STUDIA MATHEMATICA 175 (2) (2006)

Measure of non-compactness of operators interpolated by the real method

by

Radosław Szwedek (Poznań)

Abstract. We study the measure of non-compactness of operators between abstract real interpolation spaces. We prove an estimate of this measure, depending on the fundamental function of the space. An application to the spectral theory of linear operators is presented.

1. Introduction. The behavior of compact linear operators under interpolation has been studied since the 1960s. First results were established by Krasnosel'skiĭ [11], who proved that under the hypothesis of the Riesz–Thorin interpolation theorem, that is, $T : L_{p_i} \to L_{q_i}$ is bounded for $i = 0, 1$ where $1 \le p_i, q_i \le \infty$, and the additional assumption that $T : L_{p_0} \to L_{q_0}$ is compact, $q_0 < \infty$, it follows that $T : L_p \to L_q$ is also compact, where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $0 < \theta < 1$.

His results lead to the question whether similar results hold in the abstract interpolation case, when (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1}) are replaced by Banach pairs (A_0, A_1) and (B_0, B_1), respectively. The complete answer is still unknown.

The first results for the real interpolation method were obtained in 1964 by Lions–Peetre [12] for the case when $A_0 = A_1$ or $B_0 = B_1$ and by Persson [14] for the general case $A_0 \ne A_1$ and $B_0 \ne B_1$ with an approximation condition on the couple (B_0, B_1). In these cases, they showed that the operator $T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}$ is compact for $0 < \theta < 1$, $1 \le q \le \infty$. We refer to the book [2] for a more detailed history of research.

In 1969 Hayakawa [10] gave the result for the real interpolation method without the approximation hypothesis but with the assumption that $T : (A_0, A_1) \to (B_0, B_1)$ and the restrictions $T : A_0 \to B_0$, $T : A_1 \to B_1$ are both compact operators and $1 \le q < \infty$. New approaches to Hayakawa's result can be found in the paper by Cobos and Peetre [7] and the references given

2000 *Mathematics Subject Classification*: 46B70, 46M35.

Key words and phrases: linear operators, measure of non-compactness, interpolation.

there. Finally, in 1992 Cwikel [9] showed that the theorem holds whenever $T : A_0 \to B_0$ is compact and $T : A_1 \to B_1$ is bounded.

Nowadays we search for quantitative versions of interpolating compactness results. In 1999, Cobos, Fernández-Martínez and Martínez in their remarkable paper [6] obtained a logarithmic type inequality for the measure of non-compactness $\beta(\cdot)$, namely

$$\beta(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}) \leq C\beta(T : A_0 \to B_0)^{1-\theta}\beta(T : A_1 \to B_1)^{\theta}.$$

Based on some ideas from [6], we present a generalization of those results to the abstract real method of interpolation.

2. Preliminaries and notation. We use the standard notation from interpolation theory. Mostly, we follow [2] and [3], where more details are given. Let $\overline{A} = (A_0, A_1)$ be a Banach couple. As usual we let $\Delta(\overline{A}) := A_0 \cap A_1$ and $\Sigma(\overline{A}) := A_0 + A_1$. For any $t > 0$ the *K-functional* is defined by

$$K(t, a; \overline{A}) = \inf_{a=a_0+a_1} \{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\} \quad \text{for } a \in \Sigma(\overline{A}),$$

and the *J-functional* by

$$J(t, a; \overline{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\} \quad \text{for } a \in \Delta(\overline{A}).$$

A real sequence $w = \{w_n\}_{n\in\mathbb{Z}}$ is called a *weight sequence* if each w_n is positive. If E is a Banach sequence lattice modelled on $\mathbb{Z}$ and $w = \{w_n\}$ is a weight sequence, we define the weighted Banach sequence lattice $E(w)$ by setting $\|x\|_{E(w)} := \|\{x_n w_n\}\|_E$.

Recall that if a Banach sequence lattice E on $\mathbb{Z}$ is intermediate with respect to $(\ell_\infty, \ell_\infty(2^{-n}))$ (resp., $(\ell_1, \ell_1(2^{-n}))$) then the K-method space $\overline{A}_{E;K} := (A_0, A_1)_{E;K}$ consists of all $a \in A_0 + A_1$ such that $\{K(2^n, a; \overline{A})\} \in E$ with the norm

$$\|a\| = \|\{K(2^n, a; \overline{A})\}\|_E$$

while the J-method space $\overline{A}_{E;J}$ consists of all $a \in A_0 + A_1$ which can be represented in the form

$$a = \sum_{n=-\infty}^{\infty} a_n \quad (\text{convergence in } A_0 + A_1)$$

such that $\{J(2^n, a_n; \overline{A})\}_{n\in\mathbb{Z}} \in E$ with the associated norm

$$\|a\| = \inf\Big\{\|\{J(2^n, a_n; \overline{A})\}\|_E : a = \sum_{n=-\infty}^{\infty} a_n\Big\}.$$

Following the terminology of [13], the space E is said to be *K-non-trivial* (resp., *J-non-trivial*) when $\ell_\infty \cap \ell_\infty(2^{-n}) \subset E$ (resp., $E \subset \ell_1 + \ell_1(2^{-n})$).

It is well known from [13] that if E is a parameter of the real method (i.e., $\ell_\infty \cap \ell_\infty(2^{-n}) \subset E \subset \ell_1 + \ell_1(2^{-n})$) then for any Banach couple $\overline{A}$,

$$\overline{A}_{E;K} \hookrightarrow \overline{A}_{E;J}$$

and the norm of the inclusion map is less than 4. If additionally the *Calderón operator* Ω defined on $\ell_1 + \ell_1(2^{-n})$ by

$$\Omega\{\xi_n\} := \Big\{ \sum_{k=-\infty}^{\infty} \min\{1, 2^{n-k}\}|\xi_k| \Big\} \quad \text{for } \{\xi_n\} \in \ell_1 + \ell_1(2^{-n})$$

is bounded on E, then $\overline{A}_{E;J} = \overline{A}_{E;K}$. In this case we write $\overline{A}_E$ for $\overline{A}_{E;J}$ or $\overline{A}_{E;K}$.

The classical interpolation spaces play an important role from the point of view of applications. Let ϱ be a *function parameter*, i.e., $\varrho : (0,\infty) \to (0,\infty)$ is a quasi-concave function ($t \mapsto \varrho(t)$ increases and $t \mapsto \varrho(t)/t$ decreases) and

$$s_\varrho(t) = o(1) \text{ as } t \to 0 \quad \text{and} \quad s_\varrho(t) = o(t) \text{ as } t \to \infty,$$

where $s_\varrho(t) = \sup\{\varrho(tu)/\varrho(u) : u > 0\}$ for every $t > 0$. Now, if we take $E = \ell_q(1/\varrho(2^m))$ with $1 \le q \le \infty$, then

$$(A_0, A_1)_{\ell_q(1/\varrho(2^m));K} = (A_0, A_1)_{\ell_q(1/\varrho(2^m));J} = (A_0, A_1)_{\varrho,q}.$$

If $\varrho(t) = t^\theta$, $\theta \in (0,1)$, we get the classical real interpolation spaces $(A_0, A_1)_{\theta,q}$ (see, e.g., [2], [3]).

Given any sequence $\{X_m\}_{m\in\mathbb{Z}}$ of Banach spaces we denote by $(\bigoplus X_m)_E$ the space of $\{x_m\}_{m\in\mathbb{Z}} \in \prod_{m=-\infty}^{\infty} X_m$ such that $\{\|x_m\|_{X_m}\}_{m\in\mathbb{Z}} \in E$. It is a Banach space equipped with the norm

$$\|\{x_m\}\| = \|\{\|x_m\|_{X_m}\}\|_E.$$

We will need the following useful vector-valued continuous inclusions (with norms less than or equal to 1) proved in [5]:

PROPOSITION 2.1. *Let $\{X_m\}_{m\in\mathbb{Z}}$ be a sequence of Banach spaces.*

(i) *If E is K-non-trivial then*

$$((\bigoplus X_m)_{\ell_\infty}, (\bigoplus X_m)_{\ell_\infty(2^{-m})})_{E;K} \hookrightarrow (\bigoplus X_m)_E.$$

(ii) *If E is J-non-trivial then*

$$(\bigoplus X_m)_E \hookrightarrow ((\bigoplus X_m)_{\ell_1}, (\bigoplus X_m)_{\ell_1(2^{-m})})_{E;J}.$$

An immediate consequence is

COROLLARY 2.2. *Let $\{X_m\}_{m\in\mathbb{Z}}$ be a sequence of Banach spaces. If the J-method and the K-method generated by a parameter E of the real method are equivalent then*

$$((\bigoplus X_m)_{\ell_{q_0}}, (\bigoplus X_m)_{\ell_{q_1}(2^{-m})})_E = (\bigoplus X_m)_E$$

for any $1 \le q_j \le \infty$ $(j = 0, 1)$.

For a given Banach lattice E on $\mathbb{Z}$ and a subset $A \subset \mathbb{Z}$, the subspace of all sequences supported on A is denoted by $E|_A$. In particular, we may take A to be $[p,q] := \{n \in \mathbb{Z} : p \le n \le q\}$ with $p, q \in \mathbb{Z}$, $p < q$.

Let $\omega(\mathbb{Z})$ be the space of all real sequences modelled on $\mathbb{Z}$. For any $\nu \in \mathbb{Z}$, the shift operator $\tau_\nu : \omega(\mathbb{Z}) \to \omega(\mathbb{Z})$ is defined by $\tau_\nu\{\xi_m\} = \{\xi_{m+\nu}\}$.

Throughout the rest of the paper we consider Banach lattices E on $\mathbb{Z}$ such that the shift operator τ_ν is bounded in E for all $\nu \in \mathbb{Z}$. For such E we define a function $\varphi_E : (0,\infty) \times (0,\infty) \to (0,\infty)$ by $\varphi_E(2^m, 2^n) = 2^m \|\tau_{n-m}\|_{E\to E}$ for $m, n \in \mathbb{Z}$ and $\varphi_E(s,t) = \varphi_E(2^{[\log_2 s]}, 2^{[\log_2 t]})$ for any $s, t > 0$, where $[\cdot]$ denotes the greatest integer function.

We will also need the extension $\psi_E : [0,\infty) \times [0,\infty) \to [0,\infty]$ of φ_E, defined by $\psi_E(0,0) = 0$, $\psi_E(s,0) = \liminf_{v\to 0+} \varphi_E(s,v)$ for $s > 0$ and $\psi_E(0,t) = \liminf_{u\to 0+} \varphi_E(u,t)$ for $t > 0$.

We have the following technical lemma:

LEMMA 2.3. *Let E be a Banach sequence lattice on $\mathbb{Z}$ such that the shift operator τ_n is bounded in E for any $n \in \mathbb{N}$. Then the function ψ_E has the following properties*:

(a) $\psi_E(2^m s, 2^n t) \le \psi_E(2^m, 2^n)\psi_E(s,t)$ *for all* $m, n \in \mathbb{Z}$ *and* $s, t \ge 0$.
(b) *There exists a constant* $C_1 = C_1(E) \ge 1$ *such that*

$$\psi_E(su, tv) \le C_1 \psi_E(s,t)\psi_E(u,v) \quad \text{for all } s,t,u,v \ge 0.$$

(c) *If* $\sup_{s,t\in(0,1]} \psi_E(s,t) < \infty$, *then there exists a constant* $C_2 = C_2(E) \ge 1$ *such that*

$$\psi_E(s,t) \le C_2 \psi_E(u,v) \quad \text{for all } 0 \le s \le u,\ 0 \le t \le v.$$

Proof. Since $\psi_E(2^{m+k}, 2^{n+l}) \le \psi_E(2^m, 2^n)\psi_E(2^k, 2^l)$ and

$$[\log_2 2^k s] = k + [\log_2 s],$$
$$[\log_2 s] + [\log_2 t] \le [\log_2 st] \le [\log_2 s] + [\log_2 t] + 1,$$

for any $m, k, n, l \in \mathbb{Z}$ and $s, t > 0$, we get (a) and (b) with a constant $C_1 = \max_{i,j=0,1} \psi_E(2^i, 2^j)$.

For (c), since $\sup_{s,t\in(0,1]} \psi_E(s,t) < \infty$, using the previously proved inequalities, we obtain

$$\psi_E(s,t) = \psi_E\left(\frac{s}{u}\,u, \frac{t}{v}\,v\right) \le C_2 \psi_E(u,v)$$

for all $0 < s \le u, 0 < t \le v$ where $C_2 = C_1 \sup_{s,t\in(0,1]} \psi_E(s,t) < \infty$. Recalling that $\psi_E(0,t) = \liminf_{s\to 0+} \psi_E(s,t)$ and $\psi_E(s,0) = \liminf_{t\to 0+} \psi_E(s,t)$ completes the proof. ∎

LEMMA 2.4. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$ be a bounded operator.*

(i) *If E is K-non-trivial, then*

$$\|T\|_{\overline{A}_{E;K}\to\overline{B}_{E;K}} \le 2\psi_E(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}).$$

(ii) *If E is J-non-trivial, then*

$$\|T\|_{\overline{A}_{E;J}\to\overline{B}_{E;J}} \le 2\psi_E(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}).$$

Proof. (i) Since the case $\|T\|_{A_0\to B_0} = \|T\|_{A_1\to B_1} = 0$ is obvious, we may assume that $\|T\|_{A_i\to B_i} > 0$ for $i = 0$ or $i = 1$. Fix $\varepsilon, \delta \ge 0$ with $\varepsilon = 0$ if and only if $\|T\|_{A_0\to B_0} > 0$, and $\delta = 0$ if and only if $\|T\|_{A_1\to B_1} > 0$. Now take $k_0, k_1 \in \mathbb{Z}$ such that $2^{k_0-1} \le \|T\|_{A_0\to B_0} + \varepsilon < 2^{k_0}$ and $2^{k_1-1} \le \|T\|_{A_1\to B_1} + \delta < 2^{k_1}$. Let $\nu = k_1 - k_0$. Then for any $a \in \overline{A}_{E;K}$ we have

$$\begin{aligned} K(2^m, Ta; \overline{B}) &\le \inf_{a=a_0+a_1}\{\|Ta_0\|_{B_0} + 2^m\|Ta_1\|_{B_1} : a_0 \in A_0,\ a_1 \in A_1\} \\ &\le \inf_{a=a_0+a_1}\{2^{k_0}\|a_0\|_{A_0} + 2^{m+k_1}\|a_1\|_{A_1} : a_0 \in A_0,\ a_1 \in A_1\} \\ &= 2^{k_0}K(2^{m+\nu}, a; \overline{A}), \end{aligned}$$

and hence

$$\begin{aligned} \|Ta\|_{\overline{B}_{E;K}} &= \|\{K(2^m, Ta; \overline{B})\}\|_E \le 2^{k_0}\|\{K(2^{m+\nu}, a; \overline{A})\}\|_E \\ &= 2^{k_0}\|\tau_\nu\{K(2^m, a; \overline{A})\}\|_E \le 2^{k_0}\|\tau_\nu\|_{E\to E}\|\{K(2^m, a; \overline{A})\}\|_E \\ &= 2^{1+k_0-1}\|\tau_{k_1-1-(k_0-1)}\|_{E\to E}\|a\|_{\overline{A}_{E;K}} = 2\varphi_E(2^{k_0-1}, 2^{k_1-1})\|a\|_{\overline{A}_{E;K}} \\ &= 2\,\varphi_E(\|T\|_{A_0\to B_0} + \varepsilon, \|T\|_{A_1\to B_1} + \delta)\|a\|_{\overline{A}_{E;K}}. \end{aligned}$$

If $\varepsilon, \delta = 0$ then the above estimate gives the required inequality. If either $\varepsilon > 0$ or $\delta > 0$ then the inequality follows by taking the relevant limits. This completes the proof of (i).

(ii) Let $a \in \overline{A}_{E;J}$ and take any series $\sum_{m=-\infty}^{\infty} u_m$ convergent in $A_0 + A_1$ to a, where $\{u_m\} \subset A_0 \cap A_1$ is such that $\{J(2^m, u_m)\} \in E$. Since E is J-non-trivial, the series $\sum_{m=-\infty}^{\infty} u_m$ is absolutely summable in $A_0 + A_1$. Therefore, for any $\nu \in \mathbb{Z}$ we have $a = \sum_{m=-\infty}^{\infty} u_{m+\nu}$ (convergence in $A_0 + A_1$) and

$$\begin{aligned} J(2^m, Tu_{m+\nu}; \overline{B}) &\le \max\{2^{k_0}\|u_{m+\nu}\|_{A_0}, 2^{m+k_1}\|u_{m+\nu}\|_{A_1}\} \\ &= 2^{k_0}J(2^{m+\nu}, u_{m+\nu}; \overline{A}). \end{aligned}$$

This implies

$$\begin{aligned} \|Ta\|_{\overline{B}_{E;J}} &\le \|\{J(2^m, Tu_{m+\nu}; \overline{B})\}\|_E \le 2^{k_0}\|\{J(2^{m+\nu}, u_{m+\nu}; \overline{A})\}\|_E \\ &= 2^{k_0}\|\tau_\nu\{J(2^m, u_m; \overline{A})\}\|_E \le 2^{k_0}\|\tau_\nu\|_{E\to E}\|\{J(2^m, u_m; \overline{A})\}\|_E \\ &= 2^{1+k_0-1}\|\tau_{k_1-1-(k_0-1)}\|_{E\to E}\|\{J(2^m, u_m; \overline{A})\}\|_E. \end{aligned}$$

Taking the infimum over all representations $a = \sum_{m=-\infty}^{\infty} u_m$ of a, we get

$$\begin{aligned}\|Ta\|_{\overline{B}_{E;J}} &\le 2^{1+k_0-1}\|\tau_{k_1-1-(k_0-1)}\|_{E\to E}\|a\|_{\overline{A}_{E;J}}\\ &= 2\,\varphi_E(\|T\|_{A_0\to B_0}+\varepsilon, \|T\|_{A_1\to B_1}+\delta)\|a\|_{\overline{A}_{E;J}},\end{aligned}$$

which completes the proof as above. ∎

COROLLARY 2.5. *Let $\{X_m\}$ and $\{Y_m\}$ be sequences of Banach spaces, let E be a parameter of the real method, and assume that the Calderón operator Ω is bounded on E. Then there exists a constant $C = C(E) > 0$ such that for any bounded operator*

$$T : ((\bigoplus X_m)_{\ell_1}, (\bigoplus X_m)_{\ell_1(2^{-m})}) \to ((\bigoplus Y_m)_{\ell_\infty}, (\bigoplus Y_m)_{\ell_\infty(2^{-m})}),$$

we have

$$\begin{aligned}&\|T\|_{(\bigoplus X_m)_E\to(\bigoplus Y_m)_E}\\ &\qquad\le C\psi_E(\|T\|_{(\bigoplus X_m)_{\ell_1}\to(\bigoplus Y_m)_{\ell_\infty}}, \|T\|_{(\bigoplus X_m)_{\ell_1(2^{-m})}\to(\bigoplus Y_m)_{\ell_\infty(2^{-m})}}).\end{aligned}$$

Proof. Use Proposition 2.1 and Lemma 2.4. ∎

Let $\overline{A} = (A_0, A_1)$ be a Banach couple. For $m \in \mathbb{Z}$ define the Banach spaces

$$\begin{aligned}\Delta_m(\overline{A}) &= (A_0\cap A_1, J(2^m,\cdot;\overline{A})),\\ \Sigma_m(\overline{A}) &= (A_0+A_1, K(2^m,\cdot;\overline{A})).\end{aligned}$$

We note that for any sequence

$$\{a_n\} \in (\bigoplus \Delta_m(\overline{A}))_{\ell_1} + (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} = (\bigoplus \Delta_m(\overline{A}))_{\ell_1+\ell_1(2^{-m})}$$

the series $\sum_{n=-\infty}^{\infty} a_n$ is absolutely convergent in $A_0 + A_1$. This allows us to define a bounded operator $\pi : ((\bigoplus \Delta_m(\overline{A}))_{\ell_1}, (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}) \to (A_0, A_1)$ by

$$\pi\{a_n\} = \sum_{n=-\infty}^{\infty} a_n.$$

Clearly $\|\pi\| \le 1$. Moreover if E is J-non-trivial, then $\pi : (\bigoplus \Delta_m(\overline{A}))_E \to \overline{A}_{E;J}$ is a metric surjection.

We will also need the diagonal operator

$$j : (B_0, B_1) \to ((\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}, (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})})$$

defined by

$$jb = \{\ldots, b, b, b, \ldots\} \quad \text{for } b \in B_0 + B_1.$$

It is obvious that j is bounded with $\|j\| \le 1$ and $j : \overline{B}_{E;K} \to (\bigoplus \Sigma_m(\overline{B}))_E$ is a metric injection whenever E is K-non-trivial.

Let $n \in \mathbb{N}$. Following [6] we define operators P_n, Q_n^+, Q_n^- on the Banach couple $((\bigoplus \Delta_m(\overline{A}))_{\ell_1}, (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})})$ by

$$\begin{aligned} P_n\{u_m\} &= \{\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots\}, \\ Q_n^+\{u_m\} &= \{\ldots, 0, 0, u_{n+1}, u_{n+2}, \ldots\}, \\ Q_n^-\{u_m\} &= \{\ldots, u_{-n-2}, u_{-n-1}, 0, 0, \ldots\}. \end{aligned}$$

The following properties of the above operators are obvious:

- $I = P_n + Q_n^+ + Q_n^-$, where I denotes the identity operator on $(\bigoplus \Delta_m(\overline{A}))_{\ell_1} + (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}$.
- The operators P_n, Q_n^+, Q_n^-, $Q_n^+ + Q_n^-$ have norm 1 on $(\bigoplus \Delta_m(\overline{A}))_{\ell_1}$, $(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}$ and $(\bigoplus \Delta_m(\overline{A}))_E$.
- The following equalities hold:

$$\begin{aligned} \|Q_n^+\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}} &= \|Q_n^-\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Delta_m(\overline{A}))_{\ell_1}} \\ &= 2^{-(n+1)}, \\ \|P_n\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}} &= \|P_n\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Delta_m(\overline{A}))_{\ell_1}} \\ &= 2^n. \end{aligned}$$

In a similar way, we define the operators R_n, S_n^+, S_n^- on the Banach couple $((\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}, (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})})$.

We recall that if $T : X \to Y$ is an operator, then the nth *entropy number* $\varepsilon_n(T) := \varepsilon_n(T : X \to Y)$, $n \in \mathbb{N}$, is defined by

$$\varepsilon_n(T) = \inf\Big\{\varepsilon > 0 : T(B_X) \subset \bigcup_{i=1}^{n} B_Y(y_i, \varepsilon),\ y_i \in Y\Big\},$$

where for a given Banach space Z and $\varepsilon > 0$, we set $B_Z(x, \varepsilon) := \{y \in Z : \|x - y\|_Z \le \varepsilon\}$ and $B_Z = B_Z(0, 1)$.

We refer to [4] for the fundamental properties of the entropy numbers. The following facts are needed in what follows:

- $\varepsilon_n(T) > 0$ for any $0 \neq T \in L(X, Y)$,
- $\|T\| = \varepsilon_1(T) \ge \varepsilon_2(T) \ge \cdots \ge 0$ for any $T \in L(X, Y)$,
- $\varepsilon_{kn}(T_1 + T_2) \le \varepsilon_k(T_1) + \varepsilon_n(T_2)$ for any T_1, $T_2 \in L(X, Y)$,
- $\varepsilon_{kn}(RS) \le \varepsilon_k(R)\varepsilon_n(S)$ for any $S \in L(X, Y)$, $R \in L(Y, Z)$,
- $\varepsilon_n(TQ) = \varepsilon_n(T)$ for $T \in L(X, Y)$ and any metric surjection $Q : \widetilde{X} \to X$,
- $\varepsilon_n(T) \le 2\varepsilon_n(JT)$ for $T \in L(X, Y)$ and any metric injection $J : Y \to \widetilde{Y}$,

where $X, Y, Z, \widetilde{X}, \widetilde{Y}$ are arbitrary Banach spaces and $n, k \in \mathbb{N}$.

3. Main results. In this section, we prove our main results on estimates of the measure of non-compactness for special operators defined on Banach spaces and operators between Banach couples. We recall that the *measure* *of non-compactness* $\beta(T) := \beta(T : X \to Y)$ is defined for any bounded operator T between Banach spaces by

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T).$$

In what follows, without further comment, we will use C to denote positive constants (C may depend on objects that are regarded as fixed).

THEOREM 3.1. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$ be a bounded operator. Assume that E is a parameter of the real method such that the Calderón operator Ω is bounded on E. Assume that*

$$\lim_{s\to 0+} \psi_E(s,1) = \lim_{t\to 0+} \psi_E(1,t) = 0.$$

Then there exists a constant $C = C(E) > 0$ such that for any $\varepsilon > 0$ and $k_0, k_1 \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\begin{aligned}\beta((S_n^+ + S_n^-)jT\pi(Q_n^+ + Q_n^-) &: (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E) \\ &\le C\psi_E(\varepsilon_{k_0}(T : A_0 \to B_0) + \varepsilon, \varepsilon_{k_1}(T : A_1 \to B_1) + \varepsilon).\end{aligned}$$

Proof. Since for any $n \in \mathbb{N}$, we have

$$\begin{aligned}&(S_n^+ + S_n^-)jT\pi(Q_n^+ + Q_n^-) \\ &\qquad = S_n^+ jT\pi Q_n^+ + S_n^+ jT\pi Q_n^- + S_n^- jT\pi Q_n^+ + S_n^- jT\pi Q_n^-,\end{aligned}$$

it suffices to estimate the measure of non-compactness for each term.

First, note that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|S_n^+ jT\pi Q_n^-\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} &\le \|T\|_{A_0\to B_0}, \\ \|S_n^- jT\pi Q_n^+\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} &\le \|T\|_{A_1\to B_1}.\end{aligned}$$

Using the factorizations

$$\begin{array}{ccc}(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} & \xrightarrow{S_n^+ jT\pi Q_n^-} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})} \\ Q_n^- \big\downarrow & & \big\uparrow S_n^+ \\ (\bigoplus \Delta_m(\overline{A}))_{\ell_1} & \xrightarrow{jT\pi} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}\end{array}$$

and

$$\begin{array}{ccc}(\bigoplus \Delta_m(\overline{A}))_{\ell_1} & \xrightarrow{S_n^- jT\pi Q_n^+} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty} \\ Q_n^+ \big\downarrow & & \big\uparrow S_n^- \\ (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} & \xrightarrow{jT\pi} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}\end{array}$$

we obtain

$$\|S_n^+ jT\pi Q_n^-\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}\to(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} \le 2^{-2(n+1)}\|T\|_{A_0\to B_0},$$
$$\|S_n^- jT\pi Q_n^+\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}\to(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \le 2^{-2(n+1)}\|T\|_{A_1\to B_1}.$$

Combining the above estimates with Corollary 2.5 and Lemma 2.3, we find that there exists a constant $D = D(E)$ such that the measures of non-compactness of the operators $S_n^+ jT\pi Q_n^- : (\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E$ and $S_n^- jT\pi Q_n^+ : (\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E$ satisfy the following estimates:

$$\begin{aligned}\beta(S_n^+ jT\pi Q_n^-) &\le \|S_n^+ jT\pi Q_n^-\| \le D\psi_E(\|T\|_{A_0\to B_0}, 2^{-2(n+1)}\|T\|_{A_0\to B_0})\\ &\le D\|T\|_{A_0\to B_0}\psi_E(1, 2^{-2(n+1)}),\end{aligned}$$

and

$$\begin{aligned}\beta(S_n^- jT\pi Q_n^+) &\le D\psi_E(2^{-2(n+1)}\|T\|_{A_1\to B_1}, \|T\|_{A_1\to B_1})\\ &\le D\|T\|_{A_1\to B_1}\psi_E(2^{-2(n+1)}, 1).\end{aligned}$$

Now fix $\varepsilon > 0$ and $k_0, k_1 \in \mathbb{N}$, and put $\varepsilon_{k_i}(T) = \varepsilon_{k_i}(T : A_i \to B_i)$ for $i = 0, 1$. Note that there exists $N_E \in \mathbb{N}$ such that

$$\beta(S_n^+ jT\pi Q_n^- + S_n^- jT\pi Q_n^+) \le D\psi_E(\varepsilon_{k_0}(T) + \varepsilon, \varepsilon_{k_1}(T) + \varepsilon)$$

for all $n > N_E$.

In order to estimate the measures of non-compactness of the remaining two operators $S_n^+ jT\pi Q_n^+$ and $S_n^- jT\pi Q_n^-$, we first observe that the operators

$$jT\pi Q_n^+ : (\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty},$$
$$jT\pi Q_n^- : (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}$$

are bounded and the sequences $\{\|jT\pi Q_n^+\|\}_{n=1}^\infty$, $\{\|jT\pi Q_n^-\|\}_{n=1}^\infty$ of their norms are non-increasing. Let

$$\lambda_0 = \lim_{n\to\infty} \|jT\pi Q_n^+\|, \quad \lambda_1 = \lim_{n\to\infty} \|jT\pi Q_n^-\|,$$

and let $\{u_m^0\} \in B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}}$ and $\{u_m^1\} \in B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}}$ be sequences such that

$$\lambda_0 = \lim_{n\to\infty} \|jT\pi Q_n^+ u_n^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}},$$
$$\lambda_1 = \lim_{n\to\infty} \|jT\pi Q_n^- u_n^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}.$$

The properties of entropy numbers yield

$$\begin{aligned}&\varepsilon_{k_0}(jT\pi : (\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty})\\ &\qquad\le \varepsilon_{k_0}(T\pi : (\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to B_0) \le \varepsilon_{k_0}(T),\\ &\varepsilon_{k_1}(jT\pi : (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})})\\ &\qquad\le \varepsilon_{k_1}(T\pi : (\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to B_1) \le \varepsilon_{k_1}(T).\end{aligned}$$

From the definition of entropy numbers there exist sets $\{b_1^0, \ldots, b_{k_0}^0\} \subset B_0$ and $\{b_1^1, \ldots, b_{k_1}^1\} \subset B_1$ such that

$$T\pi(B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}}) \subset \bigcup_{j=1}^{k_0} B_{B_0}(b_j^0, \varepsilon_{k_0}(T) + \varepsilon/2),$$

$$T\pi(B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}}) \subset \bigcup_{j=1}^{k_1} B_{B_1}(b_j^1, \varepsilon_{k_1}(T) + \varepsilon/2).$$

Thus, for some $b_{j_i}^i$, $1 \le j_i \le k_i$, and for some subsequences $\{n_k^i\}_{k=1}^\infty$ with $i = 0, 1$, we obtain

$$T\pi Q_{n_k^0}^+ u_{n_k^0}^0 \in B_{B_0}(b_{j_0}^0, \varepsilon_{k_0}(T) + \varepsilon/2),$$
$$T\pi Q_{n_k^1}^- u_{n_k^1}^1 \in B_{B_1}(b_{j_1}^1, \varepsilon_{k_1}(T) + \varepsilon/2).$$

This implies

$$\begin{aligned} K(2^m, b_{j_0}^0; \overline{B}) &\le \|b_{j_0}^0 - T\pi Q_{n_k^0}^+ u_{n_k^0}^0\|_{B_0} + 2^m \|T\pi Q_{n_k^0}^+ u_{n_k^0}^0\|_{B_1} \\ &\le (\varepsilon_{k_0}(T) + \varepsilon/2) + 2^m \|T\|_{A_1 \to B_1} 2^{-(n_k^0+1)} \|u_{n_k^0}^0\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}} \\ &\le (\varepsilon_{k_0}(T) + \varepsilon/2) + 2^{m-n_k^0} \|T\|_{A_1 \to B_1}, \\ K(2^m, b_{j_1}^1; \overline{B}) &\le \|T\pi Q_{n_k^1}^- u_{n_k^1}^1\|_{B_0} + 2^m \|b_{j_1}^1 - T\pi Q_{n_k^1}^- u_{n_k^1}^1\|_{B_1} \\ &\le \|T\|_{A_0 \to B_0} 2^{-(n_k^1+1)} \|u_{n_k^1}^1\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}} + 2^m(\varepsilon_{k_1}(T) + \varepsilon/2) \\ &\le 2^{-n_k^1} \|T\|_{A_0 \to B_0} + 2^m(\varepsilon_{k_1}(T) + \varepsilon/2), \end{aligned}$$

and hence

$$\|jb_{j_0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} = \sup_{m \in \mathbb{Z}} K(2^m, b_{j_0}^0; \overline{B}) \le \varepsilon_{k_0}(T) + \varepsilon/2,$$
$$\|jb_{j_1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} = \sup_{m \in \mathbb{Z}} 2^{-m} K(2^m, b_{j_1}^1; \overline{B}) \le \varepsilon_{k_1}(T) + \varepsilon/2.$$

Consequently,

$$\begin{aligned} \lambda_0 &= \lim_{k \to \infty} \|jT\pi Q_{n_k^0}^+ u_{n_k^0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \\ &\le \sup_{k \in \mathbb{N}} (\|jT\pi Q_{n_k^0}^+ u_{n_k^0}^0 - jb_{j_0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} + \|jb_{j_0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}}) \\ &\le 2(\varepsilon_{k_0}(T) + \varepsilon/2) \end{aligned}$$

and

$$\begin{aligned} \lambda_1 &= \lim_{k \to \infty} \|jT\pi Q_{n_k^1}^- u_{n_k^1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} \\ &\le \sup_{k \in \mathbb{N}} (\|jT\pi Q_{n_k^1}^- u_{n_k^1}^1 - jb_{j_1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} + \|jb_{j_1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}) \\ &\le 2(\varepsilon_{k_1}(T) + \varepsilon/2). \end{aligned}$$

Hence, there is $N_a \in \mathbb{N}$ such that for $n > N_a$,

$$\|S_n^+ jT\pi Q_n^+\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \le 2(\varepsilon_{k_0}(T) + \varepsilon),$$
$$\|S_n^- jT\pi Q_n^-\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} \le 2(\varepsilon_{k_1}(T) + \varepsilon).$$

With previously chosen $\varepsilon > 0$, there exist sets $\{u_1^0, \ldots, u_{k_0}^0\} \subset B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}}$ and $\{u_1^1, \ldots, u_{k_1}^1\} \subset B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}}$ such that

$$jT\pi(B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}}) \subset \bigcup_{j=1}^{k_0} B_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}}(jT\pi u_j^0, 2\varepsilon_{k_0}(T) + \varepsilon/3),$$
$$jT\pi(B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}}) \subset \bigcup_{j=1}^{k_1} B_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}(jT\pi u_j^1, 2\varepsilon_{k_1}(T) + \varepsilon/3).$$

Set $\delta_i = \varepsilon/(\|T\|_{A_i \to B_i} + \varepsilon)$ for $i = 0, 1$. There exists $N_\varepsilon \in \mathbb{N}$ such that

$$\max_{1 \le k \le k_0} \|P_{N_\varepsilon} u_k^0 - u_k^0\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}} \le \delta_0/3,$$
$$\max_{1 \le k \le k_1} \|P_{N_\varepsilon} u_k^1 - u_k^1\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}} \le \delta_1/3.$$

This implies

$$\max_{1 \le k \le k_0} \|jT\pi P_{N_\varepsilon} u_k^0 - jT\pi u_k^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \le \varepsilon/3,$$
$$\max_{1 \le k \le k_1} \|jT\pi P_{N_\varepsilon} u_k^1 - jT\pi u_k^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} \le \varepsilon/3.$$

Thus for any $u^0 \in B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1}}$ and any $u^1 \in B_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})}}$

$$\min_{1 \le k \le k_0} \{\|jT\pi u^0 - jT\pi P_{N_\varepsilon} u_k^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}}\} \le 2\varepsilon_{k_0}(T) + 2\varepsilon/3,$$
$$\min_{1 \le k \le k_1} \{\|jT\pi u^1 - jT\pi P_{N_\varepsilon} u_k^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}\} \le 2\varepsilon_{k_1}(T) + 2\varepsilon/3.$$

The diagrams

$$\begin{array}{ccc}
(\bigoplus \Delta_m(\overline{A}))_{\ell_1} & \xrightarrow{S_n^- jT\pi P_{N_\varepsilon}} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty} \\
{\scriptstyle P_{N_\varepsilon}}\downarrow & & \uparrow{\scriptstyle S_n^-} \\
(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} & \xrightarrow{jT\pi} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}
\end{array}$$

and

$$\begin{array}{ccc}
(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} & \xrightarrow{S_n^+ jT\pi P_{N_\varepsilon}} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})} \\
{\scriptstyle P_{N_\varepsilon}}\downarrow & & \uparrow{\scriptstyle S_n^+} \\
(\bigoplus \Delta_m(\overline{A}))_{\ell_1} & \xrightarrow{jT\pi} & (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}
\end{array}$$

imply that there exists $N_b \in \mathbb{N}$ such that for $n > N_b$,

$$\max_{1\le k\le k_0}\{\|S_n^- jT\pi P_{N_\varepsilon} u_k^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}}\} \le \varepsilon/3,$$
$$\max_{1\le k\le k_1}\{\|S_n^+ jT\pi P_{N_\varepsilon} u_k^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}\} \le \varepsilon/3.$$

Thus for some $1 \le j_0 \le k_0$ and $1 \le j_1 \le k_1$, we obtain

$$\begin{aligned}\|S_n^- jT\pi u^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} &\le \|S_n^- jT\pi u^0 - S_n^- jT\pi P_{N_\varepsilon} u_{j_0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}}\\ &\quad + \|S_n^- jT\pi P_{N_\varepsilon} u_{j_0}^0\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \le 2\varepsilon_{k_0}(T) + \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|S_n^+ jT\pi u^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} &\le \|S_n^+ jT\pi u^1 - S_n^+ jT\pi P_{N_\varepsilon} u_{j_1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}\\ &\quad + \|S_n^+ jT\pi P_{N_\varepsilon} u_{j_1}^1\|_{(\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}}\\ &\le 2\varepsilon_{k_1}(T) + \varepsilon.\end{aligned}$$

This implies

$$\|S_n^- jT\pi Q_n^-\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty}} \le 2\varepsilon_{k_0}(T) + \varepsilon,$$
$$\|S_n^+ jT\pi Q_n^+\|_{(\bigoplus \Delta_m(\overline{A}))_{\ell_1(2^{-m})} \to (\bigoplus \Sigma_m(\overline{B}))_{\ell_\infty(2^{-m})}} \le 2\varepsilon_{k_1}(T) + \varepsilon.$$

Consequently, we obtain

$$\begin{aligned}&\beta(S_n^+ jT\pi Q_n^+ : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E)\\ &\qquad \le D\psi_E(2\varepsilon_{k_0}(T) + 2\varepsilon, 2\varepsilon_{k_1}(T) + 2\varepsilon) \le 2D\psi_E(\varepsilon_{k_0}(T) + \varepsilon, \varepsilon_{k_1}(T) + \varepsilon),\\ &\beta(S_n^- jT\pi Q_n^- : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E)\\ &\qquad \le 2D\psi_E(\varepsilon_{k_0}(T) + \varepsilon, \varepsilon_{k_1}(T) + \varepsilon).\end{aligned}$$

Finally, the combination of the estimates obtained yields the required estimate

$$\begin{aligned}&\beta((S_n^+ + S_n^-)jT\pi(Q_n^+ + Q_n^-) : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E)\\ &\qquad \le C\psi_E(\varepsilon_{k_0}(T) + \varepsilon, \varepsilon_{k_1}(T) + \varepsilon)\end{aligned}$$

for all $n > N = \max\{N_a, N_\varepsilon, N_b, N_E\}$, where $C = C(E) > 0$. ∎

Theorem 3.2. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let $T : \overline{A} \to \overline{B}$ be a bounded operator and E be a J-non-trivial $\mathbb{Z}$-lattice. Then for any $\varepsilon > 0$ and $n, k_0, k_1 \in \mathbb{N}$,*

$$\begin{aligned}&\beta(T\pi P_n)_{(\bigoplus \Delta_m(\overline{A}))_E \to \overline{B}_{E;J}}\\ &\qquad \le 4\psi_E(\varepsilon_{k_0}(T : A_0 \to B_0) + \varepsilon, \varepsilon_{k_1}(T : A_1 \to B_1) + \varepsilon).\end{aligned}$$

Proof. The space $E|_{[-n,n]}$ is finite-dimensional and $E|_{[-n,n]} \overset{C}{\hookrightarrow} \ell_\infty^{(2n+1)}$, where $C = C(n) > 0$. Given any $\eta > 0$, by the compactness argument there is a finite set $\{\mu_1, \ldots, \mu_s\} \subset B_{E|_{[-n,n]}}$ such that for any $\lambda \in B_{E|_{[-n,n]}}$,

$$\min_{1\le r\le s}\{\|\lambda-\mu_r\|_{E|_{[-n,n]}}\}<\eta/C.$$

Take $\varepsilon>0$ and $k_0,k_1\in\mathbb{N}$. Let $2^{l_i-1}\le\varepsilon_{k_i}(T:A_i\to B_i)+\varepsilon<2^{l_i}$ for $i=0,1$ and let $\nu=l_1-l_0$. For any $\{u_m\}_{m\in\mathbb{Z}}\in B_{(\bigoplus\Delta_m(\overline{A}))_E}$, we have

$$\begin{aligned}\|\{J(2^{-n},u_{-n}),\dots,J(2^n,u_n)\}\|_{E|_{[-n,n]}}&=\|P_n\{u_m\}\|_{(\bigoplus\Delta_m(\overline{A}))_E}\\&\le\|\{u_m\}\|_{(\bigoplus\Delta_m(\overline{A}))_E}\le 1.\end{aligned}$$

Thus there exists a positive integer $r\le s$ such that

$$\max_{i=0,1}\{2^{im}\|u_m\|_{A_i}\}=J(2^m,u_m)<\mu_m^r+\eta$$

for $m=-n,\dots,n$, where $\mu_r=\{\mu_m^r\}$. From the definition of $\varepsilon_{k_i}(T:A_i\to B_i)$, there exist finite sets $\{b_1^i,\dots,b_{k_i}^i\}\subset B_i$, $i=0,1$, such that for $m=-n,\dots,n$,

$$\min_{1\le j\le k_i}\{\|Tu_m-(\mu_m^r+\eta)b_j^i\|_{B_i}\}\le 2^{-im}(\mu_m^r+\eta)2^{l_i}=\eta_{r,m}^i.$$

For $m\in\{-n,\dots,n\}$ and positive integers $j_0\le k_0$, $j_1\le k_1$ and $r\le s$ choose elements d_m^p with $p=p(r,j_0,j_1)$ depending on r, j_0, j_1, where

$$d_m^p\in B_{B_0}((\mu_m^r+\eta)b_{j_0}^0,\eta_{r,m}^0)\cap B_{B_1}((\mu_m^r+\eta)b_{j_1}^1,\eta_{r,m}^1),$$

provided that the intersection is not empty. All elements d_m^p are in $B_0\cap B_1$, and their number may change with m (let us say it is $w=w(m)$), but it is finite. Given any $u_m\in B_{(\bigoplus\Delta_m(\overline{A}))_E}$, we can find a sequence $\{d_m^{p_m}\}_{m=-n}^n$, where $p_m\in[1,w(m)]$, such that for $m\in\{-n,\dots,n\}$,

$$\begin{aligned}J(2^{m-\nu},Tu_m-d_m^{p_m})&=\max_{i=0,1}\{2^{i(m-\nu)}\|Tu_m-d_m^{p_m}\|_{B_i}\}\\&=\max_{i=0,1}\{2^{i(m-\nu)}\|Tu_m-(\mu_m^r+\eta)b_{j_i}^i+(\mu_m^r+\eta)b_{j_i}^i-d_m^{p_m}\|_{B_i}\}\\&\le\max_{i=0,1}\{2\eta_{r,m}^0,2^{m-\nu}2\eta_{r,m}^1\}=2\eta_{r,m}^0.\end{aligned}$$

Let D_J be the collection of all sums $\sum_{m=-n}^n d_m^{p_m}$. Notice that D_J is a finite subset of $\overline{B}_{E;J}$ and for any $\{u_m\}\in B_{(\bigoplus\Delta_m(\overline{A}))_E}$ there exists one $\sum_{m=-n}^n d_m^{p_m}\in D_J$ such that

$$\begin{aligned}\Big\|T\pi P_n\{u_m\}-\sum_{m=-n}^n d_m^{p_m}\Big\|_{\overline{B}_{E;J}}&=\Big\|\sum_{m=-n}^n(Tu_m-d_m^{p_m})\Big\|_{\overline{B}_{E;J}}\\&\le\|\{J(2^m,Tu_{m+\nu}-d_{m+\nu}^{p_{m+\nu}})\}\|_{E|_{[-n-\nu,n-\nu]}}\\&\le 2^{1+l_0}\|\{\mu_{m+\nu}^r+\eta\}\|_{E|_{[-n-\nu,n-\nu]}}\le 2^{1+l_0}\|\tau_\nu\|_{E\to E}\|\{\mu_m^r+\eta\}\|_{E|_{[-n,n]}}\\&\le 2^{2+l_0-1}\Big(1+\eta\Big\|\sum_{m=-n}^n e_m\Big\|_E\Big)\|\tau_{l_1-1-l_0+1}\|_{E\to E}.\end{aligned}$$

Since $\varepsilon>0$ and η can be arbitrarily small the required estimate follows. ■

THEOREM 3.3. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let $T : \overline{A} \to \overline{B}$ be a bounded operator and E be a K-non-trivial $\mathbb{Z}$-lattice. Then for any $\varepsilon > 0$ and $n, k_0, k_1 \in \mathbb{N}$,*

$$\beta(R_n jT : \overline{A}_{E;K} \to (\bigoplus \Sigma_m(\overline{B}))_E) \\ \le 4\psi_E(\varepsilon_{k_0}(T : A_0 \to B_0) + \varepsilon, \varepsilon_{k_1}(T : A_1 \to B_1) + \varepsilon).$$

Proof. Fix $\varepsilon > 0$ and $k_0, k_1 \in \mathbb{N}$. Suppose that $2^{l_i - 1} \le \varepsilon_{k_i}(T : A_i \to B_i) + \varepsilon < 2^{l_i}$ for $i = 0, 1$ and let $\nu = l_1 - l_0$. There exists a constant $C = C(n) > 0$ such that $E|_{[-n+\nu, n+\nu]} \overset{C}{\hookrightarrow} \ell_\infty^{2n+1}$. Given any $\eta > 0$, there exists a finite set $\{\mu_1, \ldots, \mu_s\} \subset B_{E|_{[-n+\nu, n+\nu]}}$ such that for any $\lambda \in B_{E|_{[-n+\nu, n+\nu]}}$,

$$\min_{1 \le r \le s} \{\|\lambda - \mu_r\|_{E|_{[-n+\nu, n+\nu]}}\} < \eta / C.$$

For any $a \in B_{\overline{A}_{E;K}}$, we have

$$\|\{K(2^{-n+\nu}, a), \ldots, K(2^{n+\nu}, a)\}\|_{E|_{[-n+\nu, n+\nu]}} \le \|\{K(2^m, a)\}\|_E = \|a\|_{\overline{A}_{E;K}} \le 1.$$

Thus there is $\mu_r = \{\mu_m^r\}$, where $r \le s$, such that

$$K(2^{m+\nu}, a) < \mu_{m+\nu}^r + \eta \quad \text{for any } m \in \{-n, \ldots, n\}.$$

From the definition of the K-functional, there exist decompositions

$$a = a_m^0 + a_m^1, \quad \text{where } a_m^0 \in A_0,\ a_m^1 \in A_1,$$

which yields

$$\|a_m^0\|_{A_0} + 2^{m+\nu}\|a_m^1\|_{A_1} \le \mu_{m+\nu}^r + \eta$$

for $m = -n, \ldots, n$. By definition of $\varepsilon_{k_i}(T : A_i \to B_i)$, there exist finite sets $\{b_1^i, \ldots, b_{k_i}^i\} \subset B_i$, $i = 0, 1$, such that

$$\min_{1 \le j \le k_i} \{\|Ta_m^i - (\mu_{m+\nu}^r + \eta) b_j^i\|_{B_i}\} \le 2^{-i(m+\nu)}(\mu_{m+\nu}^r + \eta) 2^{l_i} = \eta_{r,m}^i.$$

Denote by D_K the collection of all vector-valued sequences $\{d_m^{p_m}\}$ defined by

$$d_m^{p_m} = \begin{cases} (\mu_{m+\nu}^r + \eta)(b_{j_m^0}^0 + b_{j_m^1}^1), & 1 \le r \le s,\ 1 \le j_m^0 \le k_0,\ 1 \le j_m^1 \le k_1 \\ & \text{for } m \in \{-n, \ldots, n\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $p_m = p_m(r, j_m^0, j_m^1)$ depends on r, j_m^0 and j_m^1. Notice that D_K is a finite subset of $(\bigoplus \Sigma_m(\overline{B}))_E$. For any $a \in B_{\overline{A}_{E;K}}$, we can choose a sequence $\{d_m^{p_m}\} \in D_K$ with

$$\|Ta_m^i - (\mu_{m+\nu}^r + \eta) b_{j_m^i}^i\|_{B_i} \le \eta_{r,m}^i \quad \text{for } i = 0, 1.$$

Thus

$$
\begin{aligned}
K(2^m, Ta - (\mu^r_{m+\nu} + \eta)(b^0_{j^0_m} + b^1_{j^1_m}))
&= K(2^m, Ta^0_m - (\mu^r_{m+\nu} + \eta) b^0_{j^0_m} + Ta^1_m - (\mu^r_{m+\nu} + \eta) b^1_{j^1_m})) \\
&\le \eta^0_{r,m} + 2^m \eta^1_{r,m} = 2\eta^0_{r,m}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|R_n jTa - \{d^{p_m}_m\}\|_{(\bigoplus \Sigma_m(\overline{B}))_E} &= \|\{K(2^m, Ta - (\mu^r_{m+\nu} + \eta)(b^0_{j^0_m} + b^1_{j^1_m})\}\|_{E|_{[-n,n]}} \\
&\le 2^{1+l_0} \|\{\mu^r_{m+\nu} + \eta\}\|_{E|_{[-n,n]}} \\
&\le 2^{1+l_0} \|\tau_\nu\|_{E\to E} \|\{\mu^r_m + \eta\}\|_{E|_{[-n+\nu, n+\nu]}} \\
&\le 2^{2+l_0-1} \Big(1 + \eta \Big\| \sum_{m=-n+\nu}^{n+\nu} e_m \Big\|_E \Big) \|\tau_{l_1 - 1 - l_0 + 1}\|_{E\to E}.
\end{aligned}
$$

Since η is arbitrary, the estimate follows. ∎

THEOREM 3.4. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let $T : \overline{A} \to \overline{B}$ be a bounded operator and E be a parameter of the real method such that the Calderón transform Ω is a bounded operator on E. Assume that $\lim_{s\to 0+} \psi_E(s,1) = \lim_{t\to 0+} \psi_E(1,t) = 0$. Then there exists a constant $C = C(E) > 0$ such that for any $\varepsilon > 0$ and $k_0, k_1 \in \mathbb{N}$,*

$$
\beta(T : \overline{A}_E \to \overline{B}_E) \le C\psi_E(\varepsilon_{k_0}(T : A_0 \to B_0) + \varepsilon, \varepsilon_{k_1}(T : A_1 \to B_1) + \varepsilon).
$$

Proof. From the properties of the operators π and j we have

$$
\begin{aligned}
\beta(T : \overline{A}_{E;J} \to \overline{B}_{E;K}) &= \beta(T\pi : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to \overline{B}_{E;K}) \\
&\le 2\beta(jT\pi : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E).
\end{aligned}
$$

We decompose the operator $jT\pi$ as

$$
\begin{aligned}
jT\pi &= (R_n + S^+_n + S^-_n) jT\pi \\
&= R_n jT\pi + (S^+_n + S^-_n) jT\pi (P_n + Q^+_n + Q^-_n) \\
&= R_n jT\pi + (S^+_n + S^-_n) jT\pi P_n + (S^+_n + S^-_n) jT\pi (Q^+_n + Q^-_n)
\end{aligned}
$$

where $n \in \mathbb{N}$. Combining the results of Theorems 3.1 (with constant D), 3.2 and 3.3, we deduce that there exists $N \in \mathbb{N}$ such that for $n > N$,

$$
\begin{aligned}
&\beta(jT\pi : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E) \\
&\quad \le \|\Omega\|_{E\to E} \beta(R_n jT : \overline{A}_{E;K} \to (\textstyle\bigoplus \Sigma_m(\overline{B}))_E) \\
&\qquad + \|\Omega\|_{E\to E} \beta(T\pi P_n : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to \overline{B}_{E;J}) \\
&\qquad + \beta((S^+_n + S^-_n) jT\pi (Q^+_n + Q^-_n) : (\textstyle\bigoplus \Delta_m(\overline{A}))_E \to (\bigoplus \Sigma_m(\overline{B}))_E) \\
&\quad \le (D + 8\|\Omega\|_{E\to E}) \psi_E(\varepsilon_{k_1}(T : A_0 \to B_0) + \varepsilon, \varepsilon_{k_1}(T : A_1 \to B_1) + \varepsilon),
\end{aligned}
$$

which yields the result with a constant $C = C(E) > 0$. ∎

We finish with some applications of the results obtained. The following is essentially a corollary of the properties of the measure of non-compactness and the results obtained above.

COROLLARY 3.5. *Under the assumptions of Theorem* 3.4,

$$\beta(T : \overline{A}_E \to \overline{B}_E) \le C\psi_E(\beta(T : A_0 \to B_0), \beta(T : A_1 \to B_1)).$$

In particular, if $T : A_0 \to B_0$ or $T : A_1 \to B_1$ is compact then $T : \overline{A}_E \to \overline{B}_E$ is also compact.

The compactness statement in Corollary 3.5 has been proved in [5, Thm. 5.4] via a direct proof without estimates of the measure of non-compactness.

COROLLARY 3.6. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$ be an operator. If ϱ is a function parameter, then there exists a constant $C > 0$ such that whenever $\beta(T : A_0 \to B_0) > 0$, we have*

$$\beta(T : \overline{A}_{\varrho,q} \to \overline{B}_{\varrho,q}) \le C\beta(T : A_0 \to B_0) s_\varrho\bigg(\frac{\beta(T : A_1 \to B_1)}{\beta(T : A_0 \to B_0)}\bigg).$$

Proof. It is easy to see that $\|\tau_n\|_{E\to E} \le s_\varrho(2^n)$ for $n \in \mathbb{Z}$, thus Corollary 3.5 applies. ∎

REMARK 3.7. After the completion of this paper we were kindly informed by Fernando Cobos that the above estimate for the measure of non-compactness for interpolation spaces generated by a function parameter was proved independently by J. M. Cordeiro [8].

REMARK 3.8. We note that if $\varrho(t) = t^\theta$ with $\theta \in (0, 1)$, then we recover the logarithmic type estimate for classical real interpolation spaces $(A_0, A_1)_{\theta,q}$ proved in [6]:

$$\beta(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}) \le C\beta(T : A_0 \to B_0)^{1-\theta}\beta(T : A_1 \to B_1)^\theta$$

for some $C > 0$.

REMARK 3.9. Let A be any Banach space and T be any bounded operator on A. It is well known (see, e.g., [4]) that the essential spectral radius of T can be expressed in terms of the measure of non-compactness:

$$r_{\mathrm{ess}}(T) = r_{\mathrm{ess}}(T : A \to A) = \lim_{n\to\infty} \beta(T^n : A \to A)^{1/n}.$$

The inequality

$$\beta(T^{kn} : A \to A) \le \beta(T^k : A \to A)^n \quad \text{for } k, n \in \mathbb{N}$$

allows us to write

$$\lim_{n\to\infty} \beta(T^n : A \to A)^{1/n} = \inf_{1\le n<\infty} \beta(T^n : A \to A)^{1/n} \le \beta(T : A \to A).$$

We conclude the paper with the following result which extends previous results by Edmunds and Teixeira [15], Albrecht [1] and Cobos, Fernández-Martínez and Martínez [6].

COROLLARY 3.10. *Let $\overline{A} = (A_0, A_1)$ be a Banach couple of complex spaces. Assume that $T : \overline{A} \to \overline{A}$ is a bounded operator and E is a parameter of the real method such that the Calderón transform Ω is a bounded operator on E and*

$$\lim_{s\to 0+} \psi_E(s,1) = \lim_{t\to 0+} \psi_E(1,t) = 0.$$

Then

$$r_{\mathrm{ess}}(T : A_E \to A_E) \le C\psi_E(r_{\mathrm{ess}}(T : A_0 \to A_0), r_{\mathrm{ess}}(T : A_1 \to A_1)),$$

with $C = C(E) > 0$.

Proof. Applying Corollary 3.5, we obtain

$$\begin{aligned}\beta(T^n : \overline{A}_E \to \overline{A}_E) &\le C\psi_E(\beta(T^n : A_0 \to A_0), \beta(T^n : A_1 \to A_1)) \\ &= C\psi_E((\beta(T^n : A_0 \to A_0)^{1/n})^n, (\beta(T^n : A_1 \to A_1)^{1/n})^n) \\ &\le CC_1^{n-1}\psi_E(\beta(T^n : A_0 \to A_0)^{1/n}, \beta(T^n : A_1 \to A_1)^{1/n})^n\end{aligned}$$

for any $n \in \mathbb{N}$. Therefore

$$\begin{aligned}&\beta(T^n : \overline{A}_E \to \overline{A}_E)^{1/n} \\ &\qquad\le C^{1/n}C_1^{1-1/n}\psi_E(\beta(T^n : A_0 \to A_0)^{1/n}, \beta(T^n : A_1 \to A_1)^{1/n}),\end{aligned}$$

which yields the result. ∎

Acknowledgments. We thank the referee for his comments.

References

[1] E. Albrecht, *Spectral interpolation*, in Oper. Theory Adv. Appl. 14, Birkhäuser, Basel, 1984, 13–37.

[2] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[3] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, Vol. 1, North-Holland, Amsterdam, 1991.

[4] B. Carl and I. Stephani, *Entropy, Compactness and the Approximation of Operators*, Cambridge Univ. Press, 1990.

[5] F. Cobos, L. M. Fernández-Cabrera and A. Martínez, *Compact operators between K- and J-spaces*, Studia Math. 166 (2005), 199–220.

[6] F. Cobos, P. Fernández-Martínez, and A. Martínez, *Interpolation of the measure of non-compactness by the real method*, ibid. 135 (1999), 24–38.

[7] F. Cobos and J. Peetre, *Interpolation of compactness using Aronszajn–Gagliardo functors*, Israel J. Math. 68 (1989), 220–240.

[8] J. M. Cordeiro, *Interpolación de ciertas clases de operadores por méthods multidimensionales*, Publicatións do Dept. Mathemática Aplicada, Sec. 1, no. 9 (1999), Univesidade de Vigo.

[9] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. 65 (1992), 333–343.

[10] K. Hayakawa, *Interpolation by the real method preserves compactness of operators*, J. Math. Soc. Japan 21 (1969), 189–199.

[11] M. A. Krasnosel'skiĭ, *On the theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229–231.

[12] J. L. Lions et J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5–68.

[13] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. 132 (1982), 291–330.

[14] A. Persson, *Compact linear mappings between interpolation spaces*, Ark. Mat. 5 (1964), 215–219.

[15] M. F. Teixeira and D. E. Edmunds, *Interpolation theory and measure of non-compactness*, Math. Nachr. 104 (1981), 129–135.

Faculty of Mathematics and Computer Science
A. Mickiewicz University
Umultowska 87
61-616 Poznań, Poland
E-mail: szwedek@amu.edu.pl

Received June 16, 2005
Revised version March 8, 2006 (5676)