Beurling algebras and uniform norms

by

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Abstract. Given a locally compact abelian group $G$ with a measurable weight $\omega$, it is shown that the Beurling algebra $L^1(G, \omega)$ admits either exactly one uniform norm or infinitely many uniform norms, and that $L^1(G, \omega)$ admits exactly one uniform norm if it admits a minimum uniform norm.

A uniform norm on a Banach algebra $A$ is a (not necessarily complete) algebra norm $|\cdot|$ satisfying the square property $|a^2| = |a|^2$ ($a \in A$). It is easy to see that any two equivalent uniform norms on a Banach algebra are identical. Thus two uniform norms are either identical or different. A Banach algebra $A$ has the unique uniform norm property (UUNP) if it admits exactly one uniform norm; in this case, the spectral radius on $A$ is the only uniform norm. Every regular, semisimple, commutative Banach algebra has UUNP; in particular, the uniform algebra $C(X)$ on a compact Hausdorff space $X$ and the group algebra $L^1(G)$ on a LCA group $G$ have UUNP. On the other hand, the disc algebra $A(D)$ has infinitely many uniform norms [BhDe1]. Banach algebras with unique uniform norm have been investigated in [BhDe2], where it is shown that the Beurling algebra $L^1(G, \omega)$ (which is always semisimple [BhDe3]) has UUNP iff $L^1(G, \omega)$ is regular. In this paper we prove that $L^1(G, \omega)$ has either exactly one uniform norm or infinitely many uniform norms; and that it has UUNP if and only if it has a minimum uniform norm. This leads to the following question that remains open: Does there exist a (necessarily commutative and semisimple) Banach algebra $A$ which admits finitely many distinct uniform norms?

Throughout let $G$ be a locally compact abelian (LCA) group, let $\lambda$ be a Haar measure, and let $\omega$ be a weight on $G$, i.e., a strictly positive measurable function $\omega$ on $G$ such that $\omega(s + t) \leq \omega(s)\omega(t)$ ($s, t \in G$). Then the Beurling algebra $L^1(G, \omega)$ consists of all complex-valued measurable functions $f$ on $G$

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such that $f \omega \in L^1(G)$ is a semisimple commutative Banach algebra with the convolution product and the norm $\|f\|_{\omega} := \int_G |f(s)|\omega(s) \, d\lambda(s)$ [BhDe3]. An $\omega$-bounded generalized character on $G$ is a continuous group homomorphism $\alpha : G \to (\mathbb{C} \setminus \{0\}, \times)$ such that $|\alpha(s)| \leq \omega(s) \, (s \in G)$. Let $H(G, \omega)$ denote the set of all $\omega$-bounded generalized characters on $G$ equipped with the compact-open topology. For $\alpha \in H(G, \omega)$, define

$$\varphi_{\alpha}(f) = \hat{f}(\alpha) = \int_G f(s)\alpha(s) \, d\lambda(s) \quad (f \in L^1(G, \omega)).$$

Then the map $T : H(G, \omega) \to \Delta(L^1(G, \omega))$ defined as $T(\alpha) = \varphi_{\alpha}$ is a homeomorphism [BhDe4]. Let $\hat{G}$ denote the dual group of $G$. Then it is easy to see that $\hat{G} \subseteq H(G, \omega)$ iff $\omega \geq 1$ on $G$. However it is always true that $\theta \alpha \in H(G, \omega)$ ($\theta \in \hat{G}$ and $\alpha \in H(G, \omega)$). For $F \subseteq H(G, \omega)$, define

$$|f|_F = \sup\{|\hat{f}(\alpha)| : \alpha \in F\} \quad (f \in L^1(G, \omega)).$$

Then $|\cdot|_F$ is a uniform seminorm on $L^1(G, \omega)$; the set $F$ is a set of uniqueness for $L^1(G, \omega)$ if $|\cdot|_F$ is a norm. For example, $\alpha\hat{G}$ is a set of uniqueness for any $\alpha \in H(G, \omega)$. For $\alpha \in H(G, \omega)$, the uniform norm $|\cdot|_{\varphi_{\alpha}\hat{G}}$ on $L^1(G, \omega)$ will be denoted by $|\cdot|_{\alpha}$. 

**THEOREM 1.** $L^1(G, \omega)$ has UUNP iff it has a minimum uniform norm.

**Proof.** If $L^1(G, \omega)$ has UUNP, then clearly it has a minimum uniform norm. Conversely, assume that it has a minimum uniform norm, say $|\cdot|_0$. Define $F = \{\alpha \in H(G, \omega) : \varphi_{\alpha}$ is $|\cdot|_0$-continuous$\}$. By taking the completion $\mathcal{A}$ of $(L^1(G, \omega), |\cdot|_0)$ and by using elementary Gelfand theory, one can see that for all $f \in L^1(G, \omega)$, $|f|_0 = |f|_F = r_{\mathcal{A}}(f)$, the spectral radius in $\mathcal{A}$. Suppose that we have proved $F = H(G, \omega)$. This implies that $|\cdot|_0 = |\cdot|_F$ is the spectral radius on $L^1(G, \omega)$ and the result is proved. So we prove $F = H(G, \omega)$ in the following two steps.

**Step 1.** $F = \hat{G}F = \{\theta \alpha : \theta \in \hat{G}$ and $\alpha \in F\}$.

Fix $\theta \in \hat{G}$. Define $|f|_{\theta F} = |\theta f|_F$ on $L^1(G, \omega)$. Then $|\cdot|_{\theta F}$ is a uniform norm on $L^1(G, \omega)$. Since $|\cdot|_F (= |\cdot|_0)$ is the minimum uniform norm on $L^1(G, \omega)$, we have

$$|f|_F \leq |f|_{\theta F} \quad (f \in L^1(G, \omega)).$$

This holds for each $\theta \in \hat{G}$. So for each $\theta \in \hat{G}$ and $f \in L^1(G, \omega)$, we have $|f|_F \leq |f|_{\theta F}$, i.e., $|f|_F \leq |\theta f|_F$. The last inequality is true for each $f$. So replacing $f$ by $\theta f$, we get $|f|_{\theta F} = |\theta f|_F \leq |f|_F$. Thus for each $\alpha \in F$, the complex homomorphism $\varphi_{\theta_{\tilde{\alpha}}}$ is $|\cdot|_0$-continuous, i.e., $\tilde{\theta}F \subseteq F$. This is true for each $\theta \in \hat{G}$. Hence $F = \hat{G}F$.

**Step 2.** $F = H(G, \omega)$. 

Suppose, if possible, $F \neq H(G, \omega)$. Choose $\alpha_1 \in F$ and $\beta_1 \in H(G, \omega) \setminus F$. Take $\alpha = |\alpha_1|$ and $\beta = |\beta_1|$. Then by Step 1, $\alpha \in F$ and $\beta \in H(G, \omega) \setminus F$. Choose $t \in G$ such that $\beta(t) < \alpha(t)$. Let $U$ be an open neighbourhood of $t$ in $G$ such that its closure $\overline{U}$ is compact and $\beta(s) < \alpha(s)$ $(s \in U)$. Take $f = \chi_U \in L^1(G, \omega)$, the characteristic function of $U$. Now
\[
|f|_\beta = \sup \{ |\hat{f}(\beta \theta)| : \theta \in \hat{G} \} = \sup \left\{ \left| \int_G f(s) \beta(s) \theta(s) \, d\lambda(s) \right| : \theta \in \hat{G} \right\}
\leq \sup \left\{ \int_U \beta(s) |\theta(s)| \, d\lambda(s) : \theta \in \hat{G} \right\} \leq \int_U \beta(s) \, d\lambda(s)
\leq \int_U \alpha(s) \, d\lambda(s) \leq \sup \{ |\hat{f}(\alpha \theta)| : \theta \in \hat{G} \} = |f|_\alpha \leq |f|_F.
\]
Thus $| \cdot |_\beta$ is a uniform norm and $|f|_\beta < |f|_F$, which is a contradiction because the latter is the minimum uniform norm on $L^1(G, \omega)$. This proves Step 2 and the result is proved.

The above result is not true in arbitrary semisimple commutative Banach algebras. For example, let $G$ be a non-discrete LCA group. Then the measure algebra $M(G)$ has a minimum uniform norm, namely $|\mu|_\infty = \sup \{ |\hat{\mu}(\theta)| : \theta \in \hat{G} \}$ ($\mu \in M(G)$) [BhDe2, Corollary 6.3]. But it does not have UUNP [BhDe2, p. 233]. Notice that both the disc algebra $A(D)$ and its variant $A_r(D) = \{ f \in C(D) : f$ is analytic on $E_r \}$ (where $0 < r < 1$ and $E_r = \{ z : r < |z| < 1 \}$) admit infinitely many uniform norms. However, $A(D)$ does not admit a minimum uniform norm, whereas $A_r(D)$ admits a minimum uniform norm, viz., $|f|_0 = \sup \{ |f(z)| : |z| \leq r \}$ ($f \in A_r(D)$).

**Theorem 2.** $L^1(G, \omega)$ admits either exactly one uniform norm or infinitely many uniform norms.

**Proof.** Assume that $L^1(G, \omega)$ has more than one distinct uniform norm. If $G$ is compact, then $L^1(G, \omega) \cong L^1(G)$ does have UUNP. Thus $G$ must be non-compact, and so its dual group $\hat{G}$ is not discrete. Now choose $\alpha \in H(G, \omega)$ and define $\tilde{\omega}(s) = \omega(s)/|\alpha(s)|$ $(s \in G)$. Then $\tilde{\omega}$ is a weight and $\tilde{\omega} \geq 1$ on $G$. It is easy to see that the map $T : L^1(G, \omega) \to L^1(G, \tilde{\omega})$ defined as $T(f) = \alpha f$ is an isometric algebra isomorphism. Hence we may assume that $\omega \geq 1$ on $G$. Therefore $\hat{G} \subseteq H(G, \omega)$. Now we define
\[
H_p(G, \omega) = \{ \alpha \in H(G, \omega) : \alpha$ is strictly positive $\}.
\]
As in the proof of Theorem 1, any two distinct elements $\alpha$ and $\beta$ in $H_p(G, \omega)$ will give distinct uniform norms on $L^1(G, \omega)$, namely $| \cdot |_\alpha$ and $| \cdot |_\beta$. Now consider the following two cases:

**Case (i):** $H_p(G, \omega)$ is not a singleton. Choose $\alpha$ and $\beta$ in $H_p(G, \omega)$ such that $\alpha \neq \beta$. Take $0 < \lambda < 1$. Define $\eta_\lambda(s) = \alpha(s)^\lambda \beta(s)^{1-\lambda}$ $(s \in G)$. Then
each \( \eta_\lambda \in H_p (G, \omega) \). Now if \( 0 < \lambda \neq \lambda' < 1 \), then \( \cdot |_{\eta_\lambda} \) and \( \cdot |_{\eta_\lambda'} \) are distinct uniform norms on \( L^1 (G, \omega) \). Thus \( L^1 (G, \omega) \) admits infinitely many uniform norms.

**Case (ii): \( H_p (G, \omega) \) is a singleton.** In this case \( H_p (G, \omega) = \{ 1_G \} \), where \( 1_G \) is the identity of \( \hat{G} \), and \( H(G, \omega) = \hat{G} \). Since \( L^1 (G, \omega) \) does not have UUNP, there exists a proper closed subset \( F \) of \( \hat{G} \) which is a set of uniqueness for \( L^1 (G, \omega) \). Since \( \hat{G} \) is not discrete, \( \hat{G} \setminus F \) is infinite. Choose \( \gamma_1, \gamma_2, \ldots \) to be different elements outside \( F \). Set \( F_0 = F \) and \( F_n = F_{n-1} \cup \{ \gamma_n \} \). For each \( n \geq 0 \), define \( |f|_n := \sup \{ |\hat{f}(\gamma)| : \gamma \in F_n \} \) \( (f \in L^1 (G, \omega)) \). Then each \( \cdot |_n \) is a uniform norm on \( L^1 (G, \omega) \). We show that they are distinct. It is enough to show that \( |g|_0 < |g|_1 \) for some \( g \in L^1 (G, \omega) \). Since \( L^1 (G) \) is regular, there exists \( f \in L^1 (G) \) such that \( \hat{f}(F) = \{ 0 \} \) and \( \hat{f}(\gamma_1) = 1 \). Fix \( 0 < \varepsilon < 1/2 \). Then there exists \( g \in C_c (G) \) such that \( \| f - g \| < \varepsilon \), where \( \| \cdot \| \) is the \( L^1 \)-norm on \( L^1 (G) \). Then \( g \in C_c (G) \subset L^1 (G, \omega) \). Also
\[
|\hat{g}(\gamma)| = |\hat{g}(\gamma) - \hat{f}(\gamma)| \leq \| \hat{f} - \hat{g} \|_{\hat{G}} \leq \| f - g \| < \varepsilon < 1/2 \quad (\gamma \in F).
\]
Moreover,
\[
|\hat{g}(\gamma_1)| = |\hat{f}(\gamma_1) - \hat{f}(\gamma_1) + \hat{g}(\gamma_1)| \geq |\hat{f}(\gamma_1)| - |\hat{f}(\gamma_1) - \hat{g}(\gamma_1)| \geq 1 - \| f - g \| > 1/2.
\]
Thus \( |g|_0 \leq 1/2 < |g|_1 \). This completes the proof. ■

Define \( \omega_1 (s) = \exp (|s|) \) and \( \omega_2 (s) = (1 + |s|)^{1/2} \) on \( \mathbb{R} \). By [D, Theorem 4.7.33], the Gelfand space of \( L^1 (\mathbb{R}, \omega_1) \) can be identified with the vertical strip \( \Pi_{-1,1} := \{ x + iy : -1 \leq x \leq 1 \} \) in the complex plane. For \( -1 \leq x \leq 1 \), define \( |f|_x = \sup \{ |\hat{f}(x + iy)| : y \in \mathbb{R} \} \) \( (f \in L^1 (\mathbb{R}, \omega_1)) \). Then each \( \cdot |_x \) is a uniform norm on \( L^1 (\mathbb{R}, \omega_1) \). By the maximum modulus principle, all of them are distinct norms. On the other hand, \( L^1 (\mathbb{R}, \omega_2) \) has exactly one uniform norm because it is regular [D, Theorem 4.3.37].

**Remark.** Assume that \( \omega \geq 1 \) on \( G \). It follows from [BhDe2, Theorem 4.1] and [Do] that \( L^1 (G, \omega) \) has UUNP if and only if \( \omega \) is non-quasi-analytic, i.e., \( \sum_{n \geq 1} (\log \omega (ns))/(1 + n^2) < \infty \) for each \( s \in G \).

References


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