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Characterizations of amenable representations of locally compact groups

by

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Abstract. Let G be a locally compact group and let π be a unitary representation. We study amenability and H-amenability of π in terms of the weak closure of $(\pi\otimes\pi)(G)$ and factorization properties of associated coefficient subspaces (or subalgebras) in $B(G)$. By applying these results, we obtain some new characterizations of amenable groups.

1. Introduction. Let G be a locally compact group and let π be a unitary representation of G. Denote by λ_2 the left regular representation of G. Bekka and Stokke defined the notion of amenability and H-amenability of π in [Bek] and [Sto], respectively. It was shown in Bekka's paper that G is amenable if and only if λ_2 is amenable. A related concept of s-amenability can be found in [C-L]. The purpose of this paper is to characterize (H-)amenability of π in two different ways.

Firstly, we characterize amenability of π using the weak closure of $(\pi\otimes\bar{\pi})(G)$ in $VN_{\pi\otimes\bar{\pi}}(G)$. Chou and Xu [C-X] showed that G is amenable if and only if 0 is not in the weak closure of $\lambda_2(G)$. In the first part of our paper, we show that this result also holds for a general representation π.

Secondly, we will characterize amenability of π using factorization properties of $A(\pi)$, the closed subalgebra generated by the set of matrix coefficients of π.

Existence of bounded approximate identities in $L^1(G)$ is an important property of $L^1(G)$. In particular, it allows us to prove a lot of results by applying Cohen's factorization theorem on $L^1(G)$ or Banach $L^1(G)$-modules. The Fourier algebra $A(G)$ is known to be the non-commutative analogue of $L^1(G)$. It is natural to ask when $A(G)$ also admits a bounded approximate identity. Leptin [Lep2] showed that this happens precisely when G is

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amenable. Thus, a lot of results which hold for $L^1(G)$ have non-commutative analogues for $A(G)$ whenever G is amenable. Surprisingly, these results or conditions for $A(G)$ actually characterize the amenability of G, or equivalently the amenability of λ_2 (see [K-L]). Since $A(G) = A(\lambda_2)$, we would like to ask if these characterizations of the amenability of λ_2 can be generalized to any unitary representation.

In the second part of this paper, we will concentrate on the study of the relationship between the existence of bounded approximate identities or factorization properties of $A(\pi \otimes \bar{\pi})$ and the amenability of π. As an application of our results, we are able to give some new characterizations of amenable groups.

This paper is organized as follows. In Section 3, we prove some basic properties that characterize the (H-)amenability of π which will be useful in the sequel. In Section 4, we prove one of the major results in this paper, namely that π is amenable if and only if 0 is not in the weak closure of $(\pi \otimes \bar{\pi})(G)$ in $VN_{\pi\otimes\bar{\pi}}(G)$. In Section 5, we prove that the (weak) factorization property of $A(\pi \otimes \bar{\pi})$ is a sufficient condition for π being amenable. As a result, we characterize the amenability of π via the (weak) factorization property of $B(\pi \otimes \bar{\pi})$. By taking $\pi = \lambda_2$, we obtain a characterization of the amenability of G using the (weak) factorization property of $B_r(G)$, the reduced Fourier–Stieltjes algebra of G. In Section 6, we prove that π is amenable if and only if $A(G)$, as a $A(\pi \otimes \bar{\pi})$- (or $B(\pi \otimes \bar{\pi})$)-bimodule, has a bounded approximate identity. As an application, we show that G is amenable if and only if $A(G)$, as a $B_r(G)$-bimodule, has a bounded approximate identity. In Section 7, under certain assumptions on π, we are able to show that the amenability of π can be characterized by the (weak) factorization property of $A(G)$ via $A(\pi \otimes \bar{\pi})$. Consequently, we obtain some new characterizations of the amenability of G.

2. Some preliminaries. In this paper, all groups will be assumed to be locally compact, and G will denote a locally compact group. A *left* (resp. *right*) *Haar measure* on G is a non-zero positive Borel regular measure μ_G on G such that μ_G is *left* (resp. *right*) *translation invariant.* Every locally compact group possesses a left (right) Haar measure, which is unique up to multiplication by a positive constant. Let m_G be a fixed left Haar measure on G. The definition of $L^p(G)$ can be found in [Fol]. The convolution operation $*$ on $L^1(G)$ is given by

$$f * g(y) = \int_G f(yx)g(x^{-1})\,dm_G(x) \qquad \text{a.e.}$$

The involution on $L^1(G)$ is given by

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})},$$

where Δ is the modular function of G. We will call the Banach *-algebra $L^1(G)$, with the above convolution product and involution, the *group algebra* of G.

Let f be a function on G and $y \in G$. We define the left translates of f through y by

$$L_y f(x) = f(y^{-1}x).$$

Let E be a Banach space. We denote the Banach dual space of E by E'.

A *unitary representation* of G is a homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology.

Let Σ_G be the class of all unitary representations of G.

Let $\lambda_2 : G \to B(L^2(G))$, $[\lambda_2(x)(f)](y) := f(x^{-1}y)$ $(x, y \in G, f \in L^2(G))$. Then λ_2 is a unitary representation of G, called the *left regular representation* of G.

For any $f \in L^1(G)$, define

$$\|f\|_{C^*(G)} := \sup_{\pi \in \Sigma_G} \|\pi(f)\|.$$

It is easily seen that $\|\cdot\|_{C^*(G)}$ is a C^*-norm on $L^1(G)$. Let $C^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C^*(G)}$. Then $C^*(G)$ is called the *full group C^*-algebra* or simply the *group C^*-algebra* of G.

Let $B(G) := \{x \mapsto \langle \pi(x)\xi, \eta\rangle : \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$ be the *Fourier–Stieltjes algebra* of G. Then $B(G)$ is a commutative Banach algebra with the pointwise multiplication and norm given by

$$\|u\|_{B(G)} = \inf\{\|\xi\|\,\|\eta\| : u(x) = \langle \pi(x)\xi,\, \eta\rangle,\ \pi \in \Sigma_G,\ \xi, \eta \in \mathcal{H}_\pi\}.$$

The *big group algebra* or the *group W^*-algebra*, $W^*(G)$, is defined as the enveloping von Neumann algebra of $C^*(G)$. Note that $B(G)' = W^*(G)$. See [Eym] for more details.

Let $VN(G)$ be the von Neumann algebra generated by the image of λ_2 in $B(L^2(G))$. It is called the *group von Neumann algebra* of G. It is proved by Eymard [Eym] that the dual Banach space of $A(G)$ is isometrically isomorphic to $VN(G)$. For $u \in A(G)$ and $T \in VN(G)$, define $u \cdot T \in VN(G)$ by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$.

Suppose that π is a unitary representation of G. Let $F_\pi(G) = \text{span}\,\{x \mapsto \langle \pi(x)\xi, \eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}$. Then $A_\pi(G)$, the *Fourier space associated to* π, is defined to be the closure of $F_\pi(G)$ in the Banach space $B(G)$. For any representation π of G, define $VN_\pi(G)$ to be the von Neumann algebra generated by $\pi(G)$ (or $\pi(L^1(G))$) in $\mathcal{L}(\mathcal{H}_\pi)$. We have $A_\pi(G)' = VN_\pi(G)$. If $\pi = \lambda_2$, then $A_\pi(G) = A(G) = F_\pi(G)$ and $VN_\pi(G) = VN(G)$. For each $u \in A_\pi(G)$, there exist some nets (ξ_n) and (η_n) in $\mathcal{H}_\pi$ such that

$$u(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \eta_n \rangle \quad \text{and} \quad \|u\| = \sum_{n=1}^{\infty} \|\xi_n\| \, \|\eta_n\|.$$

The *reduced Fourier space associated to* π, $B_\pi(G)$, is defined to be the weak* closure of $A_\pi(G)$ in $B(G)$. For further details, see [Ars].

If $\pi = \lambda_2$, then $B_r(G) = B_{\lambda_2}(G)$ is called the *reduced Fourier–Stieltjes algebra* of G.

For the definition of weak equivalence and quasi-equivalence, see [Ars], and [Dix] and [Eym] for more details.

3. Some basic characterizations of amenable and H-amenable representations. For the definitions of H-amenability and amenability of unitary representations, we refer the readers to [Bek] and [Sto]. The following proposition, which can be found in [Bek, Theorem 5.1] and [Sto, Proposition 2.6], will be useful. One can actually use these results to define H-amenability and amenability by using weak containments of representations.

PROPOSITION 3.1. *Let G be a locally compact group and π be a unitary representation of G. We have*

(a) *π is H-amenable if and only if the trivial representation 1 of G is weakly contained in π.*
(b) *π is amenable if and only if the trivial representation 1 of G is weakly contained in $\pi \otimes \bar{\pi}$.*

Let π be a unitary representation of G. Let $A(\pi)$ (resp. $B(\pi)$) be the closed subalgebra in $B(G)$ generated by $A_\pi(G)$ (resp. $B_\pi(G)$). Also, let $VN(\pi)$ be the dual space of $A(\pi)$. It is easy to see that

$$A(\pi) = A_{\tau_\pi}(G) \quad \text{where} \quad \tau_\pi = \bigoplus_{n=1}^{\infty} \pi^{\otimes n}.$$

LEMMA 3.2. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is H-amenable.*
(b) *$\bar{\pi}$ is H-amenable.*
(c) *$\pi \oplus \bar{\pi}$ is H-amenable.*

Proof. Observe that π is H-amenable if and only if $1 \in B_\pi(G)$ (see [Sto, Proposition 2.6]). Also, $B_{\bar{\pi}}(G) = \{\bar{u} : u \in B_\pi(G)\}$. The equivalence of (a) and (b) follows immediately. The rest follows from [Sto, Proposition 2.4]. ■

THEOREM 3.3. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is H-amenable.*
(b) *$\|f\|_1 = \|\pi(f)\|$ for any $f \in L^1(G)^+$.*

(c) $\|\mu\| = \|\pi(\mu)\|$ *for any* $\mu \in M(G)^+$.

(d) *There exists* $M > 0$ *such that*

$$\left|\int f\right| \le M\|\pi(f)\| \quad \text{for any } f \in L^1(G).$$

(e) *There exists* $M > 0$ *such that*

$$|\mu(G)| \le M\|\pi(\mu)\| \quad \text{for any } \mu \in M(G).$$

(f) *There exists* $M > 0$ *such that*

$$\|f\|_1 \le M\|\pi(f)\| \quad \text{for any } f \in L^1(G)^+.$$

(g) *There exists* $M > 0$ *such that*

$$\|\mu\| \le M\|\pi(\mu)\| \quad \text{for any } \mu \in M(G)^+.$$

Proof. The implication (a)⇒(e) follows from [Eym, Lemma 1.23]. Also, it is easy to check that (e)⇒(d)⇒(f) and (e)⇒(g)⇒(f).

We will first show that (f)⇒(b). Suppose that (f) holds. Let $f \in L^1(G)^+$. By replacing f by $f * f^*$, we have

$$\left|\int f\right|^2 = \left|\int f * f^*\right| \le M\|\pi(f * f^*)\| = M\|\pi(f)\|^2.$$

Therefore, by induction, we have $|\int f| \le M^{1/2n}\|\pi(f)\|$ for any natural number n. So, $|\int f| \le \|\pi(f)\|$. Since $\|\pi(f)\| \le \|f\|_1$, this implies

$$\|\pi(f)\| = \|f\|_1 \quad \text{for any } f \in L^1(G)^+.$$

Therefore, (b) holds. One can use a similar argument to show that (g)⇒(c).

The implication (c)⇒(b) is clear.

It remains to show (b)⇒(a). If $f \in L^1(G)$ is real, write $f = f^+ - f^-$. We have

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| = |\,\|\pi(f^+)\| - \|\pi(f^-)\|\,| \le \|\pi(f^+) - \pi(f^-)\| = \|\pi(f)\|.$$

Note that $\|\pi(f)\| = \|(\pi \oplus \bar{\pi})(f)\|$ for any real-valued $f \in L^1(G)$.

Also, we have $\|(\pi \oplus \bar{\pi})(f)\| = \|(\pi \oplus \bar{\pi})(\bar{f})\|$ for any $f \in L^1(G)$.

For general $f \in L^1(G)$, we have

$$\begin{aligned}
\left|\int f\right| &= \left|\int \frac{f+\bar{f}}{2} + i\int \frac{f-\bar{f}}{2i}\right| \le \left|\int \frac{f+\bar{f}}{2}\right| + \left|\int \frac{f-\bar{f}}{2i}\right| \\
&\le \left\|\pi\left(\frac{f+\bar{f}}{2}\right)\right\| + \left\|\pi\left(\frac{f-\bar{f}}{2i}\right)\right\| \\
&= \left\|(\pi \oplus \bar{\pi})\left(\frac{f+\bar{f}}{2}\right)\right\| + \left\|(\pi \oplus \bar{\pi})\left(\frac{f-\bar{f}}{2i}\right)\right\| \\
&\le \frac{1}{2}\,[2\|(\pi \oplus \bar{\pi})(f)\| + 2\|(\pi \oplus \bar{\pi})(\bar{f})\|] = 2\|(\pi \oplus \bar{\pi})(f)\|.
\end{aligned}$$

By using the same argument found in (f)⇒(b), one can show that

$$\left|\int f\right| \le \|(\pi \oplus \bar{\pi})(f)\| \quad \text{for any } f \in L^1(G).$$

This is equivalent to saying that the trivial representation is weakly contained in $\pi \oplus \bar{\pi}$ (see [Eym]). By Lemma 3.2, π is H-amenable. ■

Remark 3.4. The results are well-known if $\pi = \lambda_2$, and can be found in [Gre] for example. However, we cannot use exactly the same proofs since π is in general not self-adjoint.

Let G_d be the group G equipped with discrete topology. Recall that ([Sto, Definition 2.1]) π is H_d-*amenable* if π_d, the representation π viewed as a representation of G_d, is H-amenable. We have the following characterization of H_d-amenable representations.

Corollary 3.5. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is H_d-amenable.*
(b) *$\|f\|_1 = \|\pi(f)\|$ for any $f \in l^1(G)^+$.*
(c) *There exists $M > 0$ such that*

$$\left|\sum f(s)\right| \le M\|\pi(f)\| \quad \text{for any } f \in l^1(G).$$

(d) *There exists $M > 0$ such that*

$$\|f\|_1 \le M\|\pi(f)\| \quad \text{for any } f \in l^1(G)^+.$$

We now give another characterization of H-amenability which is of independent interest.

Lemma 3.6. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is H-amenable.*
(b) *There exists a net (ξ_α) in the unit ball of $\mathcal{H}_\pi$ such that*

$$\|\pi(\mu)\xi_\alpha - \xi_\alpha\| \to 0 \quad \text{for any } \mu \in M(G)_1^+.$$

(c) *There exists a net (ξ_α) in the unit ball of $\mathcal{H}_\pi$ such that*

$$\|\pi(f)\xi_\alpha - \xi_\alpha\| \to 0 \quad \text{for any } f \in L^1(G)_1^+.$$

(d) *There exists a net (ξ_α) in the unit ball of $\mathcal{H}_\pi$ such that*

$$\langle \pi(\mu)\xi_\alpha - \xi_\alpha, \eta\rangle \to 0 \quad \text{for any } \mu \in M(G)_1^+,\ \eta \in \mathcal{H}_\pi.$$

(e) *There exists a net (ξ_α) in the unit ball of $\mathcal{H}_\pi$ such that*

$$\langle \pi(f)\xi_\alpha - \xi_\alpha, \eta\rangle \to 0 \quad \text{for any } f \in L^1(G)_1^+,\ \eta \in \mathcal{H}_\pi.$$

Proof. We first prove (a)⇒(d). Without loss of generality, assume that μ has compact support. Let $K = \mathrm{supp}(\mu)$. By definition, there exists (ξ_α)

in the unit ball of $\mathcal{H}_\pi$ such that

$$\|\pi(x)\xi_\alpha - \xi_\alpha\| \to 0 \quad \text{uniformly on } K.$$

Therefore, for any $\eta \in \mathcal{H}_\pi$, we have

$$|\langle \pi(\mu)\xi_\alpha - \xi_\alpha, \eta\rangle| = \Big|\int_K \langle \pi(x)\xi_\alpha - \xi_\alpha, \eta\rangle\, d\mu(x)\Big| \le \|\mu\|\, \|\pi(x)\xi_\alpha - \xi_\alpha\|\, \|\eta\| \to 0.$$

It is obvious that (d)⇒(e) and (b)⇒(c). By Namioka's argument (see [Gre, Theorem 2.4.2]), one can show that (d)⇒(b) and (e)⇒(c). We will show (e)⇒(c) as an example.

Let $E = \prod \{\mathcal{H}_\pi : f \in L^1(G)_1^+\}$. Define a map $T : \mathcal{H}_\pi \to E$ by

$$T(\xi)(f) = \pi(f)(\xi) - \xi.$$

It is easy to check that $T(\mathcal{H}_{\pi,1})$ is convex. Note that the weak topology of E is the product topology of Hilbert spaces $\mathcal{H}_\pi$, and the weak closure and the norm closure of a convex subset coincide. By our assumption in (e), we have

$$0 \in \text{weak-cl}(T(\mathcal{H}_{\pi,1})) = \text{norm-cl}(T(\mathcal{H}_{\pi,1})).$$

Therefore, there exists a net (ξ_α) in the unit ball of $\mathcal{H}_\pi$ such that

$$\|\pi(f)\xi_\alpha - \xi_\alpha\| \to 0 \quad \text{for any } f \in L^1(G)_1^+.$$

Finally, suppose that (c) holds. Let w_{ξ_α} be the state on $VN_\pi(G)$ defined by

$$w_{\xi_\alpha}(T) = \langle T\xi_\alpha, \xi_\alpha\rangle \quad \text{for any } T \in VN_\pi(G).$$

Let w be a weak*-limit point of w_{ξ_α} in $VN_\pi(G)'$. Then it is clear that

$$w(\pi(f)T) = w(T) \quad \text{for any } T \in VN_\pi(G),\ f \in L^1(G).$$

Thus, $VN_\pi(G)$ has a π-TRIM as defined in [Sto, Section 3]. Therefore, π is H-amenable by [Sto, Proposition 3.1]. ■

Remark 3.7. If $\pi = \lambda_2$, the results are well-known and can be found in [Gre] or [Pat] for example.

4. Weak closure of $\pi(G)$. We quote the following result for convenience (see [Sto]).

Lemma 4.1. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is H-amenable.*
(b) *There is a state Φ on $VN_\pi(G)$ such that $\Phi(\pi(\mu)) = \mu(G)$ for any $\mu \in M(G)$.*

Let $W_\pi^*(G)$ be the enveloping von Neumann algebra of $C_\pi^*(G)$. Then $W_\pi^*(G)$ is the dual space of $B_\pi(G)$. Let ω_π be the universal representation of $C_\pi^*(G)$. We have $B_\pi(G) = A_{\omega_\pi}(G)$ and $W_\pi^*(G) = VN_{\omega_\pi}(G)$ (see [Ars]).

Lemma 4.2. *Let G be a locally compact group, and let π be a unitary representation of G. For any $u \in B_\pi(G)$, the map*

$$\Phi_u : M(G) \to \mathbb{C}, \qquad \mu \mapsto \int u\, d\mu(x),$$

is bounded if $M(G)$ is equipped with the norm topology of $VN_\pi(G)$.

Proof. Consider the map

$$\Psi_u : W^*_\pi(G) \to \mathbb{C}, \qquad T \mapsto \langle T, u\rangle.$$

The $(W^*_\pi(G), B_\pi(G)) = (VN_{\omega_\pi}(G), A_{\omega_\pi}(G))$ duality gives

$$|\Phi_u(\mu)| = \left|\int u\, d\mu\right| \le |\langle \omega_\pi(\mu), u\rangle| \le \|\omega_\pi(\mu)\|\, \|u\|_{B_\pi(G)} = \|\pi(\mu)\|\, \|u\|_{B_\pi(G)}.$$

Note that the last equality holds because π and ω_π are weakly equivalent. ∎

Theorem 4.3. *Let G be a locally compact group, and let π be a unitary representation of G. Consider the following conditions:*

(a) *π is H-amenable.*
(b) *$\|T\| = 1$ for any $T \in \mathrm{co}(\pi(G))$.*
(c) *$0 \notin$ weak-cl$(\pi(G))$.*
(d) *There exists $\epsilon_0 > 0$ and $\Phi \in VN_\pi(G)^*$ such that $\langle \Phi, \pi(x)\rangle \ge \epsilon_0$ for every $x \in G$.*
(e) *There exists $u \in B_\pi(G)$ such that $u(x_\alpha)$ is not convergent to 0 for any net (x_α) in G.*

Then (a)⇒(b)⇒(c), (a)⇒(d)⇒(c) *and* (a)⇒(e)⇒(c).

Proof. (a)⇒(b) follows from Theorem 3.3.

(a)⇒(d) follows directly from Lemma 4.1.

(a)⇒(e) is true since $1 \in B(\pi)$ if π is H-amenable.

(b)⇒(c). Suppose that (c) is not true. Then

$$0 \in \text{weak-cl}(\mathrm{co}(\pi(G)) = \text{norm-cl}(\mathrm{co}(\pi(G))).$$

This contradicts (b).

(d)⇒(c). Assume that (c) is false. There exists a net $\{x_\alpha\}$ in G such that $\pi(x_\alpha) \to 0$ weakly in $VN_\pi(G)$. This contradicts (e).

(e)⇒(c). For any $u \in B_\pi(G)$, let Φ_u be as defined in Lemma 4.2. Let Ψ_u be a Hahn–Banach extension of Φ_u to $VN_\pi(G)$. Assume that (c) is false. There exists a net $\{x_\alpha\}$ in G such that $\pi(x_\alpha) \to 0$ weakly in $VN_\pi(G)$. Then

$$u(x_\alpha) = \langle \Phi_u, x_\alpha\rangle = \langle \Psi_u, x_\alpha\rangle \to 0. \ ∎$$

Lemma 4.4. *If $\|T\| = 1$ for any $T \in \mathrm{co}(\pi(G))$, then π is H_d-amenable.*

Proof. We will show that $\|f\|_1 = \|\pi_d(f)\|$ for every $f \in l^1(G)^+$, and apply Corollary 3.5. By scaling, we may assume that $f \in l^1(G)^+_1$. Take (f_n) to be a sequence of finitely supported functions in $l^1(G)^+_1$ such that

$\|f_n - f\| \to 0$. Then each $\pi_d(f_n)$ is in $\mathrm{co}(\pi(G))$, so $\|\pi_d(f)\| = \lim \|\pi_d(f_n)\| = \lim 1 = 1$ as needed. ∎

THEOREM 4.5. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is amenable.*
(b) *$\|T\| = 1$ for any $T \in \mathrm{co}(\pi \otimes \bar{\pi}(G))$.*
(c) *$\|T\| = 1$ for any $T \in \mathrm{co}(\tau_{\pi\otimes\bar{\pi}}(G))$.*
(d) *$0 \notin \text{weak-cl}(\pi \otimes \bar{\pi}(G))$.*
(e) *$0 \notin \text{weak-cl}(\tau_{\pi\otimes\bar{\pi}}(G))$.*
(f) *There exist $\epsilon_0 > 0$ and $\Phi \in VN_{\pi\otimes\bar{\pi}}(G)^*$ such that $\langle \Phi, \pi \otimes \bar{\pi}(x)\rangle \geq \epsilon_0$ for every $x \in G$.*
(g) *There exist $\epsilon_0 > 0$ and $\Phi \in VN(\pi \otimes \bar{\pi})^*$ such that $\langle \Phi, \tau_{\pi\otimes\bar{\pi}}(x)\rangle \geq \epsilon_0$ for every $x \in G$.*
(h) *There exists $u \in B_{\pi\otimes\bar{\pi}}(G)$ such that $u(x_\alpha)$ is not convergent to 0 for any net (x_α) in G.*
(i) *There exists $u \in B(\pi \otimes \bar{\pi})$ such that $u(x_\alpha)$ is not convergent to 0 for any net (x_α) in G.*

Proof. By Theorem 4.3, we have the following implications: (a)⇒(b)⇒(d), (a)⇒(f)⇒(d), (a)⇒(h)⇒(d) and (i)⇒(e). It is easy to check that (e)⇒(d) and (h)⇒(i).

We will first prove (d)⇒(g). Since $0 \notin \text{weak-cl}(\pi \otimes \bar{\pi}(G))$, there exist $\Phi_1, \ldots, \Phi_k \in VN_{\pi\otimes\bar{\pi}}(G)_1^*$ and $\epsilon > 0$ such that

$$\pi \otimes \bar{\pi}(G) \cap \bigcap_{i=1}^{k} \{T \in VN_{\pi\otimes\bar{\pi}}(G) : |\langle \Phi_i, T\rangle| \leq \sqrt{\epsilon}\} = \emptyset.$$

Therefore,

$$G = \bigcup_{i=1}^{k} \{x \in G : |\langle \Phi_i, \pi \otimes \bar{\pi}(x)\rangle|^2 \geq \epsilon\}.$$

Hence,

$$\sum_{i=1}^{k} |\langle \Phi_i, \pi \otimes \bar{\pi}(x)\rangle|^2 \geq \epsilon \quad \text{for any } x \in G.$$

Let $(u_\alpha^i)_\alpha$ be a net in the unit ball of $A_{\pi\otimes\bar{\pi}}(G)$ such that $u_\alpha^i \to \Phi_i$ in the weak* topology. Let $v_\alpha^i = u_\alpha^i \overline{u_\alpha^i}$. Then the net $(v_\alpha^i)_\alpha$ is in the unit ball of $A(\pi \otimes \bar{\pi})$. Without loss of generality, we may assume that (v_α^i) converges to $\Psi_i \in VN(\pi \otimes \bar{\pi})$ in the weak* topology. Then

$$|\langle \Phi_i, \pi \otimes \bar{\pi}(x)\rangle|^2 = \lim_\alpha |u_\alpha^i(x)|^2 = \lim_\alpha v_{\alpha,i}(x) = \langle \Psi_i, \tau_{\pi\otimes\bar{\pi}}(x)\rangle$$

Let $\Phi = (\sum_{i=1}^{k} \Psi_i)/k$. Then

$$\langle \Phi, \tau_{\pi\otimes\bar{\pi}}(x)\rangle = \frac{1}{k}\sum_{i=1}^{k}\langle \Psi_i, \tau_{\pi\otimes\bar{\pi}}(x)\rangle = \frac{1}{k}\sum_{i=1}^{k}|\langle \Phi_i, \pi\otimes\bar{\pi}(x)\rangle|^2 \geq \frac{\epsilon}{k}$$

for any $x \in G$.

We now show that (g)⇒(c). Let $T \in \mathrm{co}(\tau_{\pi\otimes\bar{\pi}}(G))$. Without loss of generality, we may assume that $\|\Phi\| = 1$ in (g). Thus, we have

$$1 \geq \|T\| \geq \langle \Phi, \tau_{\pi\otimes\bar{\pi}}(x)\rangle \geq \epsilon_0.$$

Since $\mathrm{co}(\tau_{\pi\otimes\bar{\pi}}(G))$ is a semigroup, we have, for any natural number k,

$$1 \geq \|T\|^k \geq \|T^k\| \geq \epsilon_0.$$

Therefore, $\|T\| = 1$.

Finally, we have to show (c)⇒(a). By Lemma 4.4, $(\tau_{\pi\otimes\bar{\pi}})_d$ is amenable. Observe that $A_{\tau_\pi\otimes\tau_{\bar{\pi}}}$ is an algebra containing $u\bar{v}$ for $u, v \in F_\pi$, so $A_{\tau_{\pi\otimes\bar{\pi}}} = A(\pi\otimes\bar{\pi})$ is contained in $A_{\tau_\pi\otimes\tau_{\bar{\pi}}}$. Therefore, $(\tau_{\pi\otimes\bar{\pi}})_d$ is quasi-equivalent to a subrepresentation of $(\tau_\pi\otimes\tau_{\bar{\pi}})_d$ and hence $(\tau_\pi\otimes\tau_{\bar{\pi}})_d$ is H-amenable. However, this is equivalent to saying that π is amenable by [Sto, Corollary 2.8 and Lemma 4.2]. ■

Remark 4.6. Theorem 4.3 is a generalization of some results in [C-X]. Part of the proof of Theorem 4.5 is inspired by the proof of [C-X, Proposition 2.4].

5. Factorization properties in $A[\pi]$ and $B[\pi]$. Note that $B(G)$ has a natural involution or *-operation, namely the complex conjugation. Let $A[\pi]$ (resp. $B[\pi]$) be the closed *-subalgebra in $B(G)$ generated by $A_\pi(G)$ (resp. $B_\pi(G)$). It is not hard to see that

$$A[\pi] = A(\pi\oplus\bar{\pi}) = A_{\rho_\pi}(G) \quad \text{where} \quad \rho_\pi = \tau_{\pi\oplus\bar{\pi}} = \bigoplus_{n=1}^{\infty}(\pi\oplus\bar{\pi})^{\otimes n}.$$

The following lemma is easy to prove (see [Bek, Theorem 1.3(iii)]).

Lemma 5.1. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is amenable.*
(b) *$\bar{\pi}$ is amenable.*
(c) *$\pi\oplus\bar{\pi}$ is amenable.*

The following proposition is a direct consequence of the above lemma and [Sto, Lemma 4.2].

Proposition 5.2. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) π *is amenable.*
(b) ρ_π *is amenable.*

A Banach algebra A of complex-valued continuous functions on a topological space X is said to be *weakly self-adjoint* if there exists $K_0 > 0$ such that for each $f \in A$, we have

$$|f|^2 \in A \quad \text{and} \quad \|\,|f|^2\,\|_A \le K_0 \|f\|_A^2.$$

Put $S(A) = \{x \in X : f(x) \neq 0 \text{ for some } f \in A\}$.

Let π be a unitary representation of G. It is clear that $A[\pi]$ is a weakly self-adjoint Banach algebra on G and $S(A[\pi]) = G$.

A Banach algebra A is said to *factorize* (resp. *factorize weakly*) if $A = AA$ (resp. $A = \mathrm{span}(AA)$). We quote the following result ([F-G-L, Theorem 1.3]) for convenience:

THEOREM 5.3. *Let A be a weakly self-adjoint Banach algebra of complex-valued functions. Suppose that A factorizes weakly. Then there exists $C > 0$ such that for each compact subset $K \subseteq S(A)$ there is an $f \in A$ such that*

$$f \ge 1 \ \text{on } K, \qquad f \ge 0 \ \text{on } S(A) \quad \text{and} \quad \|f\| \le C.$$

THEOREM 5.4. *Let G be a locally compact group, and let π be a unitary representation of G. Consider the following conditions:*

(a) *$A[\pi]$ has a bounded approximate identity.*
(b) *$A[\pi]$ factorizes.*
(c) *$A[\pi]$ factorizes weakly.*
(d) *ρ_π is H-amenable.*

Then (a)⇒(b)⇒(c)⇒(d).

Proof. We only need to show (c)⇒(d). By Theorem 5.3, there exists $C > 0$ such that for each compact subset $K \subseteq G$ there is an $f \in A[\pi]$ such that

$$f \ge 1 \text{ on } K, \qquad f \ge 0 \text{ on } G \quad \text{and} \quad \|f\| \le C.$$

For any $g \in L^1(G)^+$, we have

$$\int_G fg = |\langle \rho_\pi(g), f\rangle| \le \|\rho_\pi(g)\|\,\|f\|_{A[\pi]}.$$

This implies

$$\int_K g \le C\|\rho_\pi(g)\|.$$

By regularity of the Haar measure, we have

$$\int_G g \le C\|\rho_\pi(g)\|.$$

By Theorem 3.3, ρ_π is H-amenable. ∎

REMARK 5.5. The above proof follows an idea in [Los, Proposition 2].

COROLLARY 5.6. *Let G be a locally compact group, and let π be a unitary representation of G. Consider the following conditions:*

(a) *$A(\pi \otimes \bar{\pi})$ has a bounded approximate identity.*
(b) *$A(\pi \otimes \bar{\pi})$ factorizes.*
(c) *$A(\pi \otimes \bar{\pi})$ factorizes weakly.*
(d) *π is amenable.*

Then (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).

Proof. Note that $\pi \otimes \bar{\pi}$ is quasi-equivalent to its conjugate. Also, $\tau_{\pi\otimes\bar{\pi}}$ is amenable if and only if π is amenable (see [Sto, Lemma 4.2]). ∎

REMARK 5.7. The implication (a)$\Rightarrow$(d) is given in [Sto, Theorem 4.3]. The above gives a new approach to prove this result.

Of course, $B(\pi \otimes \bar{\pi}) = B[\pi \otimes \bar{\pi}]$ since $\pi \otimes \bar{\pi}$ is weakly equivalent to its conjugate.

THEOREM 5.8. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is amenable.*
(b) *$B(\pi \otimes \bar{\pi})$ has a bounded approximate identity.*
(c) *$B(\pi \otimes \bar{\pi})$ factorizes.*
(d) *$B(\pi \otimes \bar{\pi})$ factorizes weakly.*

Proof. The implication (a)$\Rightarrow$(b) is proved in [Sto, Theorem 4.5]. Note that $B(\pi\otimes\bar{\pi}) = A_\rho(G)$ where $\rho = \omega_{\rho_{\pi\otimes\bar{\pi}}}$. By Theorem 5.4, it follows directly that (b)$\Rightarrow$(c)$\Rightarrow$(d).

We have to prove (d)$\Rightarrow$(a). By Theorem 5.4, ρ is H-amenable. Since ρ and $\rho_{\pi\otimes\bar{\pi}}$ are weakly equivalent, the trivial representation is weakly contained in $\rho_{\pi\otimes\bar{\pi}}$. Therefore, π is amenable by Lemma 5.1. ∎

The equivalence of (a) and (b) is just [Sto, Theorem 4.5]. The above provides a different proof.

COROLLARY 5.9. *Let G be a locally compact group. The following conditions are equivalent:*

(a) *G is amenable.*
(b) *$B_r(G)$ has a bounded approximate identity.*
(c) *$B_r(G)$ factorizes.*
(d) *$B_r(G)$ factorizes weakly.*

Proof. This follows from the fact that λ_2 is quasi-equivalent to $\lambda_2 \otimes \overline{\lambda_2}$. ∎

COROLLARY 5.10. *Let G be a locally compact group. The following conditions are equivalent:*

(a) *G is amenable.*
(b) *$B_r(G)$ contains a closed non-trivial translation invariant *-subalgebra which factorizes* (*weakly*).

Proof. Let A be a closed non-trivial translation invariant *-subalgebra which factorizes (weakly). Then $A = A_\pi(G) = A[\pi]$ for some unitary representation π (see [Ars]). Therefore, ρ_π is H-amenable. But ρ_π is also weakly contained in λ_2. Therefore, G is amenable since the trivial representation is weakly contained in λ_2. ∎

6. Existence of bounded approximate identities in $A(\pi)$-bimodules. We begin with the general setting: let A be a commutative Banach algebra, let X, Y be symmetric Banach A-bimodules, and put

$$M_A(X,Y) = \{T \in B(X,Y) : T(a \cdot x) = a \cdot T(x) \text{ for any } a \in A,\ x \in X\}.$$

It is straightforward to show that $M_A(X,Y)$ is a Banach subspace of $B(X,Y)$.

Lemma 6.1. *$M_A(X,Y^*)$ and $M_A(Y,X^*)$ are isometrically isomorphic. In particular, $M_A(X,A^*)$ and $M_A(A,X^*)$ are isometrically isomorphic.*

Proof. For any $\Phi \in M_A(X,Y^*)$, define $\Gamma_\Phi \in B(Y,X^*)$ by

$$\langle x, \Gamma_\Phi(y)\rangle_{(X,X^*)} = \langle \Phi(x), y\rangle_{(Y^*,Y)}.$$

For any $a \in A$, we have

$$\begin{aligned}\langle x, \Gamma_\Phi(a\cdot y)\rangle_{(X,X^*)} &= \langle \Phi(x), a\cdot y\rangle_{(Y^*,Y)} = \langle a\cdot\Phi(x), y\rangle_{(Y^*,Y)}\\ &= \langle \Phi(a\cdot x), y\rangle_{(Y^*,Y)} = \langle a\cdot x, \Gamma_\Phi(y)\rangle_{(X,X^*)} = \langle x, a\cdot\Gamma_\Phi(y)\rangle_{(X,X^*)}\end{aligned}$$

and

$$\begin{aligned}\|\Gamma_\Phi\| &= \sup\{\|\Gamma_\Phi(y)\| : y \in Y,\ \|y\| = 1\}\\ &= \sup\{|\langle x, \Gamma_\Phi(y)\rangle| : y \in Y,\ \|y\| = 1,\ x \in X,\ \|x\| = 1\}\\ &= \sup\{|\langle \Phi(x), y\rangle| : y \in Y,\ \|y\| = 1,\ x \in X,\ \|x\| = 1\}\\ &= \sup\{\|\Phi(x)\| : x \in X,\ \|x\| = 1\} = \|\Phi\|.\end{aligned}$$

By symmetry, it is easy to construct the inverse of the map $\Phi \mapsto \Gamma_\Phi$. ∎

Note that $C_0(G)$ is a symmetric Banach $A(\pi)$-bimodule via pointwise multiplications. Therefore, $M(G)$ has a natural dual module structure. Also, $VN(\pi)$, as the dual space of $A(\pi)$, has a natural dual module structure. Therefore, we obtain the following result:

Corollary 6.2. *For any unitary representation π of G, we have*

$$M_{A(\pi)}(C_0(G), VN(\pi)) \cong M_{A(\pi)}(A(\pi), M(G)).$$

We say that X, as a A-bimodule, has a *bounded approximate identity* in A if there exists a net (e_α) in A such that

$$\|e_\alpha \cdot x - x\| \to 0 \quad \text{and} \quad \sup_\alpha \|e_\alpha\|_A < \infty.$$

LEMMA 6.3. *Suppose that X, as a A-bimodule, has a bounded approximate identity in A. Then $M_A(X, A^*) = M_A(A, X^*)$ and X^* are topologically isomorphic.*

Proof. Write $M = M_A(X, A^*)$ for convenience. Let $F \in X^*$. Define $\Phi_F : X \to A^*$ by

$$\langle \Phi_F(x), a\rangle_{(A^*,A)} = F(a \cdot x).$$

It is straightforward to show that $\Phi_F \in M$. Conversely, let $\Phi \in M$. By Cohen's factorization theorem and the assumption, for any $x \in X$, there exist $a \in A$ and $y \in X$ such that $x = a \cdot y$. Define

$$F_\Phi(x) = \langle \Phi(y), a\rangle_{(A^*,A)}.$$

We need to show that F_Φ is well-defined and bounded. Let $(e_\alpha) \subseteq A$ be an approximate identity for X such that $\sup_\alpha \|e_\alpha\|_A \le C$ for some $C > 0$.

First of all, we observe that $\langle \Phi(b \cdot z), a\rangle_{(A^*,A)} = \langle a \cdot z, \Gamma_\Phi(b)\rangle_{(X,X^*)}$ for any $x, y \in A$, $z \in X$ where Γ_Φ is defined in Lemma 6.1. Now, we have

$$\begin{aligned} F_\Phi(x) = \langle \Phi(y), a\rangle &= \lim_\alpha (\langle \Phi(y - e_\alpha \cdot y), a\rangle + \langle \Phi(e_\alpha \cdot y), a\rangle) \\ &= \lim_\alpha (\langle \Phi(y - e_\alpha \cdot y), a\rangle + \langle a \cdot y, \Gamma_\Phi(e_\alpha)\rangle) \\ &= \lim_\alpha \langle x, \Gamma_\Phi(e_\alpha)\rangle, \end{aligned}$$

so F_ϕ is well-defined. Moreover, $|F_\Phi(x)| \le C\|x\| \, \|\Gamma_\Phi\| = C\|\Phi\| \, \|x\|$. ∎

The mapping $\Phi : X^* \to M_A(X, A^*)$, $F \mapsto \Phi_F$, which appears in the above proof, is called the *canonical embedding* of X^* into $M_A(X, A^*)$.

THEOREM 6.4. *Let G be a locally compact group, and let π be a unitary representation of G. The following conditions are equivalent:*

(a) *π is amenable.*
(b) *$A(G)$, as a Banach $A(\pi \otimes \bar{\pi})$-bimodule, has a bounded approximate identity.*
(c) *$A(G)$, as a Banach $B(\pi \otimes \bar{\pi})$-bimodule, has a bounded approximate identity.*
(d) *$C_0(G)$, as a Banach $A(\pi \otimes \bar{\pi})$-bimodule, has a bounded approximate identity.*
(e) *$C_0(G)$, as a Banach $B(\pi \otimes \bar{\pi})$-bimodule, has a bounded approximate identity.*
(f) *The canonical embedding $\Phi : M(G) \to M_{A(\pi\otimes\bar{\pi})}(A(\pi \otimes \bar{\pi}), M(G))$ is a continuous isomorphism.*

(g) *The canonical embedding* $\Phi : M(G) \to M_{B(\pi\otimes\bar\pi)}(B(\pi\otimes\bar\pi), M(G))$ *is a continuous isomorphism.*

Proof. (a)⇒(b)⇒(d). This follows from [Sto, Remark 3.2(3)] and the fact that $A(G)$ is norm-dense in $C_0(G)$.

(d)⇒(f) is a direct consequence of Lemma 6.3.

(f)⇒(a). Let $\mu \in M(G)^+$. Define $\Phi_\mu \in M_{A(\pi\otimes\bar\pi)}(A(G), VN(\pi\otimes\bar\pi))$ by

$$\langle \Phi_\mu(f), u\rangle = \int uf\,d\mu \quad \text{for any } f \in A(G),\ u \in A(\pi\otimes\bar\pi).$$

Write $\rho = \tau_{\pi\otimes\bar\pi}$ for convenience.

For any $\xi, \eta \in \mathcal{H}_{\rho,1}$, $f \in C_0(G)$, we have

$$\begin{aligned}
|\langle \rho(f\cdot\mu)\xi,\eta\rangle| &= \Big|\int \langle \rho(x)\xi,\eta\rangle f(x)\,d\mu(x)\Big| \le \int |\langle\rho(x)\xi,\eta\rangle|\,|f(x)|\,d\mu(x)\\
&\le \Big(\int |\langle\rho(x)\xi,\eta\rangle|^2\,d\mu(x)\Big)^{1/2}\Big(\int |f(x)|^2\,d\mu(x)\Big)^{1/2}\\
&\le \Big(\int \langle\rho(x)\xi,\eta\rangle\overline{\langle\rho(x)\xi,\eta\rangle}\,d\mu(x)\Big)^{1/2}\|f\|_\infty\|\mu\|^{1/2}\\
&= \Big(\int \langle(\rho\otimes\bar\rho)(x)\xi\otimes\xi,\eta\otimes\eta\rangle\,d\mu(x)\Big)^{1/2}\|f\|_\infty\|\mu\|^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\langle \rho(f\cdot\mu)\xi,\eta\rangle| &\le (\langle(\rho\otimes\bar\rho)(\mu)\xi\otimes\xi,\eta\otimes\eta\rangle)^{1/2}\|f\|_\infty\|\mu\|^{1/2}\\
&\le \|\rho\otimes\bar\rho)(\mu)\|^{1/2}\|f\|_\infty\|\mu\|^{1/2}\\
&\le \|\rho(\mu)\|^{1/2}\|f\|_\infty\|\mu\|^{1/2}
\end{aligned}$$

where the last inequality follows from [Eym, Lemma 1.23] and the fact that $\rho\otimes\bar\rho$ is quasi-equivalent to a subrepresentation of ρ. Therefore, we have

$$\begin{aligned}
|\langle\Phi_\mu(f),u\rangle| = \Big|\int uf\,d\mu\Big| &\le \|\rho(f\cdot\mu)\|\,\|u\|\\
&\le \|\rho(\mu)\|^{1/2}\|f\|_\infty\|\mu\|^{1/2}\|u\|,
\end{aligned}$$

and hence

$$\|\Phi_\mu\| \le \|\rho(\mu)\|^{1/2}\|\mu\|^{1/2}.$$

By the open mapping theorem and the assumption, there exists $K > 0$ such that $K\|\mu\| \le \|\Phi_\mu\|$. Therefore, there exists $M > 0$ such that

$$\|\mu\| \le M\|\rho(\mu)\|^{1/2}\|\mu\|^{1/2}.$$

Hence,

$$\|\mu\| \le M^2\|\rho(\mu)\|.$$

This completes the proof by using Theorem 3.3 and [Sto, Lemma 4.2].

The equivalence of (a), (b), (d) and (f) follows from the equivalence of (a), (c), (e) and (g) and the fact that $\tau_{\pi\otimes\bar\pi}$ and $\omega_{\tau_{\pi\otimes\bar\pi}}$ are weakly equivalent. ■

Corollary 6.5. *Let G be a locally compact group. The following conditions are equivalent:*

(a) *G is amenable.*
(b) *$A(G)$ has a bounded approximate identity.*
(c) *$A(G)$, as a Banach $B_r(G)$-bimodule, has a bounded approximate identity.*
(d) *$C_0(G)$, as a Banach $A(G)$-bimodule, has a bounded approximate identity.*
(e) *$C_0(G)$, as a Banach $B_r(G)$-bimodule, has a bounded approximate identity.*
(f) *The canonical embedding $\Phi : M_{A(G)}(A(G), M(G)) \to M(G)$ is a continuous isomorphism.*
(g) *The canonical embedding $\Phi : M_{B_r(G)}(B_r(G), M(G)) \to M(G)$ is a continuous isomorphism.*

Remark 6.6. The equivalence of (a) and (b) can be found in [Lep2]. The above corollary gives another proof of (b)⇒(a).

7. Factorization properties of $A(\pi)$-bimodules. The following proposition can be proved easily by using [Sto, Remark 3.2(3)] and simple density arguments. We outline the proof below.

Proposition 7.1. *Let G be a locally compact group, and let π be a unitary representation of G. Consider the following conditions:*

(a) *π is H-amenable.*
(b) $A(G) = A(\pi)A(G)$.
(c) $A(G) = B(\pi)A(G)$.
(d) $A(G) = \mathrm{span}(A(\pi)A(G))$.
(e) $A(G) = \mathrm{span}(B(\pi)A(G))$.
(f) $C_0(G) = A(\pi)C_0(G)$.
(g) $C_0(G) = B(\pi)C_0(G)$.
(h) $C_0(G) = \mathrm{span}(A(\pi)C_0(G))$.
(i) $C_0(G) = \mathrm{span}(B(\pi)C_0(G))$.

Then (a)⇒(b)⇒(d)⇒(h)⇒(i) *and* (a)⇒(c)⇒(e)⇒(i).

Proof. It follows from [Sto, Remark 3.2(3)] that $A(G)$, as a Banach $A(\pi)$-bimodule, has a bounded approximate identity if π is H-amenable. The rest follows from Cohen's factorization theorem and the fact that $A(G)$ is norm-dense in $C_0(G)$. ∎

Let X_π be the subspace of $C_0(G)$ defined by

$$X_\pi = \Big\{h : h = \sum_{i=1}^{\infty} u_i g_i,\ u_i \in B(\pi),\ g_i \in C_0(G),\ \sum_{i=1}^{\infty} \|u_i\|_{B(G)} \|g_i\|_\infty < \infty\Big\}.$$

We define a norm on X_π by

$$\|h\|_X = \inf\Big\{\sum_{i=1}^{\infty} \|u_i\|_{B(G)}\|g_i\|_\infty : h = \sum_{i=1}^{\infty} u_i g_i,\ u_i \in B(\pi),\ g_i \in C_0(G)\Big\}.$$

It is not hard to show that $(X_\pi, \|\cdot\|_X)$ is a Banach space.

LEMMA 7.2. *Let π be a C_0-representation of G. Suppose that $C_0(G)$ has an approximate identity contained in $B(\pi)$ which is bounded in the uniform norm. Then the space $M_{B(\pi)}(C_0(G), B(\pi)^*) = M_{B(\pi)}(B(\pi), M(G))$ is isometrically isomorphic to X_π^*.*

Proof. Write $B = B(\pi)$ and $M = M_B(C_0(G), B^*)$ for convenience.

Let $F \in C_0(G)^* = M(G)$. Define $\Phi_F : C_0(G) \to B^*$ by

$$\langle \Phi_F(x), a\rangle_{(B^*,B)} = F(a\cdot x).$$

It is straightforward to show that $\Phi_F \in M$ and $\|\Phi_F\| \le \|F\|$.

Conversely, let $\Phi \in M$. Without loss of generality, for any $x \in X_\pi$, we assume that $x = \sum_{i=1}^{n} a_i \cdot y_i$ for some finite sequences $(a_i)_{i=1}^n \subseteq B$ and $(y_i)_{i=1}^n \subseteq C_0(G)$. Define

$$F_\Phi(x) = \sum_{i=1}^{n} \langle \Phi(y_i), a_i\rangle.$$

Clearly, F_Φ is bounded by the definition of the norm of X_π. We need to show that F_Φ is well-defined. Let $(e_\alpha) \subseteq B$ be an approximate identity for $C_0(G)$ such that $\sup_\alpha \|e_\alpha\|_\infty \le C$ for some $C > 0$.

First of all, we observe that $\langle \Phi(b\cdot z), a\rangle = \langle a\cdot z, \Gamma_\Phi(b)\rangle$ for any $x, y \in B$, $z \in C_0(G)$ where Γ_Φ is defined in the proof of Lemma 6.1. Now, we have

$$\begin{aligned}
|F_\Phi(x)| &= \Big|\sum_{i=1}^{n} \langle \Phi(y_i), a_i\rangle\Big| \le \Big|\sum_{i=1}^{n} \langle \Phi(y_i - e_\alpha\cdot y_i), a_i\rangle\Big| + \Big|\sum_{i=1}^{n} \langle \Phi(e_\alpha\cdot y_i), a_i\rangle\Big| \\
&\le \|\Phi\| \sum_{i=1}^{n} \|y_i - e_\alpha\cdot y_i\|_X \|a_i\|_B + \Big|\Big\langle \sum_{i=1}^{n} a_i\cdot y_i, \Gamma_\Phi(e_\alpha)\Big\rangle\Big| \\
&= \|\Phi\| \sum_{i=1}^{n} \|y_i - e_\alpha\cdot y_i\|_X \|a_i\|_B + |\langle x, \Gamma_\Phi(e_\alpha)\rangle|.
\end{aligned}$$

By taking $\alpha \to \infty$, it follows that $F_\Phi(0) = 0$. Hence, F_Φ is well-defined.

It is clear from the definition of the norm $\|\cdot\|_X$ that $\|F_\Phi\| \le \|\Phi\|$. ∎

REMARK 7.3. The above result and part of the proof are motivated by [Fig].

A unitary representation π of G is called a *C_0-representation* if $F_\pi(G) \subseteq C_0(G)$.

THEOREM 7.4. *Let π be a C_0-representation of G. Suppose that $C_0(G)$ has a bounded approximate identity contained in $B(\pi\otimes\bar{\pi})$. Then the following conditions are equivalent:*

(a) *π is amenable.*
(b) $A(G) = A(\pi\otimes\bar{\pi})A(G)$.
(c) $A(G) = B(\pi\otimes\bar{\pi})A(G)$.
(d) $A(G) = \mathrm{span}(A(\pi\otimes\bar{\pi})A(G))$.
(e) $A(G) = \mathrm{span}(B(\pi\otimes\bar{\pi})A(G))$.
(f) $C_0(G) = A(\pi\otimes\bar{\pi})C_0(G)$.
(g) $C_0(G) = B(\pi\otimes\bar{\pi})C_0(G)$.
(h) $C_0(G) = \mathrm{span}(A(\pi\otimes\bar{\pi})C_0(G))$.
(i) $C_0(G) = \mathrm{span}(B(\pi\otimes\bar{\pi})C_0(G))$.

Proof. By Proposition 7.1, we have to show (i)⇒(a). Let $\rho = \tau_{\pi\otimes\bar{\pi}}$. If (i) holds, then $X_\rho = C_0(G)$. Therefore, by the opening mapping theorem, the norms $\|\cdot\|_X$ and $\|\cdot\|_\infty$ are equivalent. As a result, $M_{B(\rho)}(B(\rho), M(G)) = X_\rho^*$ (see Lemma 7.2) is topologically isomorphic to $M(G)$. In view of the proof of (g)⇒(a) in Theorem 6.4, the proof is complete. ∎

REMARK 7.5. The above result and part of the proof are motivated by [Neb].

By taking $\pi = \lambda_2$, we obtain some new characterizations of amenable groups.

COROLLARY 7.6. *Let G be a locally compact group. Then the following conditions are equivalent:*

(a) *G is amenable.*
(b) $A(G) = B_r(G)A(G)$.
(c) $A(G) = \mathrm{span}(B_r(G)A(G))$.
(d) $C_0(G) = B_r(G)C_0(G)$.
(e) $C_0(G) = \mathrm{span}(B_r(G)C_0(G))$.

REMARK 7.7. For general G, we have

$$A(G) = \overline{\mathrm{span}(B_r(G)A(G))} \quad \text{and} \quad C_0(G) = \overline{\mathrm{span}(B_r(G)C_0(G))}.$$

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