

When is the Haar measure a Pietsch measure for nonlinear mappings?

by

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Abstract. We show that, as in the linear case, the normalized Haar measure on a compact topological group G is a Pietsch measure for nonlinear summing mappings on closed translation invariant subspaces of $C(G)$. This answers a question posed to the authors by J. Diestel. We also show that our result applies to several well-studied classes of nonlinear summing mappings. In the final section some problems are proposed.

1. Introduction. The Haar measure on a compact topological group G is simply a Radon measure σ_G on the Borel sets of G which is translation invariant, that is, $\sigma_G(gB) = \sigma_G(B)$ for every Borel set B and every $g \in G$. A well-known fact, essentially proved by A. Haar [13] in 1933 (see also [5, 19, 24, 25]), is that there is only one normalized Haar measure on G . In the same year (in fact in the same issue of *Annals of Mathematics*), J. von Neumann [18] used Haar's Theorem to solve Hilbert's fifth problem in the case of compact groups. The uniqueness of the normalized Haar measure will be used later in this paper.

A cornerstone of the theory of absolutely summing operators, the Pietsch Domination Theorem, asserts that a continuous linear operator $v: X_1 \rightarrow X_2$ between Banach spaces is absolutely p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on the closed unit ball of the dual of X_1 , $(B_{X_1^*}, \sigma(X_1^*, X_1))$, such that

$$(1.1) \quad \|v(x)\| \leq C \left(\int_{B_{X_1^*}} |\varphi(x)|^p d\mu(\varphi) \right)^{1/p}$$

for every $x \in X_1$. Such a measure μ is said to be a *Pietsch measure* for v .

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When $X_1 = C(K)$ for some compact Hausdorff space, a simple reformulation of the Pietsch Domination Theorem tells us that a continuous linear operator $v: C(K) \rightarrow X_2$ is absolutely p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on the Borel sets of K such that

$$\|v(f)\| \leq C \left(\int_K |f(z)|^p d\mu(z) \right)^{1/p}$$

for every $f \in C(K)$. The measure μ above is also called a *Pietsch measure* for v .

The Pietsch Domination Theorem has analogs in different contexts, including versions for classes of absolutely summing nonlinear operators (see, for example, [1, 6, 7, 9, 10, 15, 22]). Recently, in [4, 21, 23] the concept of *R-S-abstract p -summing mapping* was introduced in such a way that several previous known versions of the Pietsch Domination Theorem can be regarded as particular instances of one single result.

It is interesting to mention that in most of the cases we have almost no structural information on the Pietsch measures and, as mentioned in [8, p. 56], “in general its existence is accessible only by transfinite means”. Nevertheless in the important case when $X_1 = C(G)$ and G is a compact Hausdorff topological group, the precise nature of the Pietsch measure is known: motivated by results from [12, 20], in [8, p. 56] it is proved that if G is a compact topological group, then the normalized Haar measure on G is a Pietsch measure for any translation invariant p -summing linear operator on a closed translation invariant subspace of $C(G)$ that separates points of G (cf. Theorem 2.2). The particular case in which G is the circle group $\{z \in \mathbb{C} : |z| = 1\}$ is used in [14].

J. Diestel posed to the authors the question whether, as in the linear case, in this more general context of [4, 21] the normalized Haar measure on a compact topological group G is still a Pietsch measure for any translation invariant (not necessarily linear) *R-S-abstract p -summing mapping* on a closed invariant subspace of $C(G)$. The precise meaning of the terms used will be clarified in the next section.

In this paper we solve Diestel’s question in the positive. We show in Section 3 that the answer is affirmative provided two natural and general conditions are satisfied (cf. Theorem 3.1). In Section 4 we show that several usual classes of *R-S-abstract summing* (linear and nonlinear) mappings enjoy these two conditions, confirming that Diestel’s question has a positive answer in several important cases. In particular, we improve the original linear result from [8] by showing that the assumption that F separates points of G can be dropped (cf. Theorem 4.2). However there are still some open questions which we detail in the final section.

2. Preliminaries and background. Henceforth G denotes a (not necessarily abelian) compact Hausdorff topological group. For the operation on G we use multiplicative notation. The symbol $C(G)$ stands for the Banach space of continuous functions $f: G \rightarrow \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , endowed with the usual sup norm.

DEFINITION 2.1.

- (a) A nonempty set $F \subset C(G)$ is (left) translation invariant if

$$\{T^\phi : T \in F \text{ and } \phi \in G\} \subset F,$$

where $T^\phi(\varphi) := T(\phi\varphi)$.

- (b) Let X be any set and F be a closed translation invariant subset of $C(G)$. A map $u: F \rightarrow X$ is translation invariant if $u(T) = u(T^\phi)$ for all $\phi \in G$ and $T \in F$.
- (c) A set $F \subset C(G)$ is said to separate points of G if for all $x, y \in G$, $x \neq y$, there exists $T \in F$ such that $T(x) \neq T(y)$.

The following version of the Pietsch Domination Theorem appears in [8, p. 56] (by $\pi_p(u)$ we denote the p -summing norm of the operator u):

THEOREM 2.2. *If X is a Banach space, G is a compact Hausdorff topological group, F is a closed translation invariant subspace of $C(G)$ that separates points of G , and $u: F \rightarrow X$ is a translation invariant p -summing linear operator, then the normalized Haar measure σ_G on G is a Pietsch measure for u in the sense that*

$$\|u(f)\| \leq \pi_p(u) \left(\int_G |f(x)|^p d\sigma_G(x) \right)^{1/p} \quad \text{for every } f \in F.$$

REMARK 2.3. The assumption that F separates points of G is missing in [8]. To understand where this assumption is needed in the proof provided in [8], see Subsection 4.1. In that same subsection we provide a proof that does not require this extra assumption.

Next, we describe the general Pietsch Domination Theorem proved in [4, 21]. Let X, Y and E be (arbitrary) sets, and \mathcal{H} be a family of mappings from Y to X . Let also Z be a Banach space and K be a compact Hausdorff topological space. Assume that $S: \mathcal{H} \times E \times Z \rightarrow [0, \infty)$ is an arbitrary map and $R: K \times E \times Z \rightarrow [0, \infty)$ is so that

$$R_{x,b}(\varphi) := R(\varphi, x, b)$$

is continuous on K for all $(x, b) \in E \times Z$. For $0 < p < \infty$, a mapping $f \in \mathcal{H}$ is said to be R - S -abstract p -summing if there is a constant $C > 0$ so that

$$\left(\sum_{j=1}^m S(f, x_j, b_j)^p \right)^{1/p} \leq C \left(\sup_{\varphi \in K} \sum_{j=1}^m R(\varphi, x_j, b_j)^p \right)^{1/p},$$

for all $(x_j, b_j) \in E \times Z, j = 1, \dots, m$ and $m \in \mathbb{N}$. The infimum of such constants C is denoted by $\pi_{RS,p}(f)$. The Unified Pietsch Domination Theorem [4, 21] reads as follows:

THEOREM 2.4. *Let R and S be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then f is R - S -abstract p -summing if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on K such that*

$$(2.1) \quad S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu(\varphi) \right)^{1/p}$$

for all $(x, b) \in E \times Z$. Moreover, the infimum of such constants C equals $\pi_{RS,p}(f)$. Such a measure μ is called an R - S -abstract measure for f .

Thus, J. Diestel’s question concerns the validity of Theorem 2.2 in the context of Theorem 2.4.

3. Main result. In this section we shall keep the notation of the previous one. In our main result (Theorem 3.1 below) we have identified two natural conditions on R and S under which Diestel’s question has a positive answer:

THEOREM 3.1. *Let R and S be as above, $0 < p < \infty$, and $u: Y \rightarrow X$ be a map in \mathcal{H} . Further assume that*

- (i) $K = G$ is a compact topological group and E is a closed translation invariant subspace of $C(G)$,
- (ii) $R(\varphi, T^\phi, b) \leq R(\phi\varphi, T, b)$ for all $(T, b) \in E \times Z$ and $\varphi, \phi \in G$, and
- (iii) the map $S_{u,b}: E \rightarrow (0, \infty)$ defined by $S_{u,b}(T) = S(u, T, b)$ is translation invariant for every $b \in Z$.

If u is an R - S -abstract p -summing mapping, then the normalized Haar measure on G is an R - S -abstract measure for u .

Proof. By Theorem 2.4 there exists a Borel probability measure μ on G such that

$$S(u, T, b) \leq \pi_{RS,p}(u) \left(\int_G R(\varphi, T, b)^p d\mu(\varphi) \right)^{1/p}$$

for every $(T, b) \in E \times Z$. For each $\phi \in G$ define $\mu_\phi \in C(G)^*$ by

$$\langle \mu_\phi, T \rangle := \langle \mu, T^\phi \rangle$$

for every $T \in C(G)$. Indeed, μ_ϕ is a probability measure as

$$\langle \mu_\phi, 1_G \rangle = \langle \mu, 1_G^\phi \rangle = \int_G 1_G^\phi d\mu = \mu(G) = 1,$$

where 1_G denotes the constant mapping taking the value 1. Now we prove that μ_ϕ is an R - S -abstract measure for u : given $(T, b) \in E \times Z$,

$$\begin{aligned} S(u, T, b)^p &= S_{u,b}(T)^p = S_{u,b}(T^\phi)^p = S(u, T^\phi, b)^p \\ &\leq \pi_{RS,p}(u)^p \int_G R(\varphi, T^\phi, b)^p d\mu(\varphi) \\ &\leq \pi_{RS,p}(u)^p \int_G R(\phi\varphi, T, b)^p d\mu(\varphi) \\ &= \pi_{RS,p}(u)^p \int_G R_{T,b}(\phi\varphi)^p d\mu(\varphi) \\ &= \pi_{RS,p}(u)^p \int_G R_{T,b}^\phi(\varphi)^p d\mu(\varphi) \\ &= \pi_{RS,p}(u)^p \langle \mu, (R_{T,b}^p)^\phi \rangle = \pi_{RS,p}(u)^p \langle \mu_\phi, R_{T,b}^p \rangle \\ &= \pi_{RS,p}(u)^p \int_G R(\varphi, T, b)^p d\mu_\phi(\varphi). \end{aligned}$$

Let σ_G denote the normalized Haar measure on G . As the map $\phi \mapsto \mu_\phi$ is continuous when $C(G)^*$ is endowed with the weak* topology, there exists a measure $\nu \in C(G)^*$ such that

$$\langle \nu, T \rangle = \int_G \langle \mu_\phi, T \rangle d\sigma_G(\phi)$$

for every $T \in C(G)$ (see [8, p. 57]). If $T \in E$, $T \geq 0$, then

$$\langle \nu, T \rangle = \int_G \langle \mu_\phi, T \rangle d\sigma_G(\phi) = \int_G \langle \mu, T^\phi \rangle d\sigma_G(\phi) \geq 0,$$

so ν is a non-negative regular Borel measure on G . Moreover, the following calculation shows that ν is a probability:

$$\langle \nu, 1_G \rangle = \int_G \langle \mu_\phi, 1_G \rangle d\sigma_G(\phi) = \int_G 1_G d\sigma_G = \langle \sigma_G, 1_G \rangle = 1.$$

Since

$$\frac{S(u, T, b)^p}{\pi_{RS,p}(u)^p} \leq \int_G R_{T,b}(\varphi)^p d\mu_\phi(\varphi) = \langle \mu_\phi, R_{T,b}^p \rangle$$

for every $(T, b) \in E \times Z$, it follows that

$$\begin{aligned} \frac{S(u, T, b)^p}{\pi_{RS,p}(u)^p} &= \frac{S(u, T, b)^p}{\pi_{RS,p}(u)^p} \int_G 1_G d\sigma_G = \int_G \frac{S(u, T, b)^p}{\pi_{RS,p}(u)^p} 1_G d\sigma_G \\ &\leq \int_G \langle \mu_\phi, R_{T,b}^p \rangle d\sigma_G(\phi) = \langle \nu, R_{T,b}^p \rangle = \int_G R_{T,b}(\phi)^p d\nu(\phi) \end{aligned}$$

for every $(T, b) \in E \times Z$. Therefore ν is an R - S -abstract measure for u . Next we prove that ν is translation invariant: given $T \in C(G)$ and $\phi_0 \in G$,

$$\begin{aligned} \langle \nu, T^{\phi_0} \rangle &= \int_G \langle \mu_\phi, T^{\phi_0} \rangle d\sigma_G(\phi) = \int_G \int_G T^{\phi_0}(\varphi) d\mu_\phi(\varphi) d\sigma_G(\phi) \\ &= \int_G \int_G T(\phi_0\phi\varphi) d\mu(\varphi) d\sigma_G(\phi) = \int_G \int_G T(\phi_0\phi\varphi) d\sigma_G(\phi) d\mu(\varphi) \\ &= \int_G \int_G T(\phi\varphi) d\sigma_G(\phi) d\mu(\varphi) = \int_G \int_G T(\phi\varphi) d\mu(\varphi) d\sigma_G(\phi) \\ &= \int_G \int_G T(\varphi) d\mu_\phi(\varphi) d\sigma_G(\phi) = \int_G \langle \mu_\phi, T \rangle d\sigma_G(\phi) = \langle \nu, T \rangle. \end{aligned}$$

By the uniqueness of the normalized Haar measure σ_G we conclude that $\nu = \sigma_G$. ■

Note that the translation invariance is imposed on the map $S_{u,b}(\cdot) := S(u, \cdot, b)$ instead of u . Thus, although in applications we always have the information that u is translation invariant, *a priori* our abstract result does not need this hypothesis. An extremal example shows that in fact this choice seems to be adequate: if S is the null mapping then obviously no hypothesis on $u: Y \rightarrow X$ is needed.

4. Applications. In this section we show that Theorem 3.1 applies to several usual classes of R - S -abstract summing mappings, including some well-studied classes of (linear and nonlinear) absolutely summing mappings.

We shall repeatedly use the fact that if G is a compact Hausdorff space and F is a closed subspace of $C(G)$, then

$$\sup_{\alpha \in B_{F^*}} \sum_{j=1}^m |\alpha(T_j)|^p = \sup_{\alpha \in B_{C(G)^*}} \sum_{j=1}^m |\alpha(T_j)|^p = \sup_{x \in G} \sum_{j=1}^m |T_j(x)|^p$$

for all $T_1, \dots, T_m \in F$. The first equality follows from the Hahn–Banach Theorem and the second follows from a canonical argument using *point masses* (cf. [8, p. 41]).

4.1. Absolutely summing linear operators. Let us see that, for linear operators, Theorem 3.1 recovers and generalizes Theorem 2.2. First of all let us see that the assumption that F separates points of G is crucial in the proof that [8] provides for Theorem 2.2.

PROPOSITION 4.1. *Let K a compact Hausdorff space and let F be a closed subspace of $C(K)$ that separates points of K . If Y is a Banach space and $u: F \rightarrow Y$ is a p -summing linear operator, then there is a Borel probability*

measure μ on K such that

$$\|u(f)\|^p \leq \pi_p(u)^p \int_K |f(x)|^p d\mu(x) \quad \text{for every } f \in F.$$

Proof. Consider the mapping

$$\delta: K \rightarrow F^*, \quad \delta(x) = \delta_x: F \rightarrow \mathbb{K}, \quad \delta_x(f) = f(x).$$

It is easy to see that (i) $\delta(K) \subseteq B_{F^*}$, (ii) $\delta(K)$ is norming for F , (iii) $\delta(K)$ is weak* compact in F^* . So, by the Pietsch Domination Theorem [8, Theorem 2.12] there is a Borel probability measure ν on $(\delta(K), w^*)$ such that

$$\|u(f)\|^p \leq \pi_p(u)^p \int_{\delta(K)} |y(f)|^p d\nu(y) \quad \text{for every } f \in F.$$

Since F separates points of K , it follows that δ is injective, and the inverse function $\delta^{-1}: \delta(K) \rightarrow K$ is obviously measurable with respect to the Borel sets in K and in $(\delta(K), w^*)$. Let μ be the image measure with respect to δ^{-1} , that is, μ is a Borel measure on K and $\mu(A) = \nu(\delta(A))$. It is clear that μ is a probability measure and by [11, Proposition 9.1] we have

$$\begin{aligned} \int_K |f(x)|^p d\mu(x) &= \int_{\delta(K)} |f(\delta^{-1}(y))|^p d\nu(y) = \int_{\delta(K)} |\delta(\delta^{-1}(y))(f)|^p d\nu(y) \\ &= \int_{\delta(K)} |y(f)|^p d\nu(y) \geq \frac{\|u(f)\|^p}{\pi_p(u)^p} \end{aligned}$$

for every $f \in F$. ■

The proof of the proposition above makes it clear that the assumption that F separates points of G is needed to validate the assertion made in the first three lines of the proof of [8, Theorem, p. 56]. Let us see that this assumption can be dropped with the help of Theorem 3.1:

THEOREM 4.2. *If X is a Banach space, G is a compact Hausdorff topological group, F is a closed translation invariant subspace of $C(G)$ and $u: F \rightarrow X$ is a translation invariant p -summing linear operator, then the normalized Haar measure σ_G on G is a Pietsch measure for u in the sense that*

$$\|u(f)\| \leq \pi_p(u) \left(\int_G |f(x)|^p d\sigma_G(x) \right)^{1/p} \quad \text{for every } f \in F.$$

Proof. Make the following choice for the parameters of Theorem 3.1:

$$\begin{aligned} E = F = Y, \quad K = G, \quad Z = \mathbb{K}, \quad \mathcal{H} = \mathcal{L}(F; X), \\ R: G \times F \times \mathbb{K} \rightarrow [0, \infty), \quad R(\varphi, T, b) = |T(\varphi)|, \\ S: \mathcal{L}(F; X) \times F \times \mathbb{K} \rightarrow [0, \infty), \quad S(v, T, b) = \|v(T)\|. \end{aligned}$$

Given $T_1, \dots, T_m \in F$ and $b_1, \dots, b_m \in \mathbb{K}$,

$$\begin{aligned} \sum_{j=1}^m S(u, T_j, b_j)^p &= \sum_{j=1}^m \|u(T_j)\|^p \leq \pi_p(u)^p \sup_{\alpha \in B_{F^*}} \sum_{j=1}^m |\alpha(T_j)|^p \\ &= \pi_p(u)^p \sup_{x \in G} \sum_{j=1}^m |T_j(x)|^p = \pi_p(u)^p \sup_{x \in G} \sum_{j=1}^m R(x, T_j, b_j)^p, \end{aligned}$$

which proves that u is R - S -abstract p -summing. Let us see that the conditions of Theorem 3.1 are satisfied:

$$R(\varphi, T^\phi, b) = |T^\phi(\varphi)| = |T(\phi\varphi)| = R(\phi\varphi, T, b)$$

for all $T \in F$, $\varphi, \phi \in G$ and $b \in Z$; and since u is translation invariant,

$$S_{u,b}(T) = S(u, T, b) = \|u(T)\| = \|u(T^\phi)\| = S(u, T^\phi, b) = S_{u,b}(T^\phi)$$

for all $\phi \in G$ and $b \in Z$. So the normalized Haar measure σ_G on G is an R - S -abstract measure for u . Then

$$\begin{aligned} \|u(T)\|^p &= S(u, T, b)^p \leq \pi_{RS,p}(u)^p \int_G R(\phi, T, b)^p d\sigma_G(\phi) \\ &= \pi_{RS,p}(u)^p \int_G |T(\phi)|^p d\sigma_G(\phi) \end{aligned}$$

for every $T \in F$; therefore σ_G is a Pietsch measure for u . ■

4.2. Dominated homogeneous polynomials. For the definition of dominated homogeneous polynomials and the corresponding Pietsch Domination Theorem, see [16, Definition 3.2 and Proposition 3.1] or, without proof, [2, Definition 2.1 and Theorem 3.3].

Let F be a closed translation invariant subspace of $C(G)$ and let $P : F \rightarrow X$ be a translation invariant p -dominated n -homogeneous polynomial. Choose

$$E = F = Y, \quad K = G, \quad Z = \mathbb{K},$$

\mathcal{H} to be the Banach space $\mathcal{P}(^n F; X)$ of continuous n -homogeneous polynomials from F to X with the usual sup norm, and

$$R : G \times F \times \mathbb{K} \rightarrow [0, \infty), \quad R(\varphi, T, b) = |T(\varphi)|,$$

$$S : \mathcal{P}(^n F; X) \times F \times \mathbb{K} \rightarrow [0, \infty), \quad S(Q, T, b) = \|Q(T)\|^{1/n}.$$

Since P is p -dominated, there is a constant C such that, for all $T_1, \dots, T_n \in F$ and $b_1, \dots, b_m \in \mathbb{K}$,

$$\begin{aligned} \sum_{i=1}^k S(P, T_i, b_i)^p &= \sum_{i=1}^k \|P(T_i)\|^{p/n} \leq C^p \sup_{\varphi \in B_{F^*}} \sum_{i=1}^k |\varphi(T_i)|^p \\ &= C^p \sup_{\varphi \in K} \sum_{i=1}^k |T_i(\varphi)|^p = C^p \sup_{\varphi \in K} \sum_{i=1}^k R(\varphi_i, T_i, b_i)^p. \end{aligned}$$

So P is R - S -abstract p -summing. Note also that

$$R(\varphi, T^\phi, b) = |T^\phi(\varphi)| = |T(\phi\varphi)| = R(\phi\varphi, T, b)$$

for all $T \in F$, $\varphi, \phi \in G$ and $b \in Z$, and since P is translation invariant,

$$\begin{aligned} S_{P,b}(T) &= S(P, T, b) = \|P(T)\|^{1/n} \\ &= \|P(T^\phi)\|^{1/n} = S(P, T^\phi, b) = S_{P,b}(T^\phi) \end{aligned}$$

for all $T \in F$ and $b \in Z$. By Theorem 3.1 we find that the normalized Haar measure σ_G on G is an R - S -abstract measure for P . Then

$$\begin{aligned} \|P(T)\|^p &= S(P, T, b)^p \leq \pi_{RS,p}(P)^p \int_G R(\phi, T, b)^p d\sigma_G(\phi) \\ &= \pi_{RS,p}(P)^p \int_G |T(\phi)|^p d\sigma_G(\phi) \end{aligned}$$

for every $T \in F$; therefore σ_G is a Pietsch measure for P .

4.3. α -subhomogeneous mappings. For the definition of α -subhomogeneous mappings and the corresponding Pietsch Domination Theorem we refer to [3, Definition 3.1 and Theorem 2.4].

Let F be a closed translation invariant subspace of $C(G)$ and let $f : F \rightarrow X$ be a translation invariant α -subhomogeneous $(p/\alpha, p)$ -summing mapping. Choose

$$\begin{aligned} E = F = Y, \quad K = G, \quad Z = \mathbb{K}, \quad \mathcal{H} &= \{h : F \rightarrow X : h \text{ is } \alpha\text{-subhomogeneous}\}, \\ R : G \times F \times \mathbb{K} &\rightarrow [0, \infty), \quad R(\varphi, T, b) = |T(\varphi)|, \\ S : \mathcal{H} \times F \times \mathbb{K} &\rightarrow [0, \infty), \quad S(h, T, b) = \|h(T)\|^{1/\alpha}. \end{aligned}$$

Since f is α -subhomogeneous and $(p/\alpha, p)$ -summing, there is a constant C such that, for all $T_1, \dots, T_k \in F$ and $b_1, \dots, b_k \in \mathbb{K}$,

$$\begin{aligned} \sum_{i=1}^k S(f, T_i, b_i)^p &= \sum_{i=1}^k \|f(T_i)\|^{p/\alpha} \leq C^p \sup_{\varphi \in B_{F^*}} \sum_{i=1}^k |\varphi(T_i)|^p \\ &= C^p \sup_{\varphi \in K} \sum_{i=1}^k |T_i(\varphi)|^p = C^p \sup_{\varphi \in K} \sum_{i=1}^k R(\varphi_i, T_i, b_i)^p. \end{aligned}$$

So f is R - S -abstract p -summing. Note also that

$$R(\varphi, T^\phi, b) = |T^\phi(\varphi)| = |T(\phi\varphi)| = R(\phi\varphi, T, b)$$

for every $T \in F$, $\varphi, \phi \in G$ and $b \in Z$, and as f is translation invariant,

$$S_{f,b}(T) = S(f, T, b) = \|f(T)\|^{1/\alpha} = \|f(T^\phi)\|^{1/\alpha} = S(f, T^\phi, b) = S_{f,b}(T^\phi)$$

for every $T \in F$ and $b \in Z$. By Theorem 3.1 we infer that the normalized Haar measure σ_G on G is an R - S -abstract measure for f . Then

$$\begin{aligned} \|f(T)\|^{p/\alpha} &= S(f, T, b)^p \leq \pi_{RS,p}(f)^p \int_G R(\phi, T, b)^p d\sigma_G(\phi) \\ &= \pi_{RS,p}(f)^p \int_G |T(\phi)|^p d\sigma_G(\phi) \end{aligned}$$

for every $T \in F$; therefore σ_G is a Pietsch measure for f .

4.4. Absolutely summing arbitrary mappings. Let X and F be Banach spaces. Following [4, Definition 2.1] (see also [17, Definition 3.1]), an arbitrary mapping $f: F \rightarrow X$ is *absolutely p -summing at $a \in F$* if there is a $C \geq 0$ such that

$$\sum_{j=1}^m \|f(a + x_j) - f(a)\|^p \leq C \sup_{\varphi \in B_{E'}} \sum_{j=1}^m |\varphi(x_j)|^p$$

for every natural number m and all $x_1, \dots, x_m \in F$. The Pietsch Domination Theorem for absolutely p -summing mappings at $a \in F$ can be found in [4, Theorem 4.2].

Assume that F is a closed translation invariant subspace of $C(G)$ and let $f: F \rightarrow X$ be a translation invariant absolutely p -summing mapping at $0 \in F$. Choose

$$\begin{aligned} E &= F = Y, & K &= G, & Z &= \mathbb{K}, & \mathcal{H} &= X^F = \{h: F \rightarrow X\}, \\ R: G \times F \times \mathbb{K} &\rightarrow [0, \infty), & R(\varphi, T, b) &= |T(\varphi)|, \\ S: \mathcal{H} \times F \times \mathbb{K} &\rightarrow [0, \infty), & S(h, T, b) &= \|h(T) - h(0)\|. \end{aligned}$$

Since f is absolutely p -summing at $0 \in F$, there is a constant C such that, for all $T_1, \dots, T_k \in F$ and $b_1, \dots, b_k \in \mathbb{K}$,

$$\begin{aligned} \sum_{i=1}^k S(f, T_i, b_i)^p &= \sum_{i=1}^k \|f(T_i) - f(0)\|^p \leq C^p \sup_{\varphi \in B_{F^*}} \sum_{i=1}^k |\varphi(T_i)|^p \\ &= C^p \sup_{\varphi \in K} \sum_{i=1}^k |T_i(\varphi)|^p = C^p \sup_{\varphi \in K} \sum_{i=1}^k R(\varphi_i, T_i, b_i)^p. \end{aligned}$$

So f is R - S -abstract p -summing. Note also that

$$R(\varphi, T^\phi, b) = |T^\phi(\varphi)| = |T(\phi\varphi)| = R(\phi\varphi, T, b)$$

for all $T \in F$, $\varphi, \phi \in G$ and $b \in Z$. Since f is translation invariant,

$$\begin{aligned} S_{f,b}(T^\phi) &= S(f, T^\phi, b) = \|f(T^\phi) - f(0)\| \\ &= \|f(T) - f(0)\| = S(f, T, b) = S_{f,b}(T) \end{aligned}$$

for all $T \in F$ and $b \in Z$. By Theorem 3.1 the normalized Haar measure σ_G on G is an R - S -abstract measure for f . Then

$$\begin{aligned} \|f(T) - f(0)\|^p &= S(f, T, b)^p \leq \pi_{RS,p}(f)^p \int_G R(\phi, T, b)^p d\sigma_G(\phi) \\ &= \pi_{RS,p}(f)^p \int_G |T(\phi)|^p d\sigma_G(\phi) \end{aligned}$$

for every $T \in F$; therefore σ_G is a Pietsch measure for f .

5. Open problems. For the definition of *Lipschitz p -summing mappings* and the corresponding Pietsch Domination Theorem we refer to [10, 4].

PROBLEM. Let F be a closed translation invariant subspace of $C(G)$, let X be a metric space and $f: F \rightarrow X$ be a translation invariant Lipschitz p -summing mapping. Is the Haar measure σ_G a Pietsch measure for f ?

In the case of absolutely summing arbitrary mappings (Subsection 4.4) we assumed that the translation invariant mapping $f: F \subseteq C(G) \rightarrow X$ is absolutely p -summing at the origin. What about translation invariant mappings that are absolutely p -summing at some $0 \neq T_0 \in F$? We say that a vector $T_0 \in F$ is *translation invariant* if $T_0^\phi = T_0$ for every $\phi \in G$. Let $T_0 \in F$. Define

$$f_{T_0}: F \rightarrow X, \quad f_{T_0}(T) = f(T + T_0).$$

It is easy to see that f is absolutely p -summing at T_0 if and only if f_{T_0} is absolutely p -summing at the origin. Moreover, if T_0 is translation invariant then f is translation invariant if and only if f_{T_0} is translation invariant. Thus, if we assume that f is translation invariant and absolutely p -summing at T_0 , then f_{T_0} is translation invariant and absolutely p -summing at the origin. Therefore the Haar measure is a Pietsch measure for f_{T_0} and hence for f . However, we need the extra assumption that the vector T_0 is translation invariant. Can this assumption be dropped?

PROBLEM. Let F be a closed translation invariant subspace of $C(G)$ and let $f: F \rightarrow X$ be a translation invariant mapping that is absolutely p -summing at some vector of F . Is the Haar measure σ_G a Pietsch measure for f ?

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