On unitary equivalence of invariant subspaces of the Dirichlet space

by

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Abstract. It is shown that in the Dirichlet space $D$, two invariant subspaces $M_1$, $M_2$ of the Dirichlet shift $M_z$ are unitarily equivalent only if $M_1 = M_2$.

1. Introduction. Let $\mathbb{D}$ be the open unit disk, and $dA$ the normalized Lebesgue measure on $\mathbb{D}$. The Dirichlet space $D$ consists of the analytic functions $f$ on $\mathbb{D}$ with finite Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

It is easy to verify that $D \subset H^2$, the Hardy space on $\mathbb{D}$.

Endow $D$ with norm $\| \cdot \|$, $\|

\|

f\n
\|^2 = \| f \|^2 + D(f), \quad f \in D,$

where $\| f \|_2$ is the norm of $f$ in $H^2$. Then $D$ is a reproducing kernel function space with reproducing kernel

$$K_\lambda(z) = \frac{1}{\lambda z} \log \frac{1}{1 - \lambda z^2}, \quad \lambda, z \in \mathbb{D}.$$ It is well known that $K_\lambda$ is a complete Nevanlinna–Pick kernel.

Let $M_z$ be the operator of multiplication by $\varphi(z) = z$ on $D$, called the Dirichlet shift. It is an important operator on $D$ which has been extensively studied, and there is a large literature concerning invariant subspaces of $M_z$ ($[5]-[11]$). We refer the readers to the survey paper $[12]$ for more information about the Dirichlet space $D$ and the Dirichlet shift $M_z$.

Let $\text{Lat}(M_z, D, \| \cdot \|)$ be the lattice of invariant subspaces of $M_z$ in $D$. Recall that two invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $M_z$ are unitarily equivalent if there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2} U$.

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In [7], unitary equivalence of invariant subspaces of \( M_z \) was studied, and it was shown that if two invariant subspaces \( \mathcal{M}_1, \mathcal{M}_2 \) satisfy one of the following conditions:

1. \( \mathcal{M}_1 \) contains an outer function,
2. \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \),

then \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are unitarily equivalent if and only if \( \mathcal{M}_1 = \mathcal{M}_2 \) [7, Theorem 2].

In this paper, by using some ideas of [11], we prove the following theorem.

**Theorem 1.1.** Suppose \( \mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(M_z, \mathcal{D}, \|\cdot\|) \) are unitarily equivalent. Then \( \mathcal{M}_1 = \mathcal{M}_2 \).

Also, endow \( \mathcal{D} \) with the equivalent norm \( \|\cdot\|_0 \),

(1.2) \[ \|f\|^2_0 = |f(0)|^2 + D(f), \quad f \in \mathcal{D}. \]

Then the reproducing kernel of \( \mathcal{D} \) is

\[ K^0_\lambda(z) = 1 + \log \frac{1}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D}. \]

With the norm \( \|\cdot\|_0 \) on \( \mathcal{D} \), we will show the following theorem.

**Theorem 1.2.** Let \( \mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(M_z, \mathcal{D}, \|\cdot\|_0) \). Then they are unitarily equivalent if and only if they are equal.

Let \( H^{1,2}(\mathbb{D}) \) be the completion of

\[ \left\{ u \in C^1(\mathbb{D}) : \|u\|_{1,2} = \left( \int_{\mathbb{D}} |u|^2 \, dA + \int_{\mathbb{D}} \left( \left| \frac{\partial u}{\partial z} \right|^2 + \left| \frac{\partial u}{\partial \bar{z}} \right|^2 \right) \, dA \right)^{1/2} < \infty \right\} \]

with respect to the Sobolev norm \( \|\cdot\|_{1,2} \) (see [11]). Then the Dirichlet space \( \mathcal{D} \) is a closed subspace of \( H^{1,2}(\mathbb{D}) \) with the norm \( \|\cdot\|_1 \),

(1.3) \[ \|f\|^2_1 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) + D(f), \quad f \in \mathcal{D}. \]

We have the following theorem.

**Theorem 1.3.** Let \( \mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(M_z, \mathcal{D}, \|\cdot\|_1) \). Then they are unitarily equivalent if and only if they are equal.

2. **Unitary equivalence of invariant subspaces in the norm \( \|\cdot\| \).**

In this section, the norm \( \|\cdot\| \) on \( \mathcal{D} \) is defined as

\[ \|f\|^2 = \|f\|^2_2 + D(f), \quad f \in \mathcal{D}, \]

where \( \|f\|_2 \) is the norm of \( f \) in \( H^2 \), and \( \langle \cdot, \cdot \rangle \) is the corresponding inner product on \( \mathcal{D} \).

First, we fix some notation and cite some results about invariant subspaces of \( M_z \) in \( \mathcal{D} \).
For a set $S \subset \mathcal{D}$, let $[S]$ denote the invariant subspace of the Dirichlet shift $M_z$ generated by $S$. Let $\mathbb{T} = \partial \mathbb{D}$ be the unit circle, and $dm$ the normalized Lebesgue measure on $\mathbb{T}$. We need the following results.

**Theorem 2.1.** Let $\mathcal{M} \in \text{Lat}(M_z, \mathcal{D}, \| \cdot \|)$. Then

(i) [8, Theorem 2(c)] $\mathcal{M} \ominus z\mathcal{M}$ is one-dimensional.

(ii) [5, Theorem 1] $[\mathcal{M} \ominus z\mathcal{M}] = \mathcal{M}$.

**Theorem 2.2.** Let $\mathcal{M} \in \text{Lat}(M_z, \mathcal{D}, \| \cdot \|)$. If $\phi \in \mathcal{M} \ominus z\mathcal{M}$, then

(i) [10, Theorem 3.1] $\phi$ is a multiplier of $\mathcal{D}$.

(ii) [11, Theorem 2.2(a)] $\phi' \in N(\mathcal{D})$, the Nevanlinna class of $\mathcal{D}$.

From Theorem 2.2(ii) and [4, Theorem 5.3], if $\phi \in \mathcal{M} \ominus z\mathcal{M}$, then for a.e. $\xi \in \mathbb{T}$, the nontangential limit of $\phi'$ at $\xi$, $\phi'((\xi))$, exists. For $f \in \mathcal{D}$ and $\xi \in \mathbb{T}$, let $f(\xi)$ be the nontangential limit of $f$ at $\xi$.

The following lemma comes essentially from the proof of [7, Lemma 2], which gives a necessary condition for two invariant subspaces to be unitarily equivalent.

**Lemma 2.3.** Let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Lat}(M_z, \mathcal{D}, \| \cdot \|)$ and $U : \mathcal{M}_1 \to \mathcal{M}_2$ be a unitary operator such that $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}U$. Then for any $f \in \mathcal{M}_1$ and $g = Uf$, $|f(\xi)|^2 = |g(\xi)|^2$, a.e. $\xi \in \mathbb{T}$.

For $f \in \mathcal{D}$ and $\xi \in \mathbb{T}$, define the **local Dirichlet integral** of $f$ at $\xi$ by

$$D_\xi(f) = \int_{\mathbb{T}} \left| \frac{f(\eta) - f(\xi)}{\eta - \xi} \right|^2 dm(\eta),$$

and set $D_\xi(f) = \infty$ if $f(\xi)$ does not exist. See [9] for more information about the local Dirichlet integral.


$$D(f) = \int_{\mathbb{T}} D_\xi(f) dm(\xi),$$

which implies that $D_\xi(f) \in L^1(\mathbb{T})$ whenever $f \in \mathcal{D}$.

For $f, g \in \mathcal{D}$ and $\xi \in \mathbb{T}$ such that both $D_\xi(f)$ and $D_\xi(g)$ are finite, define

$$D_\xi(f, g) = \int_{\mathbb{T}} \frac{(f(\eta) - f(\xi))(g(\eta) - g(\xi))}{|\eta - \xi|^2} dm(\eta),$$

which is called the **mixed local Dirichlet integral** of $f$ and $g$ at $\xi$ [11]. Then $D_\xi(f, g) \in L^1(\mathbb{T})$ and $D_\xi(f) = D_\xi(f, f)$.

The following lemma comes from [11, Lemma 2.1].
Lemma 2.4. Let \( f, g \in \mathcal{D}, \lambda \in \mathbb{D}, \) and \( \alpha, \beta \in \mathbb{C}. \) Then

\[
\langle \frac{1}{1 - \lambda z} f, g \rangle = \int_T \frac{1}{1 - \lambda \xi} (f(\xi) \overline{g(\xi)} + D_\xi(f, g)) \, dm(\xi)
+ \int_T \frac{\lambda \xi}{(1 - \lambda \xi)^2} f(\xi) \overline{g(\xi)} \, dm(\xi)
\]

and

\[
\int_T \frac{\lambda \xi}{(1 - \lambda \xi)^2} f(\xi) \overline{g(\xi)} \, dm(\xi) = \int_T \frac{\lambda \xi}{(1 - \lambda \xi)^2} (f(\xi) - \alpha)(g(\xi) - \beta) \, dm(\xi) + \alpha \lambda g'(\lambda).
\]

Proof of Theorem 1.1. The proof will make use of some ideas of the proof of [11, Lemma 3.1].

Let \( U : \mathcal{M}_1 \to \mathcal{M}_2 \) be a unitary operator such that \( UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2} U. \) Let \( \varphi \in \mathcal{M}_1 \ominus z\mathcal{M}_1 \) and \( \psi = U \varphi. \) Then it is easy to verify that \( \psi \in \mathcal{M}_2 \ominus z\mathcal{M}_2. \) By Lemma 2.3,

\[
|\varphi(\xi)|^2 = |\psi(\xi)|^2, \quad \text{a.e. } \xi \in \mathbb{T}.
\]

Since \( UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2} U, \) for \( \lambda \in \mathbb{D} \) we have

\[
\left\langle \frac{1}{1 - \lambda z} \varphi, \varphi \right\rangle = \left\langle \frac{1}{1 - \lambda z} \psi, \psi \right\rangle.
\]

By (2.1),

\[
\int_T \frac{1}{1 - \lambda \xi} (|\varphi(\xi)|^2 + D_\xi(\varphi)) \, dm(\xi) + \int_T \frac{\lambda \xi}{(1 - \lambda \xi)^2} |\varphi(\xi)|^2 \, dm(\xi)
= \int_T \frac{1}{1 - \lambda \xi} (|\psi(\xi)|^2 + D_\xi(\psi)) \, dm(\xi) + \int_T \frac{\lambda \xi}{(1 - \lambda \xi)^2} |\psi(\xi)|^2 \, dm(\xi).
\]

By (2.3) and (2.4), we obtain

\[
\int_T \frac{1}{1 - \lambda \xi} D_\xi(\varphi) \, dm(\xi) = \int_T \frac{1}{1 - \lambda \xi} D_\xi(\psi) \, dm(\xi).
\]

Taking \( \lambda = 0 \) in (2.5), we have

\[
\int_T D_\xi(\varphi) \, dm(\xi) = \int_T D_\xi(\psi) \, dm(\xi).
\]

Since the Poisson kernel is

\[
P_\lambda(\xi) = \frac{1 - |\lambda|^2}{|1 - \lambda \xi|} = 2 \text{Re} \left( \frac{1}{1 - \lambda \xi} \right) - 1,
\]
by (2.5) and (2.6) we get
\[ \int_{\mathbb{T}} P_\lambda(\xi) D_\xi(\varphi) \, dm(\xi) = \int_{\mathbb{T}} P_\lambda(\xi) D_\xi(\psi) \, dm(\xi). \]

Since \( D_\xi(\varphi), D_\xi(\psi) \in L^1(\mathbb{T}) \), it follows that
\[ (2.7) \quad D_\xi(\varphi) = D_\xi(\psi), \quad \text{a.e. } \xi \in \mathbb{T}. \]

On the other hand, by (2.3), we have
\[ \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\varphi(\xi)|^2 \, dm(\xi) = \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\psi(\xi)|^2 \, dm(\xi). \]

Then by (2.2), for \( \alpha, \beta \in \mathbb{C} \),
\[ \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\varphi(\xi) - \alpha|^2 \, dm(\xi) + \alpha \bar{\lambda} \varphi'(\lambda) \]
\[ = \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\psi(\xi) - \beta|^2 \, dm(\xi) + \beta \bar{\lambda} \psi'(\lambda). \]

In particular, for \( \eta \in \mathbb{T} \) such that \( \varphi(\eta), \psi(\eta), \varphi'(\eta), \psi'(\eta), D_\eta(\varphi) \) and \( D_\eta(\psi) \)
are finite, taking \( \alpha = \varphi(\eta) \) and \( \beta = \psi(\eta) \), we have
\[ (2.8) \quad \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\xi - \eta|^2 |\varphi(\xi) - \varphi(\eta)|^2 \, dm(\xi) + \varphi(\eta) \bar{\lambda} \varphi'(\lambda) \]
\[ = \int_{\mathbb{T}} \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} |\xi - \eta|^2 |\psi(\xi) - \psi(\eta)|^2 \, dm(\xi) + \psi(\eta) \bar{\lambda} \psi'(\lambda). \]

Since
\[ \frac{\bar{\lambda}}{(1 - \lambda \xi)^2} \rightarrow -1 \quad \text{as } \lambda \rightarrow \eta, \]

letting \( \lambda \rightarrow \eta \) nontangentially, it follows from (2.8) and the dominated convergence theorem that
\[ -D_\eta(\varphi) + \varphi(\eta) \bar{\eta} \varphi'(\eta) = -D_\eta(\psi) + \psi(\eta) \bar{\eta} \psi'(\eta). \]

Combining this with (2.7), we have
\[ (2.9) \quad \varphi(\eta) \bar{\eta} \varphi'(\eta) = \psi(\eta) \bar{\eta} \psi'(\eta), \quad \text{a.e. } \eta \in \mathbb{T}. \]

From (2.3) and (2.9), we get
\[ (2.10) \quad \frac{\varphi'(\xi)}{\varphi(\xi)} = \frac{\psi'(\xi)}{\psi(\xi)}, \quad \text{a.e. } \xi \in \mathbb{T}. \]

By Theorem 2.2(ii), both \( \varphi' \) and \( \psi' \) are in \( N(\mathbb{D}) \), and hence combining (2.10) and a simple reasoning shows that
\[ \frac{\varphi'(z)}{\varphi(z)} = \frac{\psi'(z)}{\psi(z)}, \quad z \in \mathbb{D}. \]
This implies that $\varphi = c\psi$ for some constant $c$. By Theorem 2.1, $\mathcal{M}_1 = \mathcal{M}_2$ as desired.

3. Unitary equivalence of invariant subspaces in the norm $\| \cdot \|_0$. In this section, we consider the Dirichlet space $\mathcal{D}$ in the norm $\| \cdot \|_0$, 
\[ \|f\|_0^2 = |f(0)|^2 + D(f), \quad f \in \mathcal{D}. \]
We use $\langle \cdot, \cdot \rangle_0$ to denote the corresponding inner product.

We begin with a discussion of Carleson’s formula. Let $f = BSF$ be the canonical factorization of $f$, where $B = \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - a_j \bar{z}}$ is a Blaschke product, $S$ is the singular part of $f$, and $\mu$ the corresponding singular measure, and $F$ is the outer part of $f$. Then Carleson’s formula [2] (see also [9, Corollary 3.6]) is
\[ D(f) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} P_{\alpha_n}(e^{is}) |f(e^{is})|^2 \, ds \]
\[ + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{2}{|e^{it} - e^{is}|^2} \, d\mu(t) |f(e^{is})|^2 \, ds \]
\[ + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \frac{(e^{2u(e^{it})} - e^{2u(e^{is})})(u(e^{it}) - u(e^{is}))}{|e^{it} - e^{is}|^2} \, dt \, ds \]
for $f \in \mathcal{D}$, where $u(e^{it}) = \log |f(e^{it})|$, and $P_{\alpha}(e^{it})$ is the Poisson kernel.

By Carleson’s formula, it is easy to verify that
\[ \|zf\|_0^2 = D(zf) = \|f\|_0^2 + D(f) = \|f\|^2, \quad f \in \mathcal{D}. \]

Proof of Theorem 1.2. By Theorem 1.1, it suffices to prove that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are unitarily equivalent in the norm $\| \cdot \|_0$, then they are unitarily equivalent in the norm $\| \cdot \|$.

Let $V : \mathcal{M}_1 \to \mathcal{M}_2$ be a unitary operator such that $VM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}V$. For $f \in \mathcal{M}_1$, let $g = Vf$. It is enough to prove that
\[ \|f\| = \|g\|. \]
Since $VM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}V$ and $V$ is unitary, we have $V(zf) = zg$ and
\[ \|zf\|_0 = \|zg\|_0. \]
By (3.1), the proof is complete.

4. Unitary equivalence of invariant subspaces in the norm $\| \cdot \|_1$. In this section, we consider the Dirichlet space $\mathcal{D}$ with the norm $\| \cdot \|_1$, 
\[ \|f\|_1^2 = \int_{\mathcal{D}} |f|^2 \, dA + D(f), \quad f \in \mathcal{D}. \]
Let $\langle \cdot, \cdot \rangle_1$ be the corresponding inner product. A direct computation shows that for nonnegative integers $n$, $m$ with $n \leq m$, and $f(z) = \sum_{l=0}^{\infty} a_l z^l \in \mathcal{D}$,

\begin{equation}
\langle z^n f, z^m f \rangle_1 = \sum_{k=m}^{\infty} a_{k-n} \overline{a}_{k-m} (z^k, z^k)_1 = \sum_{k=m}^{\infty} a_{k-n} \overline{a}_{k-m} \left( \frac{1}{k+1} + k \right).
\end{equation}

**Proof of Theorem 1.3.** Let $f(z) = \sum_{l=0}^{\infty} a_l z^l$ be in $\mathcal{D}$ and $n$, $m$ be nonnegative integers with $n \leq m$. Then by (4.1),

\begin{equation}
\langle z^n f, z^m f \rangle_1 = \sum_{k=m}^{\infty} a_{k-n} \overline{a}_{k-m} \left( \frac{1}{k+1} + k \right),
\end{equation}

\begin{equation}
\langle z^{n+1} f, z^{m+1} f \rangle_1 = \sum_{k=m+1}^{\infty} a_{k-(n+1)} \overline{a}_{k-(m+1)} \left( \frac{1}{k+1} + k \right)
= \sum_{k=m}^{\infty} a_{k-n} \overline{a}_{k-m} \left( \frac{1}{k+2} + k + 1 \right),
\end{equation}

and similarly,

\begin{equation}
\langle z^{n+2} f, z^{m+2} f \rangle_1 = \sum_{k=m}^{\infty} a_{k-n} \overline{a}_{k-m} \left( \frac{1}{k+3} + k + 2 \right).
\end{equation}

Taking (4.2) $- 2 \times (4.3) + (4.4)$ yields

\begin{equation}
\langle z^n f, z^m f \rangle_1 - 2 \langle z^{n+1} f, z^{m+1} f \rangle_1 + \langle z^{n+2} f, z^{m+2} f \rangle_1 = \sum_{k=m}^{\infty} \frac{2a_{k-n} \overline{a}_{k-m}}{(k+1)(k+2)(k+3)}.
\end{equation}

Let $U : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a unitary operator such that $UM_2|_{\mathcal{M}_1} = M_2|_{\mathcal{M}_2} U$. Suppose $f(z) = \sum_{l=0}^{\infty} a_l z^l$ in $\mathcal{M}_1$ and $g(z) = \sum_{l=0}^{\infty} b_l z^l$ in $\mathcal{M}_2$ are such that $Uf = g$. Then by (4.5), we have

\begin{equation}
\sum_{k=m}^{\infty} \frac{a_{k-n} \overline{a}_{k-m}}{(k+1)(k+2)(k+3)} = \sum_{k=m}^{\infty} \frac{b_{k-n} \overline{b}_{k-m}}{(k+1)(k+2)(k+3)}.
\end{equation}

It is obvious that $\mathcal{D} \subset L^2_{\omega}((1 - |z|^2)^2 dA(z))$, the weighted Bergman space with the orthonormal basis $\{(k+1)(k+2)(k+3) / 2 \}^{\infty}_{k=0}$, and a routine calculation shows that for $h(z) = \sum_{l=0}^{\infty} c_l z^l$ in $L^2_{\omega}((1 - |z|^2)^2 dA(z))$,

\begin{equation}
\langle z^n h, z^m h \rangle L^2_{\omega}((1 - |z|^2)^2 dA(z)) = \sum_{k=m}^{\infty} \frac{2c_{k-n} \overline{c}_{k-m}}{(k+1)(k+2)(k+3)}.
\end{equation}

By (4.6) and (4.7), for $f \in \mathcal{M}_1$ and $g \in \mathcal{M}_2$ with $Uf = g$,

\begin{equation}
\langle z^n f, z^m f \rangle L^2_{\omega}((1 - |z|^2)^2 dA(z)) = \langle z^n g, z^m g \rangle L^2_{\omega}((1 - |z|^2)^2 dA(z)).
\end{equation}
This means that \([f]\) and \([g]\), the invariant subspaces of \(L^2_a((1-|z|^2)^2dA(z))\) generated by \(f\) and \(g\), respectively, are unitarily equivalent. Therefore by Theorem 1 in [7], \(g = cf\) for some constant \(c\). This implies that \(M_1 = M_2\), completing the proof.

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