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Beurling algebras and uniform norms

by

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Abstract. Given a locally compact abelian group G with a measurable weight ω, it is shown that the Beurling algebra $L^1(G,\omega)$ admits either exactly one uniform norm or infinitely many uniform norms, and that $L^1(G,\omega)$ admits exactly one uniform norm iff it admits a minimum uniform norm.

A *uniform norm* on a Banach algebra $\mathcal{A}$ is a (not necessarily complete) algebra norm $|\cdot|$ satisfying the square property $|a^2| = |a|^2$ ($a \in \mathcal{A}$). It is easy to see that any two equivalent uniform norms on a Banach algebra are identical. Thus two uniform norms are either identical or different. A Banach algebra $\mathcal{A}$ has the *unique uniform norm property* (*UUNP*) if it admits exactly one uniform norm; in this case, the spectral radius on $\mathcal{A}$ is the only uniform norm. Every regular, semisimple, commutative Banach algebra has UUNP; in particular, the uniform algebra $C(X)$ on a compact Hausdorff space X and the group algebra $L^1(G)$ on a LCA group G have UUNP. On the other hand, the disc algebra $\mathcal{A}(\mathcal{D})$ has infinitely many uniform norms [BhDe1]. Banach algebras with unique uniform norm have been investigated in [BhDe2], where it is shown that the Beurling algebra $L^1(G,\omega)$ (which is always semisimple [BhDe3]) has UUNP iff $L^1(G,\omega)$ is regular. In this paper we prove that $L^1(G,\omega)$ has either exactly one uniform norm or infinitely many uniform norms; and that it has UUNP if and only if it has a minimum uniform norm. This leads to the following question that remains open: Does there exist a (necessarily commutative and semisimple) Banach algebra $\mathcal{A}$ which admits finitely many distinct uniform norms?

Throughout let G be a locally compact abelian (LCA) group, let λ be a Haar measure, and let ω be a *weight* on G, i.e., a strictly positive measurable function ω on G such that $\omega(s+t) \le \omega(s)\omega(t)$ ($s,t \in G$). Then the *Beurling algebra* $L^1(G,\omega)$ consists of all complex-valued measurable functions f on G

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such that $f\omega \in L^1(G)$ is a semisimple commutative Banach algebra with the convolution product and the norm $\|f\|_\omega := \int_G |f(s)|\omega(s)\,d\lambda(s)$ [BhDe3]. An *ω-bounded generalized character* on G is a continuous group homomorphism $\alpha : G \to (\mathbb{C}\setminus\{0\}, \times)$ such that $|\alpha(s)| \le \omega(s)\,(s \in G)$. Let $H(G,\omega)$ denote the set of all ω-bounded generalized characters on G equipped with the compact-open topology. For $\alpha \in H(G,\omega)$, define

$$\varphi_\alpha(f) = \widehat{f}(\alpha) = \int_G f(s)\alpha(s)\,d\lambda(s) \qquad (f \in L^1(G,\omega)).$$

Then the map $T : H(G,\omega) \to \Delta(L^1(G,\omega))$ defined as $T(\alpha) = \varphi_\alpha$ is a homeomorphism [BhDe4]. Let $\widehat{G}$ denote the dual group of G. Then it is easy to see that $\widehat{G} \subseteq H(G,\omega)$ iff $\omega \ge 1$ on G. However it is always true that $\theta\alpha \in H(G,\omega)$ ($\theta \in \widehat{G}$ and $\alpha \in H(G,\omega)$). For $F \subseteq H(G,\omega)$, define

$$|f|_F = \sup\{|\widehat{f}(\alpha)| : \alpha \in F\} \qquad (f \in L^1(G,\omega)).$$

Then $|\cdot|_F$ is a uniform seminorm on $L^1(G,\omega)$; the set F is a *set of uniqueness* for $L^1(G,\omega)$ if $|\cdot|_F$ is a norm. For example, $\alpha\widehat{G}$ is a set of uniqueness for any $\alpha \in H(G,\omega)$. For $\alpha \in H(G,\omega)$, the uniform norm $|\cdot|_{\alpha\widehat{G}}$ on $L^1(G,\omega)$ will be denoted by $|\cdot|_\alpha$.

THEOREM 1. *$L^1(G,\omega)$ has UUNP iff it has a minimum uniform norm.*

Proof. If $L^1(G,\omega)$ has UUNP, then clearly it has a minimum uniform norm. Conversely, assume that it has a minimum uniform norm, say $|\cdot|_0$. Define $F = \{\alpha \in H(G,\omega) : \varphi_\alpha$ is $|\cdot|_0$-continuous$\}$. By taking the completion $\mathcal{A}$ of $(L^1(G,\omega), |\cdot|_0)$ and by using elementary Gelfand theory, one can see that for all $f \in L^1(G,\omega)$, $|f|_0 = |f|_F = r_{\mathcal{A}}(f)$, the spectral radius in $\mathcal{A}$. Suppose that we have proved $F = H(G,\omega)$. This implies that $|\cdot|_0 = |\cdot|_F =$ the spectral radius on $L^1(G,\omega)$ and the result is proved. So we prove $F = H(G,\omega)$ in the following two steps.

STEP 1. $F = \widehat{G}F = \{\theta\alpha : \theta \in \widehat{G}$ *and* $\alpha \in F\}$.

Fix $\theta \in \widehat{G}$. Define $|f|_{\theta F} = |\theta f|_F$ on $L^1(G,\omega)$. Then $|\cdot|_{\theta F}$ is a uniform norm on $L^1(G,\omega)$. Since $|\cdot|_F$ $(= |\cdot|_0)$ is the minimum uniform norm on $L^1(G,\omega)$, we have

$$|f|_F \le |f|_{\theta F} \qquad (f \in L^1(G,\omega)).$$

This holds for each $\theta \in \widehat{G}$. So for each $\theta \in \widehat{G}$ and $f \in L^1(G,\omega)$, we have $|f|_F \le |f|_{\theta F}$, i.e., $|f|_F \le |\theta f|_F$. The last inequality is true for each f. So replacing f by $\bar{\theta} f$, we get $|f|_{\bar{\theta}F} = |\bar{\theta} f|_F \le |f|_F$. Thus for each $\alpha \in F$, the complex homomorphism $\varphi_{\bar{\theta}\alpha}$ is $|\cdot|_0$-continuous, i.e., $\bar{\theta}F \subseteq F$. This is true for each $\theta \in \widehat{G}$. Hence $F = \widehat{G}F$.

STEP 2. $F = H(G,\omega)$.

Suppose, if possible, $F \neq H(G,\omega)$. Choose $\alpha_1 \in F$ and $\beta_1 \in H(G,\omega)\setminus F$. Take $\alpha = |\alpha_1|$ and $\beta = |\beta_1|$. Then by Step 1, $\alpha \in F$ and $\beta \in H(G,\omega) \setminus F$. Choose $t \in G$ such that $\beta(t) < \alpha(t)$. Let U be an open neighbourhood of t in G such that its closure $\overline{U}$ is compact and $\beta(s) < \alpha(s)$ $(s \in U)$. Take $f = \chi_U \in L^1(G,\omega)$, the characteristic function of U. Now

$$\begin{aligned}|f|_\beta &= \sup\{|\widehat{f}(\beta\theta)| : \theta \in \widehat{G}\} = \sup\Big\{\Big|\int_G f(s)\beta(s)\theta(s)\,d\lambda(s)\Big| : \theta \in \widehat{G}\Big\}\\ &\le \sup\Big\{\int_U \beta(s)|\theta(s)|\,d\lambda(s) : \theta \in \widehat{G}\Big\} \le \int_U \beta(s)\,d\lambda(s)\\ &< \int_U \alpha(s)\,d\lambda(s) \le \sup\{|\widehat{f}(\alpha\theta)| : \theta \in \widehat{G}\} = |f|_\alpha \le |f|_F.\end{aligned}$$

Thus $|\cdot|_\beta$ is a uniform norm and $|f|_\beta < |f|_F$, which is a contradiction because the latter is the minimum uniform norm on $L^1(G,\omega)$. This proves Step 2 and the result is proved. ∎

The above result is not true in arbitrary semisimple commutative Banach algebras. For example, let G be a non-discrete LCA group. Then the measure algebra $M(G)$ has a minimum uniform norm, namely $|\mu|_\infty = \sup\{|\widehat{\mu}(\theta)| : \theta \in \widehat{G}\}$ $(\mu \in M(G))$ [BhDe2, Corollary 6.3]. But it does not have UUNP [BhDe2, p. 233]. Notice that both the disc algebra $A(\mathcal{D})$ and its variant $A_r(\mathcal{D}) = \{f \in C(\mathcal{D}) : f \text{ is analytic on } E_r\}$ (where $0 < r < 1$ and $E_r = \{z : r < |z| < 1\}$) admit infinitely many uniform norms. However, $A(\mathcal{D})$ does not admit a minimum uniform norm, whereas $A_r(\mathcal{D})$ admits a minimum uniform norm, viz., $|f|_0 = \sup\{|f(z)| : |z| \le r\}$ $(f \in A_r(\mathcal{D}))$.

THEOREM 2. *$L^1(G,\omega)$ admits either exactly one uniform norm or infinitely many uniform norms.*

Proof. Assume that $L^1(G,\omega)$ has more than one distinct uniform norm. If G is compact, then $L^1(G,\omega) \cong L^1(G)$ does have UUNP. Thus G must be non-compact, and so its dual group $\widehat{G}$ is not discrete. Now choose $\alpha \in H(G,\omega)$ and define $\widetilde{\omega}(s) = \omega(s)/|\alpha(s)|$ $(s \in G)$. Then $\widetilde{\omega}$ is a weight and $\widetilde{\omega} \ge 1$ on G. It is easy to see that the map $T : L^1(G,\omega) \to L^1(G,\widetilde{\omega})$ defined as $T(f) = \alpha f$ is an isometric algebra isomorphism. Hence we may assume that $\omega \ge 1$ on G. Therefore $\widehat{G} \subseteq H(G,\omega)$. Now we define

$$H_{\mathrm{p}}(G,\omega) = \{\alpha \in H(G,\omega) : \alpha \text{ is strictly positive}\}.$$

As in the proof of Theorem 1, any two distinct elements α and β in $H_{\mathrm{p}}(G,\omega)$ will give distinct uniform norms on $L^1(G,\omega)$, namely $|\cdot|_\alpha$ and $|\cdot|_\beta$. Now consider the following two cases:

CASE (i): *$H_{\mathrm{p}}(G,\omega)$ is not a singleton.* Choose α and β in $H_{\mathrm{p}}(G,\omega)$ such that $\alpha \neq \beta$. Take $0 < \lambda < 1$. Define $\eta_\lambda(s) = \alpha(s)^\lambda \beta(s)^{1-\lambda}$ $(s \in G)$. Then each $\eta_\lambda \in H_{\mathrm{p}}(G,\omega)$. Now if $0 < \lambda \neq \lambda' < 1$, then $|\cdot|_{\eta_\lambda}$ and $|\cdot|_{\eta_{\lambda'}}$ are distinct uniform norms on $L^1(G,\omega)$. Thus $L^1(G,\omega)$ admits infinitely many uniform norms.

Case (ii): *$H_{\mathrm{p}}(G,\omega)$ is a singleton.* In this case $H_{\mathrm{p}}(G,\omega) = \{1_G\}$, where 1_G is the identity of $\widehat{G}$, and $H(G,\omega) = \widehat{G}$. Since $L^1(G,\omega)$ does not have UUNP, there exists a proper closed subset F of $\widehat{G}$ which is a set of uniqueness for $L^1(G,\omega)$. Since $\widehat{G}$ is not discrete, $\widehat{G} \setminus F$ is infinite. Choose $\gamma_1, \gamma_2, \ldots$ to be different elements outside F. Set $F_0 = F$ and $F_n = F_{n-1} \cup \{\gamma_n\}$. For each $n \geq 0$, define $|f|_n := \sup\{|\widehat{f}(\gamma)| : \gamma \in F_n\}$ ($f \in L^1(G,\omega)$). Then each $|\cdot|_n$ is a uniform norm on $L^1(G,\omega)$. We show that they are distinct. It is enough to show that $|g|_0 < |g|_1$ for some $g \in L^1(G,\omega)$. Since $L^1(G)$ is regular, there exists $f \in L^1(G)$ such that $\widehat{f}(F) = \{0\}$ and $\widehat{f}(\gamma_1) = 1$. Fix $0 < \varepsilon < 1/2$. Then there exists $g \in C_{\mathrm{c}}(G)$ such that $\|f - g\| < \varepsilon$, where $\|\cdot\|$ is the L^1-norm on $L^1(G)$. Then $g \in C_{\mathrm{c}}(G) \subset L^1(G,\omega)$. Also

$$|\widehat{g}(\gamma)| = |\widehat{g}(\gamma) - \widehat{f}(\gamma)| \leq \|\widehat{f} - \widehat{g}\|_{\widehat{G}} \leq \|f - g\| < \varepsilon < 1/2 \quad (\gamma \in F).$$

Moreover,

$$\begin{aligned} |\widehat{g}(\gamma_1)| &= |\widehat{f}(\gamma_1) - \widehat{f}(\gamma_1) + \widehat{g}(\gamma_1)| \geq |\widehat{f}(\gamma_1)| - |\widehat{f}(\gamma_1) - \widehat{g}(\gamma_1)| \\ &\geq 1 - \|f - g\| > 1/2. \end{aligned}$$

Thus $|g|_0 \leq 1/2 < |g|_1$. This completes the proof. ∎

Define $\omega_1(s) = \exp(|s|)$ and $\omega_2(s) = (1 + |s|)^{1/2}$ on $\mathbb{R}$. By [D, Theorem 4.7.33], the Gelfand space of $L^1(\mathbb{R},\omega_1)$ can be identified with the vertical strip $\Pi_{-1,1} := \{x + iy : -1 \leq x \leq 1\}$ in the complex plane. For $-1 \leq x \leq 1$, define $|f|_x = \sup\{|\widehat{f}(x + iy)| : y \in \mathbb{R}\}$ ($f \in L^1(\mathbb{R},\omega_1)$). Then each $|\cdot|_x$ is a uniform norm on $L^1(\mathbb{R},\omega_1)$. By the maximum modulus principle, all of them are distinct norms. On the other hand, $L^1(\mathbb{R},\omega_2)$ has exactly one uniform norm because it is regular [D, Theorem 4.3.37].

Remark. Assume that $\omega \geq 1$ on G. It follows from [BhDe2, Theorem 4.1] and [Do] that $L^1(G,\omega)$ has UUNP if and only if ω is non-quasi-analytic, i.e., $\sum_{n\geq 1}(\log \omega(ns))/(1 + n^2) < \infty$ for each $s \in G$.

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