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On unitary equivalence of invariant subspaces of the Dirichlet space

by

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Abstract. It is shown that in the Dirichlet space $\mathcal{D}$, two invariant subspaces $\mathcal{M}_1$, $\mathcal{M}_2$ of the Dirichlet shift M_z are unitarily equivalent only if $\mathcal{M}_1 = \mathcal{M}_2$.

1. Introduction. Let $\mathbb{D}$ be the open unit disk, and dA the normalized Lebesgue measure on $\mathbb{D}$. The *Dirichlet space* $\mathcal{D}$ consists of the analytic functions f on $\mathbb{D}$ with finite Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

It is easy to verify that $\mathcal{D} \subset H^2$, the Hardy space on $\mathbb{D}$.

Endow $\mathcal{D}$ with norm $\|\cdot\|$,

$$\|f\|^2 = \|f\|_2^2 + D(f), \qquad f \in \mathcal{D}, \tag{1.1}$$

where $\|f\|_2$ is the norm of f in H^2. Then $\mathcal{D}$ is a reproducing kernel function space with reproducing kernel

$$K_\lambda(z) = \frac{1}{\bar{\lambda} z} \log \frac{1}{1-\bar{\lambda} z}, \qquad \lambda, z \in \mathbb{D}.$$

It is well known that K_λ is a complete Nevanlinna–Pick kernel.

Let M_z be the operator of multiplication by $\varphi(z) = z$ on $\mathcal{D}$, called the *Dirichlet shift*. It is an important operator on $\mathcal{D}$ which has been extensively studied, and there is a large literature concerning invariant subspaces of M_z ([5], [6], [7], [8]–[11]). We refer the readers to the survey paper [12] for more information about the Dirichlet space $\mathcal{D}$ and the Dirichlet shift M_z.

Let $\mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|)$ be the lattice of invariant subspaces of M_z in $\mathcal{D}$. Recall that two invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of M_z are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2} U$.

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In [7], unitary equivalence of invariant subspaces of M_z was studied, and it was shown that if two invariant subspaces $\mathcal{M}_1$, $\mathcal{M}_2$ satisfy one of the following conditions:

1. $\mathcal{M}_1$ contains an outer function,
2. $\mathcal{M}_1 \subseteq \mathcal{M}_2$,

then $\mathcal{M}_1$ and $\mathcal{M}_2$ are unitarily equivalent if and only if $\mathcal{M}_1 = \mathcal{M}_2$ [7, Theorem 2].

In this paper, by using some ideas of [11], we prove the following theorem.

THEOREM 1.1. *Suppose $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|)$ are unitarily equivalent. Then $\mathcal{M}_1 = \mathcal{M}_2$.*

Also, endow $\mathcal{D}$ with the equivalent norm $\|\cdot\|_0$,

$$\|f\|_0^2 = |f(0)|^2 + D(f), \qquad f \in \mathcal{D}. \tag{1.2}$$

Then the reproducing kernel of $\mathcal{D}$ is

$$K_\lambda^0(z) = 1 + \log \frac{1}{1 - \bar{\lambda} z}, \qquad \lambda, z \in \mathbb{D}.$$

With the norm $\|\cdot\|_0$ on $\mathcal{D}$, we will show the following theorem.

THEOREM 1.2. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|_0)$. Then they are unitarily equivalent if and only if they are equal.*

Let $H^{1,2}(\mathbb{D})$ be the completion of

$$\left\{ u \in C^1(\mathbb{D}) : \|u\|_{1,2} = \left(\int_{\mathbb{D}} |u|^2 \, dA + \int_{\mathbb{D}} \left(\left| \frac{\partial u}{\partial z} \right|^2 + \left| \frac{\partial u}{\partial \bar{z}} \right|^2 \right) dA \right)^{1/2} < \infty \right\}$$

with respect to the Sobolev norm $\|\cdot\|_{1,2}$ (see [1]). Then the Dirichlet space $\mathcal{D}$ is a closed subspace of $H^{1,2}(\mathbb{D})$ with the norm $\|\cdot\|_1$,

$$\|f\|_1^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) + D(f), \qquad f \in \mathcal{D}. \tag{1.3}$$

We have the following theorem.

THEOREM 1.3. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|_1)$. Then they are unitarily equivalent if and only if they are equal.*

2. Unitary equivalence of invariant subspaces in the norm $\|\cdot\|$.

In this section, the norm $\|\cdot\|$ on $\mathcal{D}$ is defined as

$$\|f\|^2 = \|f\|_2^2 + D(f), \qquad f \in \mathcal{D},$$

where $\|f\|_2$ is the norm of f in H^2, and $\langle \cdot, \cdot \rangle$ is the corresponding inner product on $\mathcal{D}$.

First, we fix some notation and cite some results about invariant subspaces of M_z in $\mathcal{D}$.

For a set $S \subset \mathcal{D}$, let $[S]$ denote the invariant subspace of the Dirichlet shift M_z generated by S. Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle, and dm the normalized Lebesgue measure on $\mathbb{T}$. We need the following results.

THEOREM 2.1. *Let* $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|)$. *Then*

(i) [8, Theorem 2(c)] $\mathcal{M} \ominus z\mathcal{M}$ *is one-dimensional.*
(ii) [5, Theorem 1] $[\mathcal{M} \ominus z\mathcal{M}] = \mathcal{M}$.

THEOREM 2.2. *Let* $\mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|)$. *If* $\phi \in \mathcal{M} \ominus z\mathcal{M}$, *then*

(i) [10, Theorem 3.1] ϕ *is a multiplier of* $\mathcal{D}$.
(ii) [11, Theorem 2.2(a)] $\phi' \in N(\mathbb{D})$, *the Nevanlinna class of* $\mathbb{D}$.

From Theorem 2.2(ii) and [4, Theorem 5.3], if $\phi \in \mathcal{M} \ominus z\mathcal{M}$, then for a.e. $\xi \in \mathbb{T}$, the nontangential limit of ϕ' at ξ, $\phi'(\xi)$, exists. For $f \in \mathcal{D}$ and $\xi \in \mathbb{T}$, let $f(\xi)$ be the nontangential limit of f at ξ.

The following lemma comes essentially from the proof of [7, Lemma 2], which gives a necessary condition for two invariant subspaces to be unitarily equivalent.

LEMMA 2.3. *Let* $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Lat}(M_z, \mathcal{D}, \|\cdot\|)$ *and* $U : \mathcal{M}_1 \to \mathcal{M}_2$ *be a unitary operator such that* $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}U$. *Then for any* $f \in \mathcal{M}_1$ *and* $g = Uf$, $|f(\xi)|^2 = |g(\xi)|^2$, *a.e.* $\xi \in \mathbb{T}$.

For $f \in \mathcal{D}$ and $\xi \in \mathbb{T}$, define the *local Dirichlet integral* of f at ξ by

$$D_\xi(f) = \int_{\mathbb{T}} \left| \frac{f(\eta) - f(\xi)}{\eta - \xi} \right|^2 dm(\eta),$$

and set $D_\xi(f) = \infty$ if $f(\xi)$ does not exist. See [9] for more information about the local Dirichlet integral.

A formula of J. Douglas [3] for the Dirichlet integral shows that

$$D(f) = \int_{\mathbb{T}} D_\xi(f)\, dm(\xi),$$

which implies that $D_\xi(f) \in L^1(\mathbb{T})$ whenever $f \in \mathcal{D}$.

For $f, g \in \mathcal{D}$ and $\xi \in \mathbb{T}$ such that both $D_\xi(f)$ and $D_\xi(g)$ are finite, define

$$D_\xi(f, g) = \int_{\mathbb{T}} \frac{(f(\eta) - f(\xi))\,\overline{(g(\eta) - g(\xi))}}{|\eta - \xi|^2}\, dm(\eta),$$

which is called the *mixed local Dirichlet integral* of f and g at ξ [11]. Then $D_\xi(f, g) \in L^1(\mathbb{T})$ and $D_\xi(f) = D_\xi(f, f)$.

The following lemma comes from [11, Lemma 2.1].

Lemma 2.4. *Let $f,g\in\mathcal{D}$, $\lambda\in\mathbb{D}$, and $\alpha,\beta\in\mathbb{C}$. Then*

$$\left\langle \frac{1}{1-\bar\lambda z} f, g\right\rangle = \int_{\mathbb{T}} \frac{1}{1-\bar\lambda\xi}\,(f(\xi)\,\overline{g(\xi)} + D_\xi(f,g))\,dm(\xi) + \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, f(\xi)\,\overline{g(\xi)}\,dm(\xi) \tag{2.1}$$

and

$$\int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, f(\xi)\,\overline{g(\xi)}\,dm(\xi) = \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\,(f(\xi)-\alpha)\,\overline{(g(\xi)-\beta)}\,dm(\xi) + \alpha\,\overline{\lambda\, g'(\lambda)}. \tag{2.2}$$

Proof of Theorem 1.1. The proof will make use of some ideas of the proof of [11, Lemma 3.1].

Let $U:\mathcal{M}_1\to\mathcal{M}_2$ be a unitary operator such that $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}U$. Let $\varphi\in\mathcal{M}_1\ominus z\mathcal{M}_1$ and $\psi = U\varphi$. Then it is easy to verify that $\psi\in\mathcal{M}_2\ominus z\mathcal{M}_2$. By Lemma 2.3,

$$|\varphi(\xi)|^2 = |\psi(\xi)|^2, \qquad \text{a.e. } \xi\in\mathbb{T}. \tag{2.3}$$

Since $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}U$, for $\lambda\in\mathbb{D}$ we have

$$\left\langle \frac{1}{1-\bar\lambda z}\,\varphi,\varphi\right\rangle = \left\langle \frac{1}{1-\bar\lambda z}\,\psi,\psi\right\rangle.$$

By (2.1),

$$\begin{aligned}\int_{\mathbb{T}} \frac{1}{1-\bar\lambda\xi}\,(|\varphi(\xi)|^2 + D_\xi(\varphi))\,dm(\xi) + \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\,|\varphi(\xi)|^2\,dm(\xi)\\ = \int_{\mathbb{T}} \frac{1}{1-\bar\lambda\xi}\,(|\psi(\xi)|^2 + D_\xi(\psi))\,dm(\xi) + \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\,|\psi(\xi)|^2\,dm(\xi).\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we obtain

$$\int_{\mathbb{T}} \frac{1}{1-\bar\lambda\xi}\,D_\xi(\varphi)\,dm(\xi) = \int_{\mathbb{T}} \frac{1}{1-\bar\lambda\xi}\,D_\xi(\psi)\,dm(\xi). \tag{2.5}$$

Taking $\lambda=0$ in (2.5), we have

$$\int_{\mathbb{T}} D_\xi(\varphi)\,dm(\xi) = \int_{\mathbb{T}} D_\xi(\psi)\,dm(\xi). \tag{2.6}$$

Since the Poisson kernel is

$$P_\lambda(\xi) = \frac{1-|\lambda|^2}{|1-\bar\lambda\xi|} = 2\,\mathrm{Re}\left(\frac{1}{1-\bar\lambda\xi}\right) - 1,$$

by (2.5) and (2.6) we get

$$\int_{\mathbb{T}} P_\lambda(\xi) D_\xi(\varphi)\, dm(\xi) = \int_{\mathbb{T}} P_\lambda(\xi) D_\xi(\psi)\, dm(\xi).$$

Since $D_\xi(\varphi), D_\xi(\psi) \in L^1(\mathbb{T})$, it follows that

$$D_\xi(\varphi) = D_\xi(\psi), \qquad \text{a.e. } \xi \in \mathbb{T}. \tag{2.7}$$

On the other hand, by (2.3), we have

$$\int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, |\varphi(\xi)|^2\, dm(\xi) = \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, |\psi(\xi)|^2\, dm(\xi).$$

Then by (2.2), for $\alpha, \beta \in \mathbb{C}$,

$$\int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, |\varphi(\xi)-\alpha|^2\, dm(\xi) + \alpha\, \overline{\lambda\varphi'(\lambda)} = \int_{\mathbb{T}} \frac{\bar\lambda\xi}{(1-\bar\lambda\xi)^2}\, |\psi(\xi)-\beta|^2\, dm(\xi) + \beta\, \overline{\lambda\psi'(\lambda)}.$$

In particular, for $\eta \in \mathbb{T}$ such that $\varphi(\eta)$, $\psi(\eta)$, $\varphi'(\eta)$, $\psi'(\eta)$, $D_\eta(\varphi)$ and $D_\eta(\psi)$ are finite, taking $\alpha = \varphi(\eta)$ and $\beta = \psi(\eta)$, we have

$$\int_{\mathbb{T}} \frac{\bar\lambda\xi|\xi-\eta|^2}{(1-\bar\lambda\xi)^2}\, \frac{|\varphi(\xi)-\varphi(\eta)|^2}{|\xi-\eta|^2}\, dm(\xi) + \varphi(\eta)\, \overline{\lambda\varphi'(\lambda)} = \int_{\mathbb{T}} \frac{\bar\lambda\xi|\xi-\eta|^2}{(1-\bar\lambda\xi)^2}\, \frac{|\psi(\xi)-\psi(\eta)|^2}{|\xi-\eta|^2}\, dm(\xi) + \psi(\eta)\, \overline{\lambda\psi'(\lambda)}. \tag{2.8}$$

Since

$$\frac{\bar\lambda\xi|\xi-\eta|^2}{(1-\bar\lambda\xi)^2} \to -1 \qquad \text{as } \lambda \to \eta,$$

letting $\lambda \to \eta$ nontangentially, it follows from (2.8) and the dominated convergence theorem that

$$-D_\eta(\varphi) + \varphi(\eta)\, \overline{\eta\varphi'(\eta)} = -D_\eta(\psi) + \psi(\eta)\, \overline{\eta\psi'(\eta)}.$$

Combining this with (2.7), we have

$$\varphi(\eta)\, \overline{\eta\varphi'(\eta)} = \psi(\eta)\, \overline{\eta\psi'(\eta)}, \qquad \text{a.e. } \eta \in \mathbb{T}. \tag{2.9}$$

From (2.3) and (2.9), we get

$$\frac{\varphi'(\xi)}{\varphi(\xi)} = \frac{\psi'(\xi)}{\psi(\xi)}, \qquad \text{a.e. } \xi \in \mathbb{T}. \tag{2.10}$$

By Theorem 2.2(ii), both φ' and ψ' are in $N(\mathbb{D})$, and hence combining (2.10) and a simple reasoning shows that

$$\frac{\varphi'(z)}{\varphi(z)} = \frac{\psi'(z)}{\psi(z)}, \qquad z \in \mathbb{D}.$$

This implies that $\varphi = c\psi$ for some constant c. By Theorem 2.1, $\mathcal{M}_1 = \mathcal{M}_2$ as desired.

3. Unitary equivalence of invariant subspaces in the norm $\|\cdot\|_0$. In this section, we consider the Dirichlet space $\mathcal{D}$ in the norm $\|\cdot\|_0$,

$$\|f\|_0^2 = |f(0)|^2 + D(f), \qquad f \in \mathcal{D}.$$

We use $\langle\, ,\rangle_0$ to denote the corresponding inner product.

We begin with a discussion of Carleson's formula. Let $f = BSF$ be the canonical factorization of f, where $B = \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|}\frac{a_j - z}{1-\bar{a}_j z}$ is a Blaschke product, S is the singular part of f, and μ the corresponding singular measure, and F is the outer part of f. Then Carleson's formula [2] (see also [9, Corollary 3.6]) is

$$\begin{aligned} D(f) = {} & \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=1}^{\infty} P_{\alpha_n}(e^{is})|f(e^{is})|^2\, ds \\ & + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{2}{|e^{it}-e^{is}|^2}\, d\mu(t)\, |f(e^{is})|^2\, ds \\ & + \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{(e^{2u(e^{it})} - e^{2u(e^{is})})(u(e^{it}) - u(e^{is}))}{|e^{it}-e^{is}|^2}\, dt\, ds \end{aligned}$$

for $f \in \mathcal{D}$, where $u(e^{it}) = \log|f(e^{it})|$, and $P_\alpha(e^{it})$ is the Poisson kernel.

By Carleson's formula, it is easy to verify that

$$\|zf\|_0^2 = D(zf) = \|f\|_2^2 + D(f) = \|f\|^2, \qquad f \in \mathcal{D}. \tag{3.1}$$

Proof of Theorem 1.2. By Theorem 1.1, it suffices to prove that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are unitarily equivalent in the norm $\|\cdot\|_0$, then they are unitarily equivalent in the norm $\|\cdot\|$.

Let $V : \mathcal{M}_1 \to \mathcal{M}_2$ be a unitary operator such that $VM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}V$. For $f \in \mathcal{M}_1$, let $g = Vf$. It is enough to prove that

$$\|f\| = \|g\|.$$

Since $VM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}V$ and V is unitary, we have $V(zf) = zg$ and

$$\|zf\|_0 = \|zg\|_0.$$

By (3.1), the proof is complete.

4. Unitary equivalence of invariant subspaces in the norm $\|\cdot\|_1$. In this section, we consider the Dirichlet space $\mathcal{D}$ with the norm $\|\cdot\|_1$,

$$\|f\|_1^2 = \int_{\mathbb{D}} |f|^2\, dA + D(f), \qquad f \in \mathcal{D}.$$

Let $\langle\cdot,\cdot\rangle_1$ be the corresponding inner product. A direct computation shows that for nonnegative integers n, m with $n \le m$, and $f(z) = \sum_{l=0}^{\infty} a_l z^l \in \mathcal{D}$,

$$\langle z^n f, z^m f\rangle_1 = \sum_{k=m}^{\infty} a_{k-n}\bar{a}_{k-m}\langle z^k, z^k\rangle_1 = \sum_{k=m}^{\infty} a_{k-n}\bar{a}_{k-m}\left(\frac{1}{k+1}+k\right). \tag{4.1}$$

Proof of Theorem 1.3. Let $f(z) = \sum_{l=0}^{\infty} a_l z^l$ be in $\mathcal{D}$ and n, m be nonnegative integers with $n \le m$. Then by (4.1),

$$\langle z^n f, z^m f\rangle_1 = \sum_{k=m}^{\infty} a_{k-n}\bar{a}_{k-m}\left(\frac{1}{k+1}+k\right), \tag{4.2}$$

$$\begin{aligned}\langle z^{n+1} f, z^{m+1} f\rangle_1 &= \sum_{k=m+1}^{\infty} a_{k-(n+1)}\bar{a}_{k-(m+1)}\left(\frac{1}{k+1}+k\right)\\ &= \sum_{k=m}^{\infty} a_{k-n}\bar{a}_{k-m}\left(\frac{1}{k+2}+k+1\right),\end{aligned} \tag{4.3}$$

and similarly,

$$\langle z^{n+2} f, z^{m+2} f\rangle_1 = \sum_{k=m}^{\infty} a_{k-n}\bar{a}_{k-m}\left(\frac{1}{k+3}+k+2\right). \tag{4.4}$$

Taking (4.2) – 2 × (4.3) + (4.4) yields

$$\begin{aligned}&\langle z^n f, z^m f\rangle_1 - 2\langle z^{n+1} f, z^{m+1} f\rangle_1 + \langle z^{n+2} f, z^{m+2} f\rangle_1\\ &\qquad = \sum_{k=m}^{\infty} \frac{2a_{k-n}\bar{a}_{k-m}}{(k+1)(k+2)(k+3)}.\end{aligned} \tag{4.5}$$

Let $U : \mathcal{M}_1 \to \mathcal{M}_2$ be a unitary operator such that $UM_z|_{\mathcal{M}_1} = M_z|_{\mathcal{M}_2}U$. Suppose $f(z) = \sum_{l=0}^{\infty} a_l z^l$ in $\mathcal{M}_1$ and $g(z) = \sum_{l=0}^{\infty} b_l z^l$ in $\mathcal{M}_2$ are such that $Uf = g$. Then by (4.5), we have

$$\sum_{k=m}^{\infty} \frac{a_{k-n}\bar{a}_{k-m}}{(k+1)(k+2)(k+3)} = \sum_{k=m}^{\infty} \frac{b_{k-n}\bar{b}_{k-m}}{(k+1)(k+2)(k+3)}. \tag{4.6}$$

It is obvious that $\mathcal{D} \subset L^2_a((1-|z|^2)^2 dA(z))$, the weighted Bergman space with the orthonormal basis $\{((k+1)(k+2)(k+3)/2)^{1/2} z^k\}_{k=0}^{\infty}$, and a routine calculation shows that for $h(z) = \sum_{l=0}^{\infty} c_l z^l$ in $L^2_a((1-|z|^2)^2 dA(z))$,

$$\langle z^n h, z^m h\rangle_{L^2_a((1-|z|^2)^2 dA(z))} = \sum_{k=m}^{\infty} \frac{2c_{k-n}\bar{c}_{k-m}}{(k+1)(k+2)(k+3)}. \tag{4.7}$$

By (4.6) and (4.7), for $f \in \mathcal{M}_1$ and $g \in \mathcal{M}_2$ with $Uf = g$,

$$\langle z^n f, z^m f\rangle_{L^2_a((1-|z|^2)^2 dA(z))} = \langle z^n g, z^m g\rangle_{L^2_a((1-|z|^2)^2 dA(z))}.$$

This means that $[f]$ and $[g]$, the invariant subspaces of $L_a^2((1-|z|^2)^2 dA(z))$ generated by f and g, respectively, are unitarily equivalent. Therefore by Theorem 1 in [7], $g = cf$ for some constant c. This implies that $\mathcal{M}_1 = \mathcal{M}_2$, completing the proof.

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