

Operators on a Hilbert space similar to a part of the backward shift of multiplicity one

by

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Abstract. Let $A : X \rightarrow X$ be a bounded operator on a separable complex Hilbert space X with an inner product $\langle \cdot, \cdot \rangle_X$. For $b, c \in X$, a weak resolvent of A is the complex function of the form $\langle (I - zA)^{-1}b, c \rangle_X$. We will discuss an equivalent condition, in terms of weak resolvents, for A to be similar to a restriction of the backward shift of multiplicity 1.

1. Introduction. We characterize an operator on a separable complex Hilbert space which is similar to a part of the backward shift of multiplicity 1, in terms of cyclic vectors and weak resolvents belonging to the Hardy space \mathbb{H}^∞ . We base our development on the work of Fuhrmann [6], Helton [7], Nikol'skiĭ [10], and Radjavi and Rosenthal [12]. We hope that this note would serve to unify and refine some results in the previous works mentioned above. What plays the central role is the concept of weak resolvent introduced by Fong, Nordgren, Radjavi and Rosenthal [5], [11], [13]. The weak resolvents here correspond to a class of characteristic functions of the Sz.-Nagy–Foias operator model theory [16]. Note that this operator model theory has been related to the Lax–Phillips scattering theory by Adamjan and Arov [1]. See also Helton [7] and Uetake [18].

For $A : X \rightarrow X$ a bounded operator on a complex Hilbert space X with inner product $\langle \cdot, \cdot \rangle_X$ and $b, c \in X$, we define a *weak resolvent* of A to be

$$\langle (I - zA)^{-1}b, c \rangle_X.$$

Here I denotes the identity map on X . For this see also Mlak [9] and Uetake [17], [18].

Let $\mathbb{T} = \{z : |z| = 1\}$, the unit circle in the complex plane, and $\mathbb{D} = \{z : |z| < 1\}$, the open unit disc. We use the following two Hardy spaces:

$$\mathbb{H}^2 = \left\{ f : f(z) = \sum_{n=0}^{\infty} f_n z^n, z \in \mathbb{D}, \right.$$

$$\left. \|f\|_2 = \lim_{r \rightarrow 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\}^{1/2} < \infty \right\},$$

$$\mathbb{H}^\infty = \left\{ f : f(z) = \sum_{n=0}^{\infty} f_n z^n, z \in \mathbb{D}, \right.$$

$$\left. \|f\|_\infty = \lim_{r \rightarrow 1} \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})| < \infty \right\} \subset \mathbb{H}^2.$$

Let \mathbb{L}^2 denote the Hilbert space of square integrable functions on \mathbb{T} , equipped with the inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = \sum_{n=-\infty}^{\infty} f_n \overline{g_n}$$

for $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$, $z \in \mathbb{T}$. Note that $\{z^n : z \in \mathbb{T}, n \in \mathbb{Z}\}$ is an orthonormal basis of \mathbb{L}^2 . Then \mathbb{H}^2 is isometrically isomorphic to $\{f \in \mathbb{L}^2 : f_n = 0, n < 0\} \subset \mathbb{L}^2$. Therefore we identify \mathbb{H}^2 with this subspace of \mathbb{L}^2 in the natural way and denote the inner product of \mathbb{H}^2 also by $\langle \cdot, \cdot \rangle$.

The *unilateral shift of multiplicity 1*, denoted by $S : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, is defined by $(Sf)(z) = zf(z)$, and its adjoint, the *backward shift of multiplicity 1*, denoted by $S^* : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, is defined by

$$(S^*f)(z) = \frac{f(z) - f(0)}{z} = \sum_{n=0}^{\infty} f_{n+1} z^n.$$

A subspace $K \subseteq \mathbb{H}^2$ is said to be *S-invariant* (resp. *S*-invariant*) if $SK \subseteq K$ (resp. $S^*K \subseteq K$). For an *S*-invariant* subspace $K \subseteq \mathbb{H}^2$, $S^*|_K : K \rightarrow K$ is called a *part* (or restriction) of the backward shift of multiplicity 1.

To state the main theorem we need to define some notions of cyclicity. Let $T : H \rightarrow H$ be a bounded operator on a separable complex Hilbert space H . Let $E_0 = \{(u_n)_{n=0}^{\infty} : u_n = 0 \text{ for all } n \geq N \text{ for some } N\} \subset l^2 = l^2(0, \infty)$. Given $v \in H$, define $\mathcal{C} : E_0 \rightarrow H$ by $\mathcal{C}u = \sum_{n=0}^{\infty} u_n T^n v$ for $u = (u_n)_{n=0}^{\infty} \in E_0$. Note that the sum is actually finite. Let us call \mathcal{C} the *cyclicity operator* for v and T . For this operator, see also Helton [7] and Fuhrmann [6]. Note that \mathcal{C} is not necessarily bounded.

DEFINITION 1.1. v is *cyclic* for T if $\overline{\text{span}\{T^n v : n \geq 0\}} = H$, or, equivalently, $\overline{\mathcal{C}(E_0)} = H$.

Here $\overline{\text{span}}$ denotes the closure of the linear span.

DEFINITION 1.2. v is *continuously cyclic* for T if v is cyclic for T and \mathcal{C} is continuous (i.e. bounded).

Note that if $\mathcal{C} : E_0 \rightarrow H$ is continuous, then \mathcal{C} extends to a continuous operator from l^2 to H in the natural way, since E_0 is dense in l^2 . Therefore $\mathcal{C} : E_0 \rightarrow H$ is continuous if and only if it has a continuous extension $\overline{\mathcal{C}} : l^2 \rightarrow H$. If v is continuously cyclic for T , then for the extended \mathcal{C} , $\overline{\mathcal{C}}(l^2) = H$.

DEFINITION 1.3. v is *strongly cyclic* for T if v is continuously cyclic for T and the extended $\mathcal{C} : l^2 \rightarrow H$ is surjective.

2. The main theorem. We can now state our main theorem.

THEOREM 2.1 (Main Theorem). *Let $A : X \rightarrow X$ be a bounded operator on a separable complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_X$. There exists an S^* -invariant closed subspace $K \subseteq \mathbb{H}^2$ such that $A : X \rightarrow X$ is similar to $S^*|_K : K \rightarrow K$, i.e. $A = T^{-1}(S^*|_K)T$ for some bounded bijection $T : X \rightarrow K$, if and only if there exist $b, c \in X$ such that b is continuously cyclic for A , c is strongly cyclic for A^* and the weak resolvent $\varphi(z) = \langle (I - zA)^{-1}b, c \rangle_X$ is in \mathbb{H}^∞ .*

The following corollary is an immediate consequence of the above theorem.

COROLLARY 2.2. *Let A be a bounded operator on a separable complex Hilbert space X with an inner product $\langle \cdot, \cdot \rangle_X$. If there exist $b, c \in X$ such that b is continuously cyclic for A , c is strongly cyclic for A^* and the weak resolvent $\varphi(z) = \langle (I - zA)^{-1}b, c \rangle_X$ is in \mathbb{H}^∞ , then $r(A) \leq 1$, where $r(A)$ denotes the spectral radius of A .*

Proof. By Theorem 2.1, A is similar to $S^*|_K$, a part of the backward shift of multiplicity 1 for some $K \subseteq \mathbb{H}^2$. Since $\|S^*|_K\| \leq 1$, the assertion readily follows. ■

The above theorem is an extension of the following theorem of Radjavi and Rosenthal to separable infinite-dimensional Hilbert spaces.

THEOREM 2.3 (Radjavi and Rosenthal [12, Theorem 3.27, p. 54]). *Let A be an operator on a finite-dimensional complex Hilbert space. Then A is similar to a part of S^* of multiplicity 1 if and only if A is cyclic (or, equivalently, the characteristic and minimal polynomials coincide) and $r(A) < 1$.*

Here A is said to be *cyclic* if there exists a cyclic vector for A . One can use the following theorem to deduce Theorem 2.3 from Theorem 2.1.

THEOREM 2.4 (Uetake [17, Theorem 2]). *Let A be a bounded operator on a separable complex Hilbert space X with an inner product $\langle \cdot, \cdot \rangle_X$. Let*

$b \in X$ be cyclic for A and $c \in X$ be cyclic for A^* . Suppose that $(\lambda I - A)^{-1}$ is meromorphic in an open neighborhood of λ_0 . Then $\psi(\lambda) = \langle (\lambda I - A)^{-1}b, c \rangle_X$ has a pole of order m at λ_0 if and only if $(\lambda I - A)^{-1}$ has a pole of order m at λ_0 .

Proof of Theorem 2.3. For A on a finite-dimensional space, A is cyclic if and only if A^* is cyclic. To see this consider their characteristic and minimal polynomials. Since $\varphi(z) = \lambda\psi(\lambda)$ with $z = \lambda^{-1}$, it follows that $\varphi(z)$ has no pole in the closed unit disc $\overline{\mathbb{D}}$ if and only if $\psi(\lambda)$ has no pole in $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathbb{D}}$. For a rational $\varphi = \varphi(z)$, $\varphi \in \mathbb{H}^\infty$ if and only if φ has no pole in $\overline{\mathbb{D}}$. From these observations and Theorems 2.1 and 2.4, the assertion of the theorem readily follows. ■

We should mention the following two pioneering theorems in operator model theory. For these and further developments, see Radjavi and Rosenthal [12, §3.5] and Sz.-Nagy and Foiaş [16].

THEOREM 2.5 (Rota [14]). *Let A be a bounded operator on a complex Hilbert space of dimension equal to the cardinal number α . If $r(A) < 1$, then A is similar to a part of the backward shift S^* of multiplicity α .*

THEOREM 2.6 (Foiaş [4] and de Branges and Rovnyak [2]). *Let A be a bounded operator on a complex Hilbert space X such that $\|A\| \leq 1$ and $A^n x \rightarrow 0$ for all $x \in X$, and α be the dimension of the closure of the range of $I - A^*A$. Then A is a part of the backward shift S^* of multiplicity α .*

3. Proof of the main theorem. To prove the main theorem, we need some theorems of Nikol'skiĭ and Helton. The following theorem is taken from Nikol'skiĭ [10]. We include the proof for the sake of completeness.

THEOREM 3.1 (Nicol'skiĭ [10, p. 37]). *Every S^* -invariant closed proper subspace $K \subset \mathbb{H}^2$ (i.e. $K \neq \mathbb{H}^2$) has a vector $f \in K$ that is cyclic for $S^*|_K$.*

Proof. Since $K^\perp = \mathbb{H}^2 \ominus K \neq \{0\}$ is closed and S -invariant, there exists an inner Θ (i.e. $\Theta \in \mathbb{H}^\infty$, $|\Theta(z)| = 1$ a.e. on \mathbb{T}) such that $K^\perp = \Theta\mathbb{H}^2$ by the celebrated Beurling Theorem (see e.g. [15, p. 376]). Thus $K = (\Theta\mathbb{H}^2)^\perp$. Set $f = S^*\Theta$. Then for any $h \in \mathbb{H}^2$, $\langle S^*\Theta, \Theta h \rangle = \langle \Theta, S\Theta h \rangle = \langle \Theta, z\Theta h \rangle = \langle 1, \overline{\Theta}\Theta zh \rangle = \langle 1, zh \rangle = 0$. Thus $f \in K$.

Now suppose that for some $k \in K$, $\langle (S^*|_K)^n f, k \rangle = \langle S^{*n+1}\Theta, k \rangle = 0$ for all $n \geq 0$. Note that $z^{-(n+1)} = \overline{z}^{n+1} \in \mathbb{L}^2$. Since $\langle z^{-(n+1)}, \overline{\Theta}k \rangle = \langle \overline{z}^{n+1}, \overline{\Theta}k \rangle = \langle \Theta, z^{n+1}k \rangle = \langle \Theta, S^{n+1}k \rangle = \langle S^{*n+1}\Theta, k \rangle = 0$ for all $n \geq 0$ for this k , we have $\overline{\Theta}k \in \mathbb{H}^2$. However, since $\langle \Theta h, k \rangle = 0$ for any $h \in \mathbb{H}^2$, substituting $h = \overline{\Theta}k$, we have $\langle \Theta\overline{\Theta}k, k \rangle = \langle k, k \rangle = 0$. Therefore $k = 0$. This means that f is cyclic for $S^*|_K$. ■

To proceed to Lemma 3.2, recall that

$$\text{BMO} = \left\{ f \in \mathbb{L}^1 : \sup_I \frac{1}{|I|} \int_I |f(e^{i\theta}) - f_I| d\theta < \infty, f_I = \frac{1}{|I|} \int_I f(e^{i\theta}) d\theta \right\}.$$

For BMO, see e.g. [10, p. 376].

LEMMA 3.2. *For an S^* -invariant closed subspace $K \subseteq \mathbb{H}^2$, suppose that $f \in K$ is cyclic for $S^*|K$. If $f \in \mathbb{H}^\infty$, then f is continuously cyclic for $S^*|K$.*

Proof. Since $\mathbb{H}^\infty \subset \text{BMO}$ as is easy to check, $f \in \text{BMO}$. Let $f = \sum_{n=0}^\infty f_n z^n$. Then the cyclicity operator \mathcal{C} can be represented by the infinite Hankel matrix (f_{m+n}) if one regards \mathbb{H}^2 as l^2 under the natural correspondence $\mathbb{H}^2 \ni \sum_{n=0}^\infty h_n z^n \mapsto (h_n)_{n=0}^\infty \in l^2$. Note that this correspondence gives an isometrical isomorphism between \mathbb{H}^2 and l^2 . The lemma follows from Nehari's theorem that (f_{m+n}) is bounded if and only if $f \in \text{BMO}$ (see e.g. [10, p. 376, p. 408]). ■

Throughout let $e_0 = 1 \in \mathbb{H}^2$. For a closed subspace $K \subseteq \mathbb{H}^2$, let P_K denote the orthogonal projection of \mathbb{H}^2 onto K .

LEMMA 3.3. *For an S^* -invariant closed subspace $K \subseteq \mathbb{H}^2$, set $g = P_K e_0$. Then g is strongly cyclic for $(S^*|K)^*$.*

Proof. Define the cyclicity operator $\mathcal{O} : l^2 \rightarrow K$ for g and $(S^*|K)^*$ by

$$\mathcal{O}y = \sum_{n=0}^\infty y_n (S^*|K)^{*n} g$$

for $y = (y_n)_{n=0}^\infty \in l^2$. We show that \mathcal{O} is continuous and surjective.

For any $k \in K$, $\sum_{n=0}^\infty \langle k, S^n e_0 \rangle S^n e_0 = k$. Note that $S^n e_0 = z^n \in \mathbb{H}^2$. Since $P_K k = k$, we have $\langle k, S^n e_0 \rangle = \langle P_K k, S^n e_0 \rangle = \langle k, P_K S^n e_0 \rangle$ and so

$$\sum_{n=0}^\infty \langle k, P_K S^n e_0 \rangle P_K S^n e_0 = k.$$

Recall that K is S^* -invariant and hence is $(S^*|K)^*$ -invariant as well. So for any $h \in \mathbb{H}^2$ we have

$$\begin{aligned} \langle P_K S^n e_0, h \rangle &= \langle S^n e_0, P_K h \rangle = \langle e_0, S^{*n} P_K h \rangle = \langle e_0, P_K (S^*|K)^n P_K h \rangle \\ &= \langle P_K e_0, (S^*|K)^n P_K h \rangle = \langle g, (S^*|K)^n P_K h \rangle \\ &= \langle (S^*|K)^{*n} g, P_K h \rangle = \langle P_K (S^*|K)^{*n} g, h \rangle = \langle (S^*|K)^{*n} g, h \rangle. \end{aligned}$$

Therefore we have

$$(S^*|K)^{*n} g = P_K S^n e_0.$$

Now take an arbitrary $y = (y_n)_{n=0}^\infty \in l^2$. Then $h = h(z) = \sum_{n=0}^\infty y_n z^n \in \mathbb{H}^2$. Recall that this correspondence gives the natural isometrical isomor-

phism between l^2 and \mathbb{H}^2 . Recall also that $S^n e_0 = z^n \in \mathbb{H}^2$. Since

$$\sum_{n=0}^{\infty} y_n S^n e_0 = \sum_{n=0}^{\infty} y_n z^n = h,$$

we have

$$\sum_{n=0}^{\infty} y_n (S^*|K)^{*n} g = \sum_{n=0}^{\infty} y_n P_K S^n e_0 = P_K h \in K.$$

From this we see that \mathcal{O} is continuous and surjective. Therefore by definition, g is strongly cyclic for $(S^*|K)^*$. ■

LEMMA 3.4. *Let K be an S^* -invariant closed subspace of \mathbb{H}^2 . For $f \in K$ and $g = P_K e_0$, we have $\langle (I_K - z(S^*|K))^{-1} f, g \rangle = f(z)$, $z \in \mathbb{D}$. Here I_K denotes the identity map on K .*

Proof. Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Since $S^*|K$ is a contraction, for $z \in \mathbb{D}$ we have

$$\begin{aligned} \langle (I_K - z(S^*|K))^{-1} f, g \rangle &= \left\langle \sum_{n=0}^{\infty} (zS^*|K)^n f, P_K e_0 \right\rangle = \left\langle P_K \sum_{n=0}^{\infty} (zS^*|K)^n f, e_0 \right\rangle \\ &= \left\langle \sum_{n=0}^{\infty} (zS^*|K)^n f, e_0 \right\rangle = \sum_{n=0}^{\infty} \langle (S^*|K)^n f, e_0 \rangle z^n \\ &= \sum_{n=0}^{\infty} f_n z^n = f(z). \end{aligned}$$

Here we view $(I_K - z(S^*|K))^{-1} f = \sum_{n=0}^{\infty} (zS^*|K)^n f$ as an element of $K \subseteq \mathbb{H}^2$ which depends on the parameter z . ■

To give the proof of the main theorem, we need the following theorem of Helton. Again for the sake of completeness, we give a detailed proof, focusing on the construction of the similarity operator.

THEOREM 3.5 (Helton [7]). *For $i = 1, 2$, let X_i be a separable complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{X_i}$. Let bounded operators $A_i : X_i \rightarrow X_i$ and vectors $b_i, c_i \in X_i$, $i = 1, 2$, be such that b_i is continuously cyclic for A_i , c_i is strongly cyclic for A_i^* and $\langle A_1^n b_1, c_1 \rangle_{X_1} = \langle A_2^n b_2, c_2 \rangle_{X_2}$ for all $n \geq 0$. Then A_2 is similar to A_1 .*

Proof. Throughout this proof let $i = 1$ or 2 . Let $\mathcal{C}_i u = \sum_{n=0}^{\infty} u_n A_i^n b_i$ for $u = (u_n)_{n=0}^{\infty} \in l^2$ and $\mathcal{O}_i y = \sum_{n=0}^{\infty} y_n A_i^{*n} c_i$ for $y = (y_n)_{n=0}^{\infty} \in l^2$. Note that \mathcal{C}_i and \mathcal{O}_i are continuous by assumption. Thus $\mathcal{O}_i^* : X_i \rightarrow l^2$ is also continuous (see e.g. [3, p. 31]). It is easy to see that $\mathcal{O}_i^* x_i = (\langle A_i^n x_i, c_i \rangle_{X_i})_{n=0}^{\infty}$

for $x_i \in X_i$. Thus it is easy to check that

$$\mathcal{O}_i^* \mathcal{C}_i u = \left(\left\langle A_i^n \sum_{m=0}^{\infty} u_m A_i^m b_i, c_i \right\rangle_{X_i} \right)_{n=0}^{\infty} = \left(\sum_{m=0}^{\infty} \langle A_i^{n+m} b_i, c_i \rangle_{X_i} u_m \right)_{n=0}^{\infty}.$$

Therefore $\mathcal{O}_1^* \mathcal{C}_1 = \mathcal{O}_2^* \mathcal{C}_2$. Note that $\text{im } \mathcal{O}_1^* \mathcal{C}_1 = \text{im } \mathcal{O}_2^* \mathcal{C}_2 \subseteq \text{im } \mathcal{O}_1^* \cap \text{im } \mathcal{O}_2^* \subseteq \text{im } \mathcal{O}_i^*$.

Since \mathcal{O}_1 is a bounded surjection and $\ker \mathcal{O}_1^* = (\text{im } \mathcal{O}_1)^\perp = \{0\}$, it follows that $\mathcal{O}_1^* : X_1 \rightarrow l^2$ is a bounded injection. Note that $\text{im } \mathcal{O}_1^* = (\ker \mathcal{O}_1)^\perp$ and so $\text{im } \mathcal{O}_1^*$ is closed. Therefore by the Inverse Mapping Theorem (see e.g. [3, p. 94]), there exists a bounded bijection $\mathcal{O}_1^{*L} : \text{im } \mathcal{O}_1^* \rightarrow X_1$ such that $\mathcal{O}_1^{*L} \mathcal{O}_1^* = I_{X_1}$. Thus $\mathcal{C}_1 = \mathcal{O}_1^{*L} \mathcal{O}_2^* \mathcal{C}_2$. There exists a bijection $\mathcal{C}_2^R : \text{im } \mathcal{C}_2 \rightarrow (\ker \mathcal{C}_2)^\perp$ such that $\mathcal{C}_2 \mathcal{C}_2^R = I : \text{im } \mathcal{C}_2 \rightarrow \text{im } \mathcal{C}_2$. Note that \mathcal{C}_2^R need not be bounded. Since $\mathcal{O}_2^* : X_2 \rightarrow \text{im } \mathcal{O}_2^*$ and $\mathcal{O}_1^{*L} : \text{im } \mathcal{O}_1^* \rightarrow X_1$ have bounded inverses, it follows that $\mathcal{C}_1 \mathcal{C}_2^R = \mathcal{O}_1^{*L} \mathcal{O}_2^* : \text{im } \mathcal{C}_2 \rightarrow \mathcal{O}_1^{*L} \mathcal{O}_2^*(\text{im } \mathcal{C}_2)$ is bounded and has a bounded inverse.

Since b_1 is cyclic for A_1 , for every $x_1 \in X_1$ there exists a sequence $u^{(n)} \in l^2$ such that $\text{im } \mathcal{C}_1 \ni x_1^{(n)} = \mathcal{C}_1 u^{(n)} \rightarrow x_1$. Since $\mathcal{O}_1^* \mathcal{C}_1 = \mathcal{O}_2^* \mathcal{C}_2$, by setting $x_2^{(n)} = \mathcal{C}_2 u^{(n)} \in \text{im } \mathcal{C}_2$ we have $\mathcal{O}_1^* x_1^{(n)} = \mathcal{O}_2^* x_2^{(n)}$. Thus $x_1^{(n)} = \mathcal{O}_1^{*L} \mathcal{O}_2^* x_2^{(n)}$. This means that $\mathcal{O}_1^{*L} \mathcal{O}_2^*(\text{im } \mathcal{C}_2)$ is dense in X_1 . Of course $\text{im } \mathcal{C}_2$ is also dense in X_2 by assumption. Therefore $T = \mathcal{C}_1 \mathcal{C}_2^R = \mathcal{O}_1^{*L} \mathcal{O}_2^*$ extends to an invertible bounded bijection from X_2 to X_1 in the natural way.

It is also easily checked that $\mathcal{O}_1^* A_1 \mathcal{C}_1 = \mathcal{O}_2^* A_2 \mathcal{C}_2$. Thus $A_1 T = T A_2$ and so $A_2 = T^{-1} A_1 T$. ■

Now we can give the proof of the main theorem.

Proof of Theorem 2.1. (If) Set $f_0 = \varphi \in \mathbb{H}^\infty$ and let $K = \overline{\text{span}}\{S^{*n} f_0 : n \geq 0\} \subseteq \mathbb{H}^2$. Here the closure is taken in the topology of \mathbb{H}^2 . Obviously, K is S^* -invariant and closed, and f_0 is cyclic for $S^*|K$. Moreover, by Lemma 3.2, f_0 is continuously cyclic for $S^*|K$.

Set $g = P_K e_0$. By Lemma 3.3, g is strongly cyclic for $(S^*|K)^*$. By Lemma 3.4, $\langle (I_K - z(S^*|K))^{-1} f_0, g \rangle = f_0(z) = \varphi(z)$. Thus all the conditions in Theorem 3.5 are satisfied for $A_1 = S^*|K$, $b_1 = f_0 \in X_1$, $c_1 = g \in X_1$ where $X_1 = K$ and $A_2 = A$, $b_2 = b \in X_2$, $c_2 = c \in X_2$ where $X_2 = X$, and so A can be recovered from $S^*|K$ by $T : X \rightarrow K$ as constructed in the proof of Theorem 3.5.

(Only if) Let $T : X \rightarrow K$ be a bounded bijection such that $K \subseteq \mathbb{H}^2$ is S^* -invariant and closed, and $A = T^{-1}(S^*|K)T$.

If $K = \mathbb{H}^2$, then, for example, $f = f(z) = e^{1/(z-2)} \in \mathbb{H}^\infty$ is cyclic for $S^* = S^*|\mathbb{H}^2$, since f does not have a meromorphic pseudocontinuation into $(\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathbb{D}}$. See [10, pp. 30–32]. As $f \in \mathbb{H}^\infty$, it is continuously cyclic for S^* by Lemma 3.2.

If $K \neq \mathbb{H}^2$, then we can use Theorem 3.1 and set $f = S^*\Theta$ as in the proof of that theorem. Since $\Theta \in \mathbb{H}^\infty$ and $f(z) = (S^*\Theta)(z) = (\Theta(z) - \Theta(0))/z$, we see that $f \in \mathbb{H}^\infty$. Thus by Lemma 3.2, f is continuously cyclic for $S^*|K$.

In both cases, set $g = P_Ke_0$. By Lemma 3.3, g is strongly cyclic for $(S^*|K)^*$. Now let $b = T^{-1}f, c = T^*g$. Then it is clear that b is continuously cyclic for A , and c is strongly cyclic for A^* . By Lemma 3.4, we have

$$\begin{aligned} \langle (I - zA)^{-1}b, c \rangle_X &= \langle (I - zT^{-1}(S^*|K)T)^{-1}T^{-1}f, T^*g \rangle_X \\ &= \langle (I_K - z(S^*|K))^{-1}f, g \rangle = f(z) \in \mathbb{H}^\infty. \blacksquare \end{aligned}$$

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