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Local integrability of strong and iterated maximal functions

by

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Abstract. Let $M_{\rm S}$ denote the strong maximal operator. Let M_x and M_y denote the one-dimensional Hardy–Littlewood maximal operators in the horizontal and vertical directions in $\mathbb{R}^2$. A function h supported on the unit square $Q = [0,1] \times [0,1]$ is exhibited such that $\int_Q M_y M_x h < \infty$ but $\int_Q M_x M_y h = \infty$. It is shown that if f is a function supported on Q such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, then there exists a set A of finite measure in $\mathbb{R}^2$ such that $\int_A M_{\rm S} f = \infty$.

We begin by listing some basic definitions.

DEFINITION 1. Let f be a measurable function defined on $\mathbb{R}^n$. Denote by $B(p,r)$ the Euclidean ball in $\mathbb{R}^n$ centered at p of radius r, and by $|B(p,r)|$ the Lebesgue measure of $B(p,r)$. The *Hardy–Littlewood maximal function* of f is defined on $\mathbb{R}^n$ by

$$M_{\rm HL} f(p) = \sup_{r>0} \frac{1}{|B(p,r)|} \int\limits_{B(p,r)} |f(z)|\, dz. \tag{1}$$

DEFINITION 2. Let f be a measurable function defined on $\mathbb{R}^2$. The *strong maximal function* of f is defined on $\mathbb{R}^2$ by

$$M_{\rm S} f(x,y) = \sup_{\substack{x_1<x<x_2\\ y_1<y<y_2}} \frac{1}{(x_2-x_1)(y_2-y_1)} \int\limits_{x_1}^{x_2} \int\limits_{y_1}^{y_2} |f(u,v)|\, dv\, du. \tag{2}$$

DEFINITION 3. Let f be a measurable function defined on $\mathbb{R}^2$. The *horizontal maximal function* of f is defined on $\mathbb{R}^2$ by

$$M_x f(u,v) = \sup_{u_1<u<u_2} \frac{1}{u_2-u_1} \int\limits_{u_1}^{u_2} |f(w,v)|\, dw. \tag{3}$$

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Definition 4. Let f be a measurable function defined on $\mathbb{R}^2$. The *vertical maximal function* of f is defined on $\mathbb{R}^2$ by

$$M_y f(u,v) = \sup_{v_1<v<v_2} \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} |f(u,w)|\,dw. \tag{4}$$

The following Orlicz spaces will be very useful to us.

Definition 5. Let $I^{(n)}$ denote the unit n-cube $[0,1]\times\ldots\times[0,1]$ in $\mathbb{R}^n$. $L(\log L)^k(I^{(n)})$ is the Lebesgue space of functions on $I^{(n)}$ with norm

$$\|f\|_{L(\log L)^k(I^{(n)})} = \inf\left\{c>0 : \int_{I^{(n)}} \frac{|f|}{c}\log\left(3+\frac{|f|}{c}\right)^k \le 1\right\} < \infty. \tag{5}$$

In the following we will denote the unit square $I^{(2)}$ in $\mathbb{R}^2$ by Q. Also, the k-fold iteration $M_{\mathrm{HL}}\circ\ldots\circ M_{\mathrm{HL}}$ of the maximal operator M_{HL} will be denoted by M_{HL}^k.

We now recall the following theorem of E. M. Stein:

Theorem 6 ([11]). *Let k be a positive integer. There exist constants $0<c=c(k,n)<C=C(k,n)<\infty$ such that if f is supported on $I^{(n)}$, then*

$$c\int_{I^{(n)}} M_{\mathrm{HL}}^k f \le \|f\|_{L(\log L)^k(I^{(n)})} \le C\int_{I^{(n)}} M_{\mathrm{HL}}^k f. \tag{6}$$

Inequalities such as (6) in the case $k=1$ will often be denoted by

$$\|f\|_{L\log L(I^{(n)})} \sim \int_{I^{(n)}} M_{\mathrm{HL}} f$$

for the remainder of this paper.

We now show that there exists a function h supported on Q such that $\int_Q M_y M_x h < \infty$, but $\int_Q M_x M_y h = \infty$.

Theorem 7. *There exists a function h, supported on Q, such that $\int_Q M_x M_y h = \infty$, but $\int_Q M_y M_x h < \infty$.*

Proof. We define the functions h_{2^n} as follows:

$$h_{2^n}(x,y) = \sum_{m=0}^{2^n-1} 2^{2^n-m-1}\chi_{[0,2^{-2^n+m+1}]}(x)\cdot\chi_{[m\cdot 2^{-n},(m+1)\cdot 2^{-n}]}(y). \tag{7}$$

The functions h_2, h_4, and h_8 are depicted in Figure 1.

Lemma 8.

$$\int_Q M_x M_y h_{2^n} \sim n\cdot 2^n. \tag{8}$$

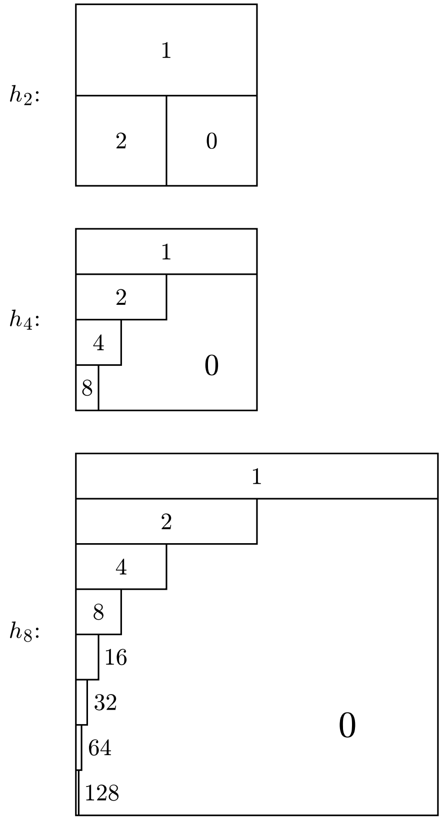

Fig. 1. Functions h_2, h_4, and h_8

Proof. Divide Q into 2^{2n} regions $I_{j,k}$, $j = 1, \ldots, 2^n$, $k = 1, \ldots, 2^n$, where $I_{j,k}$ is defined as follows:

$$I_{1,k} = [0, 2^{-2^n+1}] \times [(k-1) \cdot 2^{-n}, k \cdot 2^{-n}], \qquad 1 \le k \le 2^n;$$

$$I_{j,k} = [2^{-2^n+j-1}, 2^{-2^n+j}] \times [(k-1) \cdot 2^{-n}, k \cdot 2^{-n}], \qquad 2 \le j \le 2^n,\ 1 \le k \le 2^n.$$

Let $\widetilde{I}_{j,k} = \int_{I_{j,k}} M_y h_{2^n}$.

Note that, in Q, $M_y h_{2^n}(u,v)$ is nonincreasing in u for any fixed $v \in [0,1]$. Note also that if $(u,v) \in I^\circ_{j,k}$ and $(w,v) \in I^\circ_{j,k}$, then $M_y h_{2^n}(u,v) = M_y h_{2^n}(w,v)$.

Now, if $(u,v) \in I_{1,k}$, then $M_x M_y h_{2^n}(u,v) = M_y h_{2^n}(u,v)$. Hence

$$\int\limits_{I_{1,k}} M_x M_y h_{2^n}(u,v)\, du\, dv = \widetilde{I}_{1,k}, \qquad 1 \le k \le 2^n.$$

Suppose $2 \leq j \leq 2^n$. Then

$$\int_{I_{j,k}} M_x M_y h_{2^n} = \int_{I_{j,k}} \frac{1}{x}\Big[\int_0^x M_y h_{2^n}(u,v)\,du\Big]\,dv\,dx \tag{9}$$

$$= \int_{I_{j,k}} \frac{1}{x}\Big[\frac{\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k}}{2^{-n}} + \int_{2^{-2^n+j-1}}^{x} M_y h_{2^n}(u,v)\,du\Big]\,dv\,dx$$

$$= (\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k})\log 2 + \int_{I_{j,k}} \frac{1}{x}\int_{2^{-2^n+j-1}}^{x} M_y h_{2^n}(u,v)\,du\,dv\,dx$$

$$= (\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k})\log 2 + \int_{I_{j,k}} \frac{1}{x}(M_y h_{2^n}(x,v))(x - 2^{-2^n+j-1})\,dv\,dx$$

$$= (\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k})\log 2 + \int_{I_{j,k}} M_y h_{2^n}(x_0,v)\Big(1 - \frac{2^{-2^n+j-1}}{x}\Big)\,dv\,dx$$

(where x_0 is an arbitrary element of $(2^{-2^n+j-1}, 2^{-2^n+j})$)

$$= (\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k})\log 2 + \widetilde{I}_{j,k}$$

$$- (\log 2)(2^{-2^n+j-1})\Big(\int_{(k-1)\cdot 2^{-n}}^{k\cdot 2^{-n}} M_y h_{2^n}(x_0,v)\,dv\Big)$$

$$= (\widetilde{I}_{1,k} + \ldots + \widetilde{I}_{j-1,k})\log 2 + \widetilde{I}_{j,k} - \widetilde{I}_{j,k}\cdot \log 2.$$

So,

$$\int_Q M_x M_y h_{2^n} = \sum_{j,k=1}^{2^n} \int_{I_{j,k}} M_x M_y h_{2^n} \tag{10}$$

$$= \widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n}$$

$$+ (\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n})\log 2 + (\widetilde{I}_{2,1} + \ldots + \widetilde{I}_{2,2^n}) - (\widetilde{I}_{2,1} + \ldots + \widetilde{I}_{2,2^n})\log 2$$

$$+ (\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n} + \widetilde{I}_{2,1} + \ldots + \widetilde{I}_{2,2^n})\log 2 + (\widetilde{I}_{3,1} + \ldots + \widetilde{I}_{3,2^n})$$

$$- (\widetilde{I}_{3,1} + \ldots + \widetilde{I}_{3,2^n})\log 2 + \ldots$$

$$+ (\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n} + \ldots + \widetilde{I}_{2^n-1,1} + \ldots + \widetilde{I}_{2^n-1,2^n})\log 2$$

$$+ (\widetilde{I}_{2^n,1} + \ldots + \widetilde{I}_{2^n,2^n}) - (\widetilde{I}_{2^n,1} + \ldots + \widetilde{I}_{2^n,2^n})\log 2$$

$$= (\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n}) + (\widetilde{I}_{2,1} + \ldots + \widetilde{I}_{2,2^n}) + \ldots + (\widetilde{I}_{2^n,1} + \ldots + \widetilde{I}_{2^n,2^n})$$

$$+ (2^n - 1)(\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n})\log 2$$

$$
\begin{aligned}
&+\ (2^n-3)(\widetilde{I}_{2,1}+\ldots+\widetilde{I}_{2,2^n})\log 2\\
&+\ (2^n-4)(\widetilde{I}_{3,1}+\ldots+\widetilde{I}_{3,2^n})\log 2+\ldots\\
&+\ 1\cdot(\widetilde{I}_{2^n-2,1}+\ldots+\widetilde{I}_{2^n-2,2^n})\log 2\\
&+\ 0\cdot(\widetilde{I}_{2^n-1,1}+\ldots+\widetilde{I}_{2^n-1,2^n})\log 2\\
&+\ (-1)(\widetilde{I}_{2^n,1}+\ldots+\widetilde{I}_{2^n,2^n})\log 2.
\end{aligned}
$$

PROPOSITION 9. *If* $1\le j\le 2^n$, *then* $\widetilde{I}_{j,1}+\ldots+\widetilde{I}_{j,2^n}\sim n/2^n$.

Proof. Suppose $2\le j\le 2^n$. Then

$$
\begin{aligned}
(11)\qquad \widetilde{I}_{j,1}+\ldots+\widetilde{I}_{j,2^n} &= \int\limits_{2^{-2^n+j-1}}^{2^{-2^n+j}}\int\limits_0^1 M_y h_{2^n}(x,z)\,dz\,dx\\
&= 2^{-2^n+j-1}\int\limits_0^1 M_y h_{2^n}(2^{-2^n+j-1/2},z)\,dz\\
&\sim 2^{-2^n+j-1}\cdot\alpha_j,
\end{aligned}
$$

where α_j satisfies (by the fact that $\int_0^1 M_y h_{2^n}(u,z)\,dz\sim\|h_{2^n}(u,\cdot)\|_{L\log L}$) the equation

$$
(12)\qquad \frac{1}{2^n}\biggl[\frac{1}{\alpha_j}\log\biggl(3+\frac{1}{\alpha_j}\biggr)+\frac{2}{\alpha_j}\log\biggl(3+\frac{2}{\alpha_j}\biggr)
+\ldots+\frac{2^{2^n-j}}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)\biggr]=1.
$$

Now,

$$
\begin{aligned}
(13)\qquad &\frac{1}{\alpha_j}\log\biggl(3+\frac{1}{\alpha_j}\biggr)+\ldots+\frac{2^{2^n-j}}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)\\
&\le\frac{1}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)+\ldots+\frac{2^{2^n-j}}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)\\
&\le\frac{2^{2^n-j+1}}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)\\
&\le 2\biggl[\frac{1}{\alpha_j}\log\biggl(3+\frac{1}{\alpha_j}\biggr)+\ldots+\frac{2^{2^n-j}}{\alpha_j}\log\biggl(3+\frac{2^{2^n-j}}{\alpha_j}\biggr)\biggr].
\end{aligned}
$$

So $\alpha_j\sim\widetilde{\alpha}_j$, where $\widetilde{\alpha}_j$ is defined by

$$
(14)\qquad \frac{2^{2^n-j+1}}{\widetilde{\alpha}_j}\log\biggl(3+\frac{2^{2^n-j}}{\widetilde{\alpha}_j}\biggr)=2^n.
$$

Then for $2 \le j,k \le 2^n$ we have

$$\widetilde{I}_{j,1} + \ldots + \widetilde{I}_{j,2^n} \sim \widetilde{I}_{k,1} + \ldots + \widetilde{I}_{k,2^n}, \tag{15}$$

since

$$\begin{aligned}\widetilde{I}_{j,1} + \ldots + \widetilde{I}_{j,2^n} &\sim 2^{-2^n+j-1} \cdot \widetilde{\alpha}_j = 2^{-2^n+j-1} \cdot 2^{k-j} \cdot \widetilde{\alpha}_k \\ &= 2^{-2^n+k-1} \cdot \widetilde{\alpha}_k \sim \widetilde{I}_{k,1} + \ldots + \widetilde{I}_{k,2^n}.\end{aligned}$$

Clearly $\widetilde{I}_{1,1} + \ldots + \widetilde{I}_{1,2^n} \sim \widetilde{I}_{2,1} + \ldots + \widetilde{I}_{2,2^n}$, so (15) holds for $1 \le j,k \le 2^n$. As

$$\widetilde{I}_{2^n,1} + \ldots + \widetilde{I}_{2^n,2^n} = \frac{1}{2}\Big(2^{-n} + \int_{2^{-n}}^{1} \frac{2^{-n}}{x}\,dx\Big) = 2^{-(n+1)}(1 + \log 2^n) \sim n/2^n,$$

we get the assertion. ■

We now finish the proof of Lemma 8. The proposition and (10) imply

$$\begin{aligned}\int_Q M_x M_y h_{2^n} &\sim \frac{n}{2^n} \cdot 2^n + (2^n - 2 + 2^n - 3 + \ldots + 1) \cdot \frac{n}{2^n} \cdot \log 2 \\ &= n + \frac{1}{2}(2^n - 1)(2^n - 2) \cdot \frac{n}{2^n} \cdot \log 2 \\ &\sim n + \frac{1}{2} \cdot \log 2 \cdot n \cdot 2^n \sim n \cdot 2^n. \ ■\end{aligned}$$

Lemma 10.

$$\int_Q M_y M_x h_{2^n} \le 10 \cdot 2^n.$$

Proof. Clearly if $(u,v) \in Q$, then

$$M_x h_{2^n}(u,v) \le \begin{cases} 2^{2^n-1}, & u \le 1/2^{2^n-1}, \\ 1/u, & u > 1/2^{2^n-1}. \end{cases}$$

So

$$M_y M_x h_{2^n}(u,v) \le \begin{cases} 2^{2^n-1}, & u \le 1/2^{2^n-1}, \\ 1/u, & u > 1/2^{2^n-1}. \end{cases}$$

Then

$$\int_Q M_y M_x h_{2^n} \le 2^{2^n-1}\Big(\frac{1}{2^{2^n-1}}\Big) + \int_{2^{1-2^n}}^{1} \frac{1}{x}\,dx = 1 + \log(2^{2^n-1}) \le 10 \cdot 2^n. \ ■$$

We now define the function h by

$$h = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^{k-1}}} h_{2^{4^{k-1}}}. \tag{16}$$

Then

$$\int_Q M_y M_x h \le \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^{k-1}}} \int_Q M_y M_x h_{2^{4^{k-1}}} \le 10 \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^{k-1}}} \cdot 2^{4^{k-1}} = 5, \tag{17}$$

but

$$\int_Q M_x M_y h \ge \lim_{k\to\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^{k-1}}} \int_Q M_x M_y h_{2^{4^{k-1}}} \sim \lim_{k\to\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^{k-1}}} \cdot 2^{4^{k-1}} \cdot 4^{k-1} = \lim_{k\to\infty} 2^{k-3} = \infty. \ \blacksquare \tag{18}$$

We now show that if f is a function such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, then there exists a set A of finite measure in $\mathbb{R}^2$ such that $\int_A M_{\mathrm{S}} f = \infty$. This result is particularly interesting in view of the fact that M. E. Gomez has constructed a function g supported on Q such that $\int_Q M_x M_y g$ and $\int_Q M_y M_x g$ are infinite, but $M_{\mathrm{S}} g$ is integrable over every set of finite measure in $\mathbb{R}^2$ (see [5]). Such a construction is also implicit in the work of Bagby and Jawerth and Morrow ([1], [9]).

THEOREM 11. *Suppose that f is a measurable function supported on Q, $\int_Q M_x M_y f = \infty$, and $\int_Q M_y M_x f < \infty$. Then there exists a set A of finite measure in $\mathbb{R}^2$ such that $\int_A M_{\mathrm{S}} f = \infty$.*

Proof. We first recall a theorem due to Fava, Gatto, and Gutiérrez.

THEOREM 12 ([3]). *Suppose f is a measurable function supported in Q. Then $M_x M_y f$ is integrable over every set of finite measure in $\mathbb{R}^2$ if and only if $\|f\|_{L(\log L)^2(Q)} < \infty$.*

The maximal operators M, $\overline{M}$, defined as follows, will be very useful to us:

DEFINITION 13. Let f be a measurable function supported on Q. The associated maximal function Mf is defined on Q by

$$Mf(p_1, p_2) = \sup_{x_1 < p_1 < x_2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \int_0^1 |f(x, y)| \, dy \, dx.$$

The associated maximal function $\overline{M} f$ is defined on Q by

$$\overline{M} f(p_1, p_2) = \sup_{y_1 < p_2 < y_2} \frac{1}{y_2 - y_1} \int_0^1 \int_{y_1}^{y_2} |f(x, y)| \, dy \, dx.$$

Now, we define the auxiliary functions g and h as follows. If $p = (p_1, p_2) \in \mathbb{R}^2$, let

$$g(p) = \int_0^1 |f(p_1, y)|\, dy \cdot \chi_Q(p), \qquad h(p) = \int_0^1 |f(x, p_2)|\, dx \cdot \chi_Q(p).$$

Note that if $p \in \mathbb{R}^2$, then

$$M_{\mathrm{S}} f(p) \geq \tfrac{1}{2}[M_{\mathrm{S}} g(p) + M_{\mathrm{S}} h(p)]. \tag{19}$$

Since g and h are tensors on Q, also note that

$$M_{\mathrm{S}} g(p) = M_x M_y g(p) = M_y M_x g(p), \tag{20}$$

$$M_{\mathrm{S}} h(p) = M_x M_y h(p) = M_y M_x h(p). \tag{21}$$

Now, by Theorem 12, $M_x M_y g$ is integrable over every set of finite measure in $\mathbb{R}^2$ if and only if $g \in L(\log L)^2(Q)$, and similarly for h in place of g. If $\|g + h\|_{L(\log L)^2(Q)} = \infty$, equations (19)–(21) imply the existence of a set of finite measure in $\mathbb{R}^2$ over which $M_{\mathrm{S}} f$ is not integrable. So it suffices to show $\|g + h\|_{L(\log L)^2(Q)} = \infty$.

An application of the Fubini Theorem and Theorem 6 yields that

$$\|g\|_{L(\log L)^2(Q)} \sim \int_Q MMf, \qquad \|h\|_{L(\log L)^2(Q)} \sim \int_Q \overline{M}\,\overline{M} f.$$

So it is enough to show $\int_Q MMf + \int_Q \overline{M}\,\overline{M} f = \infty$. As $\int_Q M_x M_y f \lesssim \|f\|_{L(\log L)^2(Q)}$, we see that the proof reduces to proving the following.

Theorem 14. *Let f be a measurable function supported on Q. Then*

$$\|f\|_{L(\log L)^2(Q)} \lesssim \int_Q MMf + \int_Q M_y M_x f + \int_Q \overline{M}\,\overline{M} f.$$

Proof. It will be technically convenient to work with the dyadic analogues of the maximal operators M_{HL}, M, $\overline{M}$, M_x, and M_y. Recall that a *dyadic interval* in $[0,1]$ is an interval of the form $[k \cdot 2^j, (k+1) \cdot 2^j]$, where j is a nonpositive integer and k is a nonnegative integer such that $(k+1) \cdot 2^j \leq 1$. We denote the set of dyadic subintervals of $[0,1]$ by $\mathcal{I}^{\Delta}$. A *dyadic square* in Q is a set of the form $I \times J$, where I and J are dyadic intervals in $[0,1]$ of the same length. We denote the set of dyadic squares in Q by $\mathcal{S}^{\Delta}$. We formally define the dyadic maximal operators M_{HL}^{Δ}, M^{Δ}, $\overline{M}^{\Delta}$, M_x^{Δ}, and M_y^{Δ} as follows.

Definition 15. Let f be a measurable function supported on Q. The dyadic Hardy–Littlewood maximal function $M_{\mathrm{HL}}^{\Delta} f$ is defined on Q by

$$M_{\mathrm{HL}}^{\Delta} f(p) = \sup_{p \in S \in \mathcal{S}^{\Delta}} \frac{1}{|S|} \int_S |f|.$$

The maximal function $M^{\Delta} f$ is defined by

$$M^{\Delta} f(p_1, p_2) = \sup_{p_1 \in I \in \mathcal{I}^{\Delta}} \frac{1}{|I|} \int_I \int_0^1 |f(x,y)| \, dy \, dx.$$

$\overline{M}^{\Delta} f$ is defined by

$$\overline{M}^{\Delta} f(p_1, p_2) = \sup_{p_2 \in I \in \mathcal{I}^{\Delta}} \frac{1}{|I|} \int_0^1 \int_I |f(x,y)| \, dy \, dx.$$

$M_x^{\Delta} f$ is defined by

$$M_x^{\Delta} f(p_1, p_2) = \sup_{p_1 \in I \in \mathcal{I}^{\Delta}} \frac{1}{|I|} \int_I |f(x, p_2)| \, dx.$$

Similarly, $M_y^{\Delta} f$ is defined by

$$M_y^{\Delta} f(p_1, p_2) = \sup_{p_2 \in I \in \mathcal{I}^{\Delta}} \frac{1}{|I|} \int_I |f(p_1, y)| \, dy.$$

If f is a measurable function supported on $[0,1]$, then the dyadic Hardy–Littlewood maximal function $M_{\mathrm{HL}}^{\Delta} f$ is defined on $[0,1]$ by

$$M_{\mathrm{HL}}^{\Delta} f(p) = \sup_{p \in I \in \mathcal{I}^{\Delta}} \frac{1}{|I|} \int_I |f|.$$

We will also need the following results found in [7].

Lemma 16 ([7]). *Let f be a measurable function supported on Q. Then*

$$\int_Q M_{\mathrm{HL}} f \sim \int_Q (M_x f + M_y f). \tag{22}$$

Moreover,

$$\int_Q M_{\mathrm{HL}} f \sim \int_Q (M f + M_y f). \tag{23}$$

Lemma 17 ([7]). *Let f be a measurable function supported on Q. Then*

$$\int_Q M^{\Delta} M_x^{\Delta} f \lesssim \int_Q M^{\Delta} M^{\Delta} f + \int_Q M_y^{\Delta} M_x^{\Delta} f. \tag{24}$$

Furthermore,

$$\int_Q M_x M_x f \lesssim \int_Q M M f + \int_Q M_y M_x f. \tag{25}$$

Lemma 18 ([7]). *Let f be a nonnegative measurable function supported on Q. Let $\widetilde{f}(x,y)$ be the function supported on Q which is nonincreasing in x*

(i.e. $\widetilde{f}(x_1, y) \geq \widetilde{f}(x_2, y)$ *whenever* $0 \leq x_1 \leq x_2 \leq 1$ *and* $0 \leq y \leq 1$*) and such that, for each* $y \in [0,1]$, $\widetilde{f}(\cdot, y)$ *and* $f(\cdot, y)$ *are equidistributed. Then*

$$\int_Q M_{\mathrm{HL}} M_y \widetilde{f} \lesssim \int_Q M_{\mathrm{HL}} M_y f.$$

LEMMA 19. *Suppose* $f(x,y)$ *is a nonnegative measurable function supported on* Q *which is nonincreasing in* x. *Then*

$$\int_Q M_x M_y f \lesssim \int_Q MMf + \int_Q M_y M_x f.$$

Proof. As f is nonincreasing in x, $\int_Q MMf = \int_Q M_x M_x f$. So, letting f' be a function supported on Q such that $f'(x, \cdot)$ and $f(x, \cdot)$ are equidistributed for each $x \in [0,1]$ and also such that $f'(x,y)$ is nonincreasing in y, we see that

$$\begin{aligned}
\int_Q (MMf + M_y M_x f) &= \int_Q (M_x M_x f + M_y M_x f) \\
&\sim \int_Q M_{\mathrm{HL}} M_x f \qquad \text{(by Lemma 16)} \\
&\gtrsim \int_Q M_{\mathrm{HL}} M_x f' \qquad \text{(by Lemma 18)} \\
&\sim \int_Q M_x M_x f' + \int_Q M_y M_x f' \\
&\gtrsim \int_Q M_y M_x f' \\
&= \int_Q M_x M_y f' \qquad \text{(since } f' \text{ is nonincreasing in both variables)} \\
&\sim \int_Q M_x M_y f \qquad \text{(by Theorem 6 and the Fubini Theorem).} \ \blacksquare
\end{aligned}$$

LEMMA 20. *Let* $B = [0, 2^{-n}] \times [0,1]$ *be a subset of* Q. *Suppose* f *is a measurable function supported in* B. *Then*

$$\int_B M_{\mathrm{HL}} f \lesssim \int_B (M_x + M_y) f.$$

Proof. Let $g(x,y) = f(x \bmod 2^{-n}, y)$. As

$$|\{p \in Q : M_{\mathrm{HL}} g(p) > \alpha\}| \geq 2^n |\{p \in B : M_{\mathrm{HL}} f(p) > \alpha\}|,$$

we see that $\int_Q M_{\mathrm{HL}} g \geq |B|^{-1} \int_B M_{\mathrm{HL}} f$. As

$$\int_Q M_{\mathrm{HL}} g \sim \int_Q (M_x + M_y) g \sim \frac{1}{|B|} \int_B (M_x + M_y) f,$$

we see that $\int_B M_{\mathrm{HL}} f \lesssim \int_B (M_x + M_y) f$. ■

Corollary 21. *Let R be a rectangle supported in Q of width or height one. Let f be a measurable function supported on R. Then*

$$\frac{1}{|R|} \int_R M_{\mathrm{HL}} f \lesssim \frac{1}{|R|} \int_R (M_x + M_y) f.$$

Proof. This follows from Lemma 20 by symmetry arguments. ■

Lemma 22. *Suppose f is a nonnegative measurable function supported on Q which is nonincreasing in x. Then*

$$\|f\|_{L(\log L)^2(Q)} \lesssim \int_Q (MMf + \overline{M}\,\overline{M} f + M_y M_x f).$$

Proof. We assume without loss of generality that $f \in C^\infty(Q)$. Note that

$$|\{p \in Q : M_{\mathrm{HL}} f(p) > \alpha\}| \leq 1000 |\{p \in Q : M_{\mathrm{HL}}^{\Delta} f(p) > \alpha/1000\}|$$

for all $\alpha > 0$. Then

$$\int_Q M_{\mathrm{HL}} M_{\mathrm{HL}} f \sim \int_Q M_{\mathrm{HL}}^{\Delta} M_{\mathrm{HL}}^{\Delta} f.$$

Hence

$$\begin{aligned} \|f\|_{L(\log L)^2(Q)} &\sim \int_Q M_{\mathrm{HL}}^{\Delta} M_{\mathrm{HL}}^{\Delta} f \\ &\sim \int_Q \overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f + \int_Q M_x^{\Delta} M_{\mathrm{HL}}^{\Delta} f \quad \text{(by Lemma 16)}. \end{aligned}$$

It is then enough to show

(i) $$\int_Q \overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f \lesssim \int_Q (MMf + \overline{M}\,\overline{M} f + M_y M_x f),$$

(ii) $$\int_Q M_x^{\Delta} M_{\mathrm{HL}}^{\Delta} f \lesssim \int_Q (MMf + \overline{M}\,\overline{M} f + M_y M_x f).$$

To prove (i), let ℓ be a unit horizontal line segment through Q. Let $p \in \ell$. Let B be a horizontal dyadic band through Q such that $\overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f(p) \sim |B|^{-1} \int_B M_{\mathrm{HL}}^{\Delta} f$. Let now $f_{\mathrm{int}} = f \cdot \chi_B$, $f_{\mathrm{ext}} = f \cdot \chi_{B^c}$. Now by Corollary 21,

$$\frac{1}{|B|} \int_B M_{\mathrm{HL}}^{\Delta} f_{\mathrm{int}} \lesssim \frac{1}{|B|} \int_B (M_x^{\Delta} + M_y^{\Delta}) f_{\mathrm{int}} \lesssim \overline{M}^{\Delta} (M_x^{\Delta} + M_y^{\Delta}) f(p).$$

Also,

$$\frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f_{\mathrm{ext}} = \int_\ell M_{\mathrm{HL}}^{\Delta} f_{\mathrm{ext}} \lesssim \int_\ell M_y M_x f_{\mathrm{ext}} \lesssim \int_\ell M_y M_x f.$$

(The equality above holds because $M_{\mathrm{HL}}^{\Delta} f_{\mathrm{ext}}$ is constant on vertical slices of B.)

So

$$\int_\ell \overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f \lesssim \int_\ell \overline{M}^{\Delta}(M_x^{\Delta}+M_y^{\Delta})f + \int_\ell M_y M_x f.$$

Hence

$$\int_Q \overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f \lesssim \int_Q \overline{M}^{\Delta}(M_x^{\Delta}+M_y^{\Delta})f + \int_Q M_y M_x f.$$

Now $\int_Q \overline{M}^{\Delta}(M_x^{\Delta}+M_y^{\Delta})f \lesssim \int_Q \overline{M}^{\Delta} M_x^{\Delta} f + \int_Q \overline{M}^{\Delta} M_y^{\Delta} f$. It is clear that $\int_Q \overline{M}^{\Delta} M_x^{\Delta} f \le \int_Q M_y M_x f$. We also have

$$\begin{aligned}\int_Q \overline{M}^{\Delta} M_y^{\Delta} f &\lesssim \int_Q \overline{M}\,\overline{M} f + \int_Q M_x M_y f && \text{(by Lemma 17)}\\ &\lesssim \int_Q \overline{M}\,\overline{M} f + \int_Q (MMf + M_y M_x f) && \text{(by Lemma 19).}\end{aligned}$$

Hence

$$\int_Q \overline{M}^{\Delta}(M_x^{\Delta}+M_y^{\Delta})f \lesssim \int_Q (MMf + \overline{M}\,\overline{M} f + M_y M_x f),$$

and thus

$$\int_Q \overline{M}^{\Delta} M_{\mathrm{HL}}^{\Delta} f \lesssim \int_Q (MMf + \overline{M}\,\overline{M} f + M_y M_x f).$$

So (i) is proved.

We now prove (ii). Since f is nonincreasing in x, we have $\int_Q M_x^{\Delta} M_{\mathrm{HL}}^{\Delta} f \sim \int_Q M^{\Delta} M_{\mathrm{HL}}^{\Delta} f$. Let ℓ now denote a vertical unit segment in Q, and B the part of Q to the left of ℓ. We want to show

$$\int_\ell M^{\Delta} M_{\mathrm{HL}}^{\Delta} f \sim \frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f \lesssim \int_\ell (MMf + \overline{M}\,\overline{M} f + M_y M_x f).$$

Let $f_{\mathrm{int}} = f\cdot\chi_B$, $f_{\mathrm{ext}} = f\cdot\chi_{B^{\mathrm{c}}}$. Now

$$\frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f \lesssim \frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f_{\mathrm{int}} + \frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f_{\mathrm{ext}}.$$

By Corollary 21 we have

$$\frac{1}{|B|}\int_B M_{\mathrm{HL}}^{\Delta} f_{\mathrm{int}} \lesssim \frac{1}{|B|}\int_B (M_x+M_y) f_{\mathrm{int}} \lesssim M(M_x+M_y)f(p)$$

for any $p \in \ell$. So

$$\frac{1}{|B|}\int_B M^{\Delta}_{\mathrm{HL}} f_{\mathrm{int}} \lesssim \int_\ell M(M_x + M_y)f.$$

Now if $p \in B$, then $M^{\Delta}_{\mathrm{HL}} f_{\mathrm{ext}}(p) \le M_y f(p)$ since $f(x,y)$ is nonincreasing in x and f_{ext} is supported to the right of B. Hence

$$\frac{1}{|B|}\int_B M^{\Delta}_{\mathrm{HL}} f_{\mathrm{ext}} \lesssim \frac{1}{|B|}\int_B M_y f \lesssim M(M_y f)(q)$$

for any point $q \in \ell$. So

$$\frac{1}{|B|}\int_B M^{\Delta}_{\mathrm{HL}} f_{\mathrm{ext}} \lesssim \int_\ell M M_y f.$$

Hence $\int_\ell M^{\Delta} M^{\Delta}_{\mathrm{HL}} f \lesssim \int_\ell M(M_x + M_y)f$. This implies

$$\int_Q M M^{\Delta}_{\mathrm{HL}} f \lesssim \int_Q M(M_x + M_y)f.$$

As $\int_Q M M_x f \sim \int_Q M M f$, since f is nonincreasing in x, and

$$\int_Q M M_y f \le \int_Q M_x M_y f \lesssim \int_Q (M M f + M_y M_x f)$$

by Lemma 19, we get the desired result. ∎

We now complete the proof of Theorem 14, and hence of Theorem 11. f is a measurable function supported on Q. Without loss of generality we assume f is a nonnegative function as well. Let $\widetilde{f}(x,y)$ be a function supported on Q which is nonincreasing in x and such that $\widetilde{f}(\cdot,y)$ and $f(\cdot,y)$ are equidistributed for each $y \in [0,1]$. Now

$$\begin{aligned}
\|f\|_{L(\log L)^2(Q)} &= \|\widetilde{f}\|_{L(\log L)^2(Q)} \\
&\lesssim \int_Q (M M \widetilde{f} + M_y M_x \widetilde{f} + \overline{M}\,\overline{M}\widetilde{f}) \quad \text{(Lemma 22)} \\
&\sim \int_Q (M_x M_x \widetilde{f} + M_y M_x \widetilde{f}) + \int_Q \overline{M}\,\overline{M}\widetilde{f} \\
&\sim \int_Q M_{\mathrm{HL}} M_x \widetilde{f} + \int_Q \overline{M}\,\overline{M}\widetilde{f} \quad \text{(Lemma 16)} \\
&\sim \int_Q M_{\mathrm{HL}} M_x f + \int_Q \overline{M}\,\overline{M} f \quad (\text{as } \|M_x f\|_{L\log L(Q)} \sim \|M_x \widetilde{f}\|_{L\log L(Q)})
\end{aligned}$$

$$\sim \int_Q M_x M_x f + \int_Q M_y M_x f + \int_Q \overline{M}\,\overline{M} f \qquad \text{(Lemma 16)}$$

$$\lesssim \int_Q MMf + \int_Q M_y M_x f + \int_Q \overline{M}\,\overline{M} f \qquad \text{(Lemma 17)}$$

as desired. ∎

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