4.

PROPOSITION 9. *There exists a constant $C > 0$, which depends only on n, such that for any $t > 0$ and $j \in \mathbb{Z}$,*

$$\|S_t\varphi_j\|_{L^\infty(\mathbf{H}_n)} \le C\,2^{(N-2)j}t^{-1/2}. \tag{32}$$

Moreover, there exist $C' > 0$, $t_0 \ge 1$ and $j_0 \in \mathbb{Z}_+$, which depend only on n, *such that for any* $t \geq t_0$ *and* $j \geq j_0$,

$$\left|(S_t\varphi_j)\left(0, -\frac{2^{3+2j}t}{n+1}\right)\right| \geq C' 2^{(N-2)j} t^{-1/2}. \tag{33}$$

Proof. In this proof we will denote by C a positive constant which will not necessarily be the same at each occurrence, with the convention that C can depend only on n.

First we prove (32). Fix $t > 0$ and $j \in \mathbb{Z}$. By (10) and (12) we have

$$\widehat{S_t\varphi_j}(m, \lambda) = e^{-it(4(2m+n)|\lambda|+\lambda^2)} R((2m+n)2^{-2j}\lambda).$$

So, for $(z, s) \in \mathbf{H}_n$ fixed, by (8) and (5) we have

$$(S_t\varphi_j)(z, s) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{\infty} \int_{\mathbb{R}^*} e^{-i\lambda s} e^{-it(4(2m+n)|\lambda|+\lambda^2)} R((2m+n)2^{-2j}\lambda) \times e^{-|\lambda|\,|z|^2} L_m^{(n-1)}(2|\lambda|\,|z|^2)|\lambda|^n\, d\lambda.$$

Put $s' = s/t$ and perform the change of variable $x = (2m+n)2^{-2j}\lambda$. Then

$$(S_t\varphi_j)(z, s) = \frac{2^{n-1}}{\pi^{n+1}} 2^{Nj} \sum_{m=0}^{\infty} \int_{1/2<|x|<4} e^{-i2^{2j}tg_{s',m}(x)} h_{z,m}(x)\, dx \tag{34}$$

where

$$g_{s',m}(x) = \frac{2^{2j}x^2}{(2m+n)^2} + 4|x| + \frac{s'x}{2m+n} \tag{35}$$

and

$$h_{z,m}(x) = R(x)e^{-\frac{2^{2j}|x||z|^2}{2m+n}} L_m^{(n-1)}\left(\frac{2^{1+2j}|x|\,|z|^2}{2m+n}\right)\frac{|x|^n}{(2m+n)^{n+1}}. \tag{36}$$

So $g_{s',m}(-x) = g_{-s',m}(x)$ and $h_{z,m}(-x) = h_{z,m}(x)$. Therefore, by symmetry we can consider only the integral

$$I_m = \int_{1/2}^{4} e^{-i2^{2j}tg_{s',m}(x)} h_{z,m}(x)\, dx$$

and our assertion simply reads:

$$\sum_{m=0}^{\infty} |I_m| \leq C\, 2^{-2j} t^{-1/2}. \tag{37}$$

For the sake of simplicity, from now on we will write g_m and h_m instead of $g_{s',m}$ and $h_{z,m}$, respectively. Moreover, we put $M = 2m+n$. For $1/2 \leq x \leq 4$ we have

$$g_m'(x) = \frac{2^{1+2j}}{M^2}\, x + 4 + \frac{s'}{M} \tag{38}$$

and

$$g_m''(x) = \frac{2^{1+2j}}{M^2}. \tag{39}$$

Furthermore, by Lemma 7 one can verify (see also [BGX]) that

$$\|h_m\|_{L^\infty([1/2,4])} + \|h_m'\|_{L^1([1/2,4])} \leq CM^{-2}. \tag{40}$$

We now have three possible ways to estimate $|I_m|$: the first and most direct is

$$|I_m| \leq C\|h_m\|_{L^\infty([1/2,4])} \leq CM^{-2}, \tag{41}$$

which sums independently of j and t. The second, using Lemma 5 with the second derivative (given by (39)) and estimate (40), is

$$|I_m| \leq C\,2^{-2j}t^{-1/2}M^{-1}, \tag{42}$$

which does not sum. The third exploits Lemma 5 with the first derivative, but only when this is possible, i.e. when there exists $\delta_m > 0$ such that $|g_m'(x)| \geq \delta_m$ for any $x \in [1/2,4]$, and in this case by (40) we have

$$|I_m| \leq C\,2^{-2j}t^{-1}M^{-2}\delta_m^{-1}. \tag{43}$$

We are going to estimate $\sum_{m=0}^{\infty}|I_m|$ by splitting it into the sum of three terms and by applying to every term the most convenient among the three estimates above, after choosing a suitable δ_m. More precisely,

$$\sum_{m=0}^{\infty}|I_m| = \sum_{m\in A_1}|I_m| + \sum_{m\in A_2}|I_m| + \sum_{m\in A_3}|I_m|$$

where

$$\begin{aligned}
A_1 &= \{m \in \mathbb{N} : M \geq 2^{2j}t^{1/2}\},\\
A_2 &= \{m \in \mathbb{N} : M < 2^{2j}t^{1/2}\\
&\qquad\qquad \text{and } |g_m'(x)| \geq 2^{j-1}t^{-1/4}M^{-3/2} \text{ for any } x \in [1/2,4]\},\\
A_3 &= \{m \in \mathbb{N} : M < 2^{2j}t^{1/2}\\
&\qquad\qquad \text{and } |g_m'(x)| < 2^{j-1}t^{-1/4}M^{-3/2} \text{ for some } x \in [1/2,4]\}.
\end{aligned}$$

If $m \in A_1$ then estimates (41) and (29) yield the desired estimate

$$\sum_{m\in A_1}|I_m| \leq C\,2^{-2j}t^{-1/2}. \tag{44}$$

If $m \in A_2$ then, by applying (43) with $\delta_m = 2^{j-1}t^{-1/4}M^{-3/2}$, we have

$$|I_m| \leq C\,2^{-3j}t^{-3/4}M^{-1/2}$$

and hence estimate (30) yields the desired estimate

$$\sum_{m\in A_2}|I_m| \leq C\,2^{-2j}t^{-1/2}. \tag{45}$$

If $m \in A_3$ then estimate (42) yields

$$\sum_{m\in A_3} |I_m| \le C\, 2^{-2j} t^{-1/2} \sum_{m\in A_3} M^{-1}. \tag{46}$$

So we only have to prove that

$$\sum_{m\in A_3} M^{-1} \le C. \tag{47}$$

In fact, once we prove (47), estimate (37) follows directly from (44)–(46).

It is not restrictive to suppose $2^{2j}t^{1/2} > n$, otherwise $A_3 = \emptyset$. By (38) we can immediately verify that for $1/2 \le x \le 4$ the inequality $|g'_m(x)| < 2^{j-1}t^{-1/4}M^{-3/2}$ is satisfied if and only if

$$-\frac{2^{j-1}t^{-1/4}M^{1/2} + 4M^2 + s'M}{2^{1+2j}} < x < \frac{2^{j-1}t^{-1/4}M^{1/2} - 4M^2 - s'M}{2^{1+2j}}.$$

So $m \in A_3$ if and only if M satisfies the following system of inequalities:

$$\begin{cases} n \le M < 2^{2j}t^{1/2}, \\ -2^{-1-2j}(2^{j-1}t^{-1/4}M^{1/2} + 4M^2 + s'M) < 4, \\ 2^{-1-2j}(2^{j-1}t^{-1/4}M^{1/2} - 4M^2 - s'M) > 1/2. \end{cases} \tag{48}$$

By multiplying the last two inequalities by 2^{2+2j} and isolating the term with fractional exponent, we rewrite (48) in the following form:

$$\begin{cases} n \le M < 2^{2j}t^{1/2}, \\ 2^{j}t^{-1/4}M^{1/2} > -8M^2 - 2s'M - 2^{4+2j}, \\ 2^{j}t^{-1/4}M^{1/2} > 8M^2 + 2s'M + 2^{1+2j}. \end{cases} \tag{49}$$

But the first inequality implies

$$2^{j}t^{-1/4}M^{1/2} < 2^{2j}$$

and hence the solutions of (49) are also solutions of the following system:

$$\begin{cases} M > 0, \\ 8M^2 + 2s'M + 17\cdot 2^{2j} > 0, \\ 8M^2 + 2s'M + 2^{2j} < 0. \end{cases} \tag{50}$$

The solutions of (50) can be found by a direct calculation. If

$$s' \ge -2^{3/2+j}$$

the system (50) does not have solutions, so $A_3 = \emptyset$. If

$$-\sqrt{17}\cdot 2^{3/2+j} < s' < -2^{3/2+j}$$

the solutions of (50) are given by

$$\frac{-s' - \sqrt{s'^2 - 2^{3+2j}}}{8} < M < \frac{-s' + \sqrt{s'^2 - 2^{3+2j}}}{8}.$$

In this case, since

$$\frac{-s' + \sqrt{s'^2 - 2^{3+2j}}}{-s' - \sqrt{s'^2 - 2^{3+2j}}} \le 33 + 8\sqrt{17},$$

by (31) we get (47). Finally, if

$$s' \le -\sqrt{17} \cdot 2^{3/2+j}$$

the solutions of (50) are given by

$$\frac{-s' - \sqrt{s'^2 - 2^{3+2j}}}{8} < M < \frac{-s' - \sqrt{s'^2 - 17 \cdot 2^{3+2j}}}{8}$$

and by

$$\frac{-s' + \sqrt{s'^2 - 17 \cdot 2^{3+2j}}}{8} < M < \frac{-s' + \sqrt{s'^2 - 2^{3+2j}}}{8}.$$

Also in this case, since

$$\frac{-s' - \sqrt{s'^2 - 17 \cdot 2^{3+2j}}}{-s' - \sqrt{s'^2 - 2^{3+2j}}} \le 17 + 4\sqrt{17}$$

and

$$\frac{-s' + \sqrt{s'^2 - 2^{3+2j}}}{-s' + \sqrt{s'^2 - 17 \cdot 2^{3+2j}}} \le 1 + \frac{4}{\sqrt{17}},$$

by (31) we get (47). This concludes the argument.

Now we prove (33). From now on, we suppose $t \ge 1$ and $j \in \mathbb{Z}_+$. By (34)–(36) we have

$$(S_t\varphi_j)\Big(0, -\frac{2^{3+2j}t}{n+1}\Big) = \frac{2^{n-1}}{\pi^{n+1}}\, 2^{Nj} \sum_{m=0}^{\infty} \int\limits_{1/2<|x|<4} e^{-i2^{2j}tG_m(x)} H_m(x)\, dx$$

where

$$G_m(x) = \frac{2^{2j}x^2}{(2m+n)^2} + 4|x| - \frac{2^{3+2j}x}{(n+1)(2m+n)}$$

and

$$H_m(x) = \binom{m+n-1}{m} R(x)\, \frac{|x|^n}{(2m+n)^{n+1}}.$$

For any $m \in \mathbb{N}$ we put

$$J_m = \int_{1/2}^{4} e^{-i2^{2j}tG_m(x)} H_m(x)\, dx, \qquad K_m = \int_{-4}^{-1/2} e^{-i2^{2j}tG_m(x)} H_m(x)\, dx.$$

Thus

$$\Big|(S_t\varphi_j)\Big(0, -\frac{2^{3+2j}t}{n+1}\Big)\Big| \ge C\, 2^{Nj}\Big(|J_0| - \sum_{m=0}^{\infty} |K_m| - \sum_{m\in B_1} |J_m| - \sum_{m\in B_2} |J_m|\Big) \tag{51}$$

where

$$B_1 = \{m \in \mathbb{Z}_+ : |G'_m(x)| \geq t^{-1/4} \text{ for any } x \in [1/2, 4]\},$$
$$B_2 = \{m \in \mathbb{Z}_+ : |G'_m(x)| < t^{-1/4} \text{ for some } x \in [1/2, 4]\}.$$

We will estimate separately the four terms on the right-hand side of (51). We put again $M = 2m + n$.

In order to estimate $|J_0|$ from below, we need one more hypothesis about the function R, besides the assumptions made at the beginning of Section 4:

$$R(\tau) = 1 \quad \text{if } \frac{4n+3}{4(n+1)} \leq |\tau| \leq \frac{4n+2}{n+1}.$$

This is possible by taking for example

$$R(\tau) = \begin{cases} 1 - \chi(4|\tau|) & \text{if } |\tau| \leq (4n+3)/(4(n+1)), \\ \chi(|\tau|) & \text{if } |\tau| > (4n+3)/(4(n+1)) \end{cases}$$

where χ is a function in $C^\infty(\mathbb{R})$ such that $\chi(\tau) = 1$ if $\tau \leq (4n+2)/(n+1)$ and $\chi(\tau) = 0$ if $\tau \geq (4n+3)/(n+1)$. In this way, if we suppose $j \geq j_1$ large enough, we can apply Lemma 6 with $\sigma = 2^{4j}t/n^2$ and $\gamma = 4n/(n+1) - n^2 2^{1-2j}$ and we obtain

$$|J_0| \geq C_1 2^{-2j} t^{-1/2}. \tag{52}$$

In order to estimate $\sum_{m=0}^\infty |K_m|$ from above we first observe that

$$G'_m(x) < -4 \quad \text{for } x < 0.$$

Thus Lemma 5 applied with $k = 1$ and estimate (40) yield

$$|K_m| \leq C\, 2^{-2j} t^{-1} M^{-2}.$$

So

$$\sum_{m=0}^{\infty} |K_m| \leq C_2\, 2^{-2j} t^{-1}. \tag{53}$$

If $m \in B_1$ then Lemma 5 applied again with $k = 1$ and estimate (40) yield

$$|J_m| \leq C\, 2^{-2j} t^{-3/4} M^{-2}.$$

So

$$\sum_{m \in B_1} |J_m| \leq C_3\, 2^{-2j} t^{-3/4}. \tag{54}$$

If $m \in B_2$ then Lemma 5 applied with $k = 2$ and estimate (40) yield

$$|J_m| \leq C\, 2^{-2j} t^{-1/2} M^{-1}. \tag{55}$$

We now want to estimate $\sum_{m \in B_2} M^{-1}$. As in the first part of the proof, it is straightforward to verify that $m \in B_2$ if and only if M satisfies the following system of inequalities:

$$\begin{cases} M \geq n+2, \\ (n+1)(4+t^{-1/4})M^2 - 2^{3+2j}M + (n+1)2^{3+2j} > 0, \\ (n+1)(4-t^{-1/4})M^2 - 2^{3+2j}M + (n+1)2^{2j} < 0. \end{cases} \tag{56}$$

If we consider only the system of the last two inequalities of (56), we find by a direct calculation that the set of solutions may be the union of two distinct intervals. But if we take $j \geq j_2$ large enough, the solutions M belonging to one of the two intervals do not satisfy the condition $M \geq n+2$. So the only solutions of (56) are given by

$$\alpha(j,t) < M < \beta(j,t)$$

where

$$\alpha(j,t) = \frac{2^{2+2j}(1+\sqrt{1-(n+1)^2(4+t^{-1/4})2^{-1-2j}})}{(n+1)(4+t^{-1/4})},$$

$$\beta(j,t) = \frac{2^{2+2j}(1+\sqrt{1-(n+1)^2(4-t^{-1/4})2^{-4-2j}})}{(n+1)(4-t^{-1/4})}.$$

We get

$$\begin{aligned} \beta(j,t)-\alpha(j,t) &= \frac{2^{2+2j}}{(n+1)(16-t^{-1/2})} \\ &\quad \times (2t^{-1/4} + (4+t^{-1/4})\sqrt{1-(n+1)^2(4-t^{-1/4})2^{-4-2j}} \\ &\quad - (4-t^{-1/4})\sqrt{1-(n+1)^2(4+t^{-1/4})2^{-1-2j}}) \\ &\leq C\,2^{2j}(t^{-1/4} + \sqrt{1-(n+1)^2(4-t^{-1/4})2^{-4-2j}} \\ &\quad - \sqrt{1-(n+1)^2(4+t^{-1/4})2^{-1-2j}}) \\ &\leq C(1+2^{2j}t^{-1/4}) \end{aligned}$$

since for $j \geq j_2$ we have

$$\sqrt{1-(n+1)^2(4-t^{-1/4})2^{-4-2j}} - \sqrt{1-(n+1)^2(4+t^{-1/4})2^{-1-2j}} \leq C\,2^{-2j}$$

as is easy to verify by multiplying and dividing by

$$\sqrt{1-(n+1)^2(4-t^{-1/4})2^{-4-2j}} + \sqrt{1-(n+1)^2(4+t^{-1/4})2^{-1-2j}}.$$

On the other hand

$$M > \alpha(j,t) \geq C\,2^{2j}$$

and hence, using the fact that the number of integer points in an interval (α,β) is smaller than $1+\beta-\alpha$, we obtain

$$\sum_{m\in B_2} M^{-1} \leq C \sum_{m\in B_2} 2^{-2j} \leq C(2^{-2j}+t^{-1/4}). \tag{57}$$

Then, for $j \geq j_2$ and $t \geq 1$, (55) and (57) yield

$$\sum_{m\in B_2} |J_m| \leq C_4(2^{-4j}t^{-1/2} + 2^{-2j}t^{-3/4}). \tag{58}$$

The constants C_1, C_2, C_3, C_4 in (52), (53), (54) and (58) depend only on n, as well as the integers j_1 and j_2. Take $t_0 \geq 1$ such that

$$C_2 t_0^{-1/2} + (C_3 + C_4)t_0^{-1/4} \leq C_1/2 \tag{59}$$

and $j_0 \geq \max\{j_1, j_2\}$ such that

$$C_4 2^{-2j_0} \leq C_1/4. \tag{60}$$

Then, putting together (52), (53), (54) and (58) and using (59) and (60), for any $t \geq t_0$ and $j \geq j_0$ we obtain

$$|J_0| - \sum_{m=0}^{\infty} |K_m| - \sum_{m\in B_1} |J_m| - \sum_{m\in B_2} |J_m| \geq \frac{C_1}{4}\, 2^{-2j}t^{-1/2}. \tag{61}$$

So (33) follows directly from (51) and (61). ∎

From Proposition 9 it is easy to obtain our sharp dispersive inequality:

Corollary 10. *There exists a constant $C > 0$, which depends only on n, such that for any $t > 0$ and $u_0 \in \mathcal{S}(\mathbf{H}_n)$,*

$$\|S_t u_0\|_{L^\infty(\mathbf{H}_n)} \leq C t^{-1/2} \|u_0\|_{\dot{B}_1^{N-2,1}(\mathbf{H}_n)}. \tag{62}$$

Moreover, there exist $C' > 0$, $t_0 \geq 1$ and $j_0 \in \mathbb{Z}_+$, which depend only on n, such that for any $t \geq t_0$ and $j \geq j_0$,

$$\|S_t \varphi_j\|_{L^\infty(\mathbf{H}_n)} \geq C' t^{-1/2} \|\varphi_j\|_{\dot{B}_1^{N-2,1}(\mathbf{H}_n)}. \tag{63}$$

Proof. In order to obtain (62), we apply (16), (18) and then Young's inequality and estimate (32) for the functions $\widetilde{\varphi}_j$; we get

$$\begin{aligned}
\|S_t u_0\|_{L^\infty(\mathbf{H}_n)} &\leq \sum_{j\in\mathbb{Z}} \|\Delta_j S_t u_0\|_{L^\infty(\mathbf{H}_n)} \\
&= \sum_{j\in\mathbb{Z}} \|(\Delta_j S_t u_0) * \widetilde{\varphi}_j\|_{L^\infty(\mathbf{H}_n)} \\
&= \sum_{j\in\mathbb{Z}} \|(\Delta_j u_0) * (S_t \widetilde{\varphi}_j)\|_{L^\infty(\mathbf{H}_n)} \\
&\leq \sum_{j\in\mathbb{Z}} \|\Delta_j u_0\|_{L^1(\mathbf{H}_n)} \|S_t \widetilde{\varphi}_j\|_{L^\infty(\mathbf{H}_n)} \\
&\leq C t^{-1/2} \sum_{j\in\mathbb{Z}} 2^{(N-2)j} \|\Delta_j u_0\|_{L^1(\mathbf{H}_n)} \\
&= C t^{-1/2} \|u_0\|_{\dot{B}_1^{N-2,1}(\mathbf{H}_n)}.
\end{aligned}$$

In order to obtain (63), we observe that the homogeneity property (13) implies that the L^1 norm of φ_j is invariant with respect to j. So by (14) we have

$$\|\varphi_j\|_{\dot{B}_1^{N-2,1}(\mathbf{H}_n)} \le C\, 2^{(N-2)j}. \tag{64}$$

Then estimate (63) is a straightforward consequence of (33) and (64). ∎

6. Strichartz inequalities. We are now in a position to prove our Strichartz inequalities. We will denote by $L_T^p(X)$ the space $L^p((0,T),X)$ and by $L_I^p(X)$ the space $L^p(I,X)$ for $I \subseteq \mathbb{R}$.

THEOREM 11. *Let $r_1, r_2 \in [2,\infty]$. Let $\varrho_1, \varrho_2 \in \mathbb{R}$ and p_1, $p_2 \in [1,\infty]$ such that*:

(a) $2/p_i = 1/2 - 1/r_i$ *for* $i = 1,2$,
(b) $\varrho_i = -(N-2)(1/2 - 1/r_i)$ *for* $i = 1,2$.

Let r_i', p_i' be such that $1/r_i' + 1/r_i = 1$ and $1/p_i' + 1/p_i = 1$ for $i = 1,2$. Then

$$\|S_t u_0\|_{L_{\mathbb{R}}^{p_1}(\dot{B}_{r_1}^{\varrho_1,2}(\mathbf{H}_n))} \le C\|u_0\|_{L^2(\mathbf{H}_n)}, \tag{65}$$

$$\Big\|\int_0^t S_{t-\sigma} f(\sigma)\,d\sigma\Big\|_{L_T^{p_1}(\dot{B}_{r_1}^{\varrho_1,2}(\mathbf{H}_n))} \le C\|f\|_{L_T^{p_2'}(\dot{B}_{r_2'}^{-\varrho_2,2}(\mathbf{H}_n))} \tag{66}$$

where the constant $C > 0$ depends neither on u_0, f nor on T.

Proof. Once we have obtained (32), the procedure is classical and a good reference is, for example, the papers by Ginibre and Velo ([GV2]) or by Ginibre ([Gin]). We will recall the main steps for the reader's convenience.

STEP 1. As in (17), let $\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ such that $\Delta_j u_0 = \Delta_j u_0 * \widetilde{\varphi}_j$ for all $j \in \mathbb{Z}$. The operator S_t being unitary on $L^2(\mathbf{H}_n)$, we have

$$\|\Delta_j S_t u_0\|_{L^2(\mathbf{H}_n)} = \|\Delta_j u_0\|_{L^2(\mathbf{H}_n)}, \quad t > 0,\ j \in \mathbb{Z}. \tag{67}$$

Moreover, in the proof of Corollary 10 we have shown that

$$\|\Delta_j S_t u_0\|_{L^\infty(\mathbf{H}_n)} \le C\, 2^{(N-2)j} t^{-1/2} \|\Delta_j u_0\|_{L^1(\mathbf{H}_n)}, \quad t > 0,\ j \in \mathbb{Z}. \tag{68}$$

By interpolating (67) and (68), for $r \ge 2$ and $1/r + 1/r' = 1$ we get

$$\|\Delta_j S_t u_0\|_{L^r(\mathbf{H}_n)} \le C\, \frac{2^{2(N-2)(1/2-1/r)j}}{t^{1/2-1/r}}\, \|\Delta_j u_0\|_{L^{r'}(\mathbf{H}_n)}, \quad t > 0,\ j \in \mathbb{Z},$$

which for $q \in [1,\infty]$ yields

$$\|S_t u_0\|_{\dot{B}_r^{-(N-2)(1/2-1/r),q}(\mathbf{H}_n)} \le \frac{C}{t^{1/2-1/r}}\, \|u_0\|_{\dot{B}_{r'}^{(N-2)(1/2-1/r),q}(\mathbf{H}_n)}.$$

Step 2. We prove (66) in the particular case $r_1 = r_2 = r$, $p_1 = p_2 = p$ such that $r \geq 2$, $2/p = 1/2 - 1/r$ and $\varrho_1 = \varrho_2 = -(N-2)(1/2 - 1/r)$. From the Minkowski inequality and, for $r > 2$, the Hardy–Littlewood–Sobolev theorem (see e.g. [Ste, page 354]), for every $q \in [1, \infty]$ we obtain

$$\begin{aligned}(69)\qquad &\Big\| \int_0^t S_{t-\sigma} f(\sigma)\, d\sigma \Big\|_{L^p_T(\dot{B}_r^{-(N-2)(1/2-1/r),q}(\mathbf{H}_n))} \\ &\leq C \Big\| \int_0^t \frac{1}{(t-\sigma)^{1/2-1/r}} \|f(\sigma)\|_{\dot{B}_{r'}^{(N-2)(1/2-1/r),q}(\mathbf{H}_n)}\, d\sigma \Big\|_{L^p([0,T])} \\ &\leq C \|f\|_{L^{p'}_T(\dot{B}_{r'}^{(N-2)(1/2-1/r),q}(\mathbf{H}_n))}.\end{aligned}$$

Step 3. With the same procedure as in Step 2, we also prove that for any interval $I \subseteq \mathbb{R}$ one has

$$(70)\qquad \Big\| \int_I S_{t-\sigma} f(\sigma)\, d\sigma \Big\|_{L^p_I(\dot{B}_r^{-(N-2)(1/2-1/r),q}(\mathbf{H}_n))} \leq C \|f\|_{L^{p'}_I(\dot{B}_{r'}^{(N-2)(1/2-1/r),q}(\mathbf{H}_n))}.$$

By a standard duality argument (see for example [BGX] or [GV2]) and for the only case $q = 2$ as the second index in the Besov space, from (70) we get (65) for the particular case $I = \mathbb{R}$.

Step 4. We consider for any interval $I \subseteq \mathbb{R}$ the bounded operator

$$A : L^1_I(L^2(\mathbf{H}_n)) \to L^2(\mathbf{H}_n), \qquad f(z,s,\sigma) \mapsto \int_I S_{-\sigma} f(z,s,\sigma)\, d\sigma,$$

whose adjoint (with duality in $L^2(\mathbf{H}_n)$ defined by the scalar product) is

$$A^* : L^2(\mathbf{H}_n) \to L^\infty_I(L^2(\mathbf{H}_n)), \qquad u_0(z,s) \mapsto S_t u_0(z,s).$$

The composition of these two operators gives

$$A^*A : L^1_I(L^2(\mathbf{H}_n)) \to L^\infty_I(L^2(\mathbf{H}_n)), \qquad f(z,s,\sigma) \mapsto \int_I S_{t-\sigma} f(z,s,\sigma)\, d\sigma,$$

and by the factorization of A^*A, using the result obtained in Step 3, we obtain the result in (66) for this operator. Unfortunately, this is not actually the integral operator appearing in Theorem 11 because of the "retardation" in time, following Ginibre's notation. We can write explicitly, for f defined on $[0,T] \times \mathbf{H}_n$,

$$\int_0^t S_{t-\sigma} f(\sigma)\, d\sigma = \int_I \chi_{\mathbb{R}^+}(t-\sigma) S_{t-\sigma} \widetilde{f}(\sigma)\, d\sigma, \qquad t \in [0,T],$$

where I is any interval containing $[0,T]$, χ is the characteristic function and $\widetilde{f}$ is the continuation of f by zero over $\mathbb{R} \times \mathbf{H}_n$. Thanks to the fact that the spaces $L^p_I(\dot B^{\varrho,q}_r(\mathbf{H}_n))$ are stable under restriction in time (which means that the multiplication by the characteristic function of an interval in time is a bounded operator with norm uniformly bounded with respect to the interval), it is possible to overcome this technical complication and to proceed as follows.

Step 5. Following Lemma 2.3 in [GV2], we get

$$\int_0^t S_{t-\sigma} f(\sigma)\,d\sigma : L^{p'}_T(\dot B^{(N-2)(1/2-1/r),2}_{r'}(\mathbf{H}_n)) \to L^\infty_T(L^2(\mathbf{H}_n)) \tag{71}$$

with norm independent of T. Therefore, by duality we also have

$$\int_0^t S_{t-\sigma} f(\sigma)\,d\sigma :\ L^1_T(L^2(\mathbf{H}_n)) \to L^p_T(\dot B^{-(N-2)(1/2-1/r),2}_{r}(\mathbf{H}_n)) \tag{72}$$

with norm independent of T.

Step 6. Finally, remembering the "diagonal" case (69) and interpolating in every possible way with (71) and (72) we obtain (66). ∎

Going back to (19) and (20), by Theorem 11 we have the following

Corollary 12. *Under the same hypotheses as in Theorem* 11, *the solution u of the Cauchy problem* (4) *satisfies the estimate*

$$\|u\|_{L^{p_1}_T(\dot B^{\varrho_1,2}_{r_1}(\mathbf{H}_n))} \le C(\|u_0\|_{L^2(\mathbf{H}_n)} + \|f\|_{L^{p_2'}_T(\dot B^{-\varrho_2,2}_{r_2'}(\mathbf{H}_n))})$$

where the constant $C>0$ does not depend on T.

To conclude our paper, we would like to make some remarks, especially in comparison with the real setting.

Remarks. 1. In the Heisenberg case, we are not faced with any limiting case as it happens on the contrary in the real framework. This is due to the structure of the Gelfand transform of a radial function in $\mathbf{H}_n$ which involves only a one-dimensional oscillating integral and, as a consequence, the Hardy–Littlewood–Sobolev theorem always applies.

2. Unlike the real case, the Schrödinger operator (for the full Laplacian) always determines a loss of regularity, except in the $L^1_T(L^2(\mathbf{H}_n)) \to L^\infty_T(L^2(\mathbf{H}_n))$ case. However, the singular Besov spaces involved in Theorem 11 are not in the scale of the Sobolev embedding for the $L^2(\mathbf{H}_n)$ space and therefore the result might not be simply obtained by inclusions. For the wave operator (with the sub-Laplacian) in $\mathbf{H}_n$, the analogy with the real setting is much closer, as pointed out in [BGX]. One might wonder if the full Laplacian wave operator in $\mathbf{H}_n$ would behave differently; in our opinion, that is not the case and it will be the object of further investigation.

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Received July 22, 2002 (4997)