Local integrability of strong and iterated maximal functions

by

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Abstract. Let $M_S$ denote the strong maximal operator. Let $M_x$ and $M_y$ denote the one-dimensional Hardy–Littlewood maximal operators in the horizontal and vertical directions in $\mathbb{R}^2$. A function $h$ supported on the unit square $Q = [0,1] \times [0,1]$ is exhibited such that $\int_Q M_y M_x h < \infty$ but $\int_Q M_x M_y h = \infty$. It is shown that if $f$ is a function supported on $Q$ such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, then there exists a set $A$ of finite measure in $\mathbb{R}^2$ such that $\int_A M_S f = \infty$.

We begin by listing some basic definitions.

DEFINITION 1. Let $f$ be a measurable function defined on $\mathbb{R}^n$. Denote by $B(p, r)$ the Euclidean ball in $\mathbb{R}^n$ centered at $p$ of radius $r$, and by $|B(p, r)|$ the Lebesgue measure of $B(p, r)$. The Hardy–Littlewood maximal function of $f$ is defined on $\mathbb{R}^n$ by

$$M_{HL} f(p) = \sup_{r > 0} \frac{1}{|B(p, r)|} \int_{B(p, r)} |f(z)| \, dz.$$  

DEFINITION 2. Let $f$ be a measurable function defined on $\mathbb{R}^2$. The strong maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_S f(x, y) = \sup_{x_1 < x < x_2, y_1 < y < y_2} \frac{1}{(x_2 - x_1)(y_2 - y_1)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} |f(u, v)| \, dv \, du.$$  

DEFINITION 3. Let $f$ be a measurable function defined on $\mathbb{R}^2$. The horizontal maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_x f(u, v) = \sup_{u_1 < u < u_2} \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} |f(w, v)| \, dw.$$  

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Definition 4. Let $f$ be a measurable function defined on $\mathbb{R}^2$. The vertical maximal function of $f$ is defined on $\mathbb{R}^2$ by

$$M_y f(u,v) = \sup_{v_1 < v < v_2} \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} |f(u,w)|\,dw.$$  

The following Orlicz spaces will be very useful to us.

Definition 5. Let $I(n)$ denote the unit $n$-cube $[0,1] \times \ldots \times [0,1]$ in $\mathbb{R}^n$. $L(\log L)^k(I(n))$ is the Lebesgue space of functions on $I(n)$ with norm

$$\|f\|_{L(\log L)^k(I(n))} = \inf \left\{ c > 0 : \int_{I(n)} \frac{|f|}{c} \log \left( 3 + \frac{|f|}{c} \right)^k \leq 1 \right\} < \infty.$$  

In the following we will denote the unit square $I(2)$ in $\mathbb{R}^2$ by $Q$. Also, the $k$-fold iteration $M_{HL} \circ \ldots \circ M_{HL}$ of the maximal operator $M_{HL}$ will be denoted by $M_{HL}^k$.

We now recall the following theorem of E. M. Stein:

Theorem 6 ([11]). Let $k$ be a positive integer. There exist constants $0 < c = c(k,n) < C = C(k,n) < \infty$ such that if $f$ is supported on $I(n)$, then

$$c \int_{I(n)} M_{HL}^k f \leq \|f\|_{L(\log L)^k(I(n))} \leq C \int_{I(n)} M_{HL}^k f.$$  

Inequalities such as (6) in the case $k = 1$ will often be denoted by

$$\|f\|_{L\log L(I(n))} \sim \int_{I(n)} M_{HL} f$$  

for the remainder of this paper.

We now show that there exists a function $h$ supported on $Q$ such that $\int_Q M_y M_x h < \infty$, but $\int_Q M_x M_y h = \infty$.

Theorem 7. There exists a function $h$, supported on $Q$, such that $\int_Q M_x M_y h = \infty$, but $\int_Q M_y M_x h < \infty$.

Proof. We define the functions $h_{2^n}$ as follows:

$$h_{2^n}(x,y) = \sum_{m=0}^{2^n-1} 2^{2^n-m-1} \chi_{[0,2^{-2^n+m+1}]}(x) \cdot \chi_{[m2^{-n},(m+1)2^{-n}]}(y).$$  

The functions $h_2$, $h_4$, and $h_8$ are depicted in Figure 1.

Lemma 8.

$$\int_Q M_x M_y h_{2^n} \sim n \cdot 2^n.$$  

Proof. Divide $Q$ into $2^{2n}$ regions $I_{j,k}$, $j = 1, \ldots, 2^n$, $k = 1, \ldots, 2^n$, where $I_{j,k}$ is defined as follows:

$$I_{1,k} = [0, 2^{-2^n+1}] \times [(k - 1) \cdot 2^{-n}, k \cdot 2^{-n}], \quad 1 \leq k \leq 2^n;$$

$$I_{j,k} = [2^{-2^n+j-1}, 2^{-2^n+j}] \times [(k - 1) \cdot 2^{-n}, k \cdot 2^{-n}], \quad 2 \leq j \leq 2^n, \quad 1 \leq k \leq 2^n.$$

Let $\tilde{I}_{j,k} = \int_{I_{j,k}} M_y h_{2^n}.$

Note that, in $Q$, $M_y h_{2^n}(u, v)$ is nonincreasing in $u$ for any fixed $v \in [0, 1]$. Note also that if $(u, v) \in I_{j,k}^o$ and $(w, v) \in I_{j,k}^o$, then $M_y h_{2^n}(u, v) = M_y h_{2^n}(w, v)$.

Now, if $(u, v) \in I_{1,k}$, then $M_x M_y h_{2^n}(u, v) = M_y h_{2^n}(u, v)$. Hence

$$\int_{I_{1,k}} M_x M_y h_{2^n}(u, v) \, du \, dv = \tilde{I}_{1,k}, \quad 1 \leq k \leq 2^n.$$
Suppose $2 \leq j \leq 2^n$. Then

$$\int_{I_{j,k}} M_x M_y h_{2^n} = \int_{I_{j,k}} \frac{1}{x} \left[ \int_{0}^{x} M_y h_{2^n}(u, v) \, du \right] \, dv \, dx$$

$$= \left( \int_{0}^{x} \frac{1}{x} \left[ \tilde{I}_{1,k} + \ldots + \tilde{I}_{j-1,k} \right] 2^{-n} \, dv \, dx \right) + \left( \int_{2^{-n} + j-1}^{2^n} M_y h_{2^n}(u, v) \, du \, dv \, dx \right)$$

$$= (\tilde{I}_{1,k} + \ldots + \tilde{I}_{j-1,k}) \log 2 + \int_{I_{j,k}} \frac{1}{x} M_y h_{2^n}(x, v)(x - 2^{-n} + j-1) \, dv \, dx$$

$$= (\tilde{I}_{1,k} + \ldots + \tilde{I}_{j-1,k}) \log 2 + \int_{I_{j,k}} M_y h_{2^n}(x, v) \left( 1 - \frac{2^{-n} + j-1}{x} \right) \, dv \, dx$$

(where $x_0$ is an arbitrary element of $(2^{-n} + j-1, 2^{-n} + j)$)

$$= (\tilde{I}_{1,k} + \ldots + \tilde{I}_{j-1,k}) \log 2 + \tilde{I}_{j,k}$$

$$- (\log 2)(2^{-n} + j-1) \left( \int_{(k-1)2^{-n}}^{k2^{-n}} M_y h_{2^n}(x, v) \, dv \right)$$

$$= (\tilde{I}_{1,k} + \ldots + \tilde{I}_{j-1,k}) \log 2 + \tilde{I}_{j,k} - \tilde{I}_{j,k} \cdot \log 2.$$

So,

$$\int_{Q} M_x M_y h_{2^n} = \sum_{j,k=1}^{2^n} \int_{I_{j,k}} M_x M_y h_{2^n}$$

$$= \tilde{I}_{1,1} + \ldots + \tilde{I}_{1,2^n}$$

$$+ (\tilde{I}_{1,1} + \ldots + \tilde{I}_{1,2^n}) \log 2 + (\tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}) - (\tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}) \log 2$$

$$+ (\tilde{I}_{1,1} + \ldots + \tilde{I}_{2,2^n} + \tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}) \log 2 + (\tilde{I}_{3,1} + \ldots + \tilde{I}_{3,2^n})$$

$$- (\tilde{I}_{3,1} + \ldots + \tilde{I}_{3,2^n}) \log 2 + \ldots$$

$$+ (\tilde{I}_{1,1} + \ldots + \tilde{I}_{2,2^n} + \ldots + \tilde{I}_{2^n-1,1} + \ldots + \tilde{I}_{2^n-1,2^n}) \log 2$$

$$+ (\tilde{I}_{2,1} + \ldots + \tilde{I}_{2^n,2^n}) - (\tilde{I}_{2,1} + \ldots + \tilde{I}_{2^n,2^n}) \log 2$$

$$= (\tilde{I}_{1,1} + \ldots + \tilde{I}_{1,2^n}) + (\tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}) + \ldots + (\tilde{I}_{2^n,1} + \ldots + \tilde{I}_{2^n,2^n})$$

$$+ (2^n - 1)(\tilde{I}_{1,1} + \ldots + \tilde{I}_{1,2^n}) \log 2.$$
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\[ + (2^n - 3)(\tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}) \log 2 \]
\[ + (2^n - 4)(\tilde{I}_{3,1} + \ldots + \tilde{I}_{3,2^n}) \log 2 + \ldots \]
\[ + 1 \cdot (\tilde{I}_{2^n-2,1} + \ldots + \tilde{I}_{2^n-2,2^n}) \log 2 \]
\[ + 0 \cdot (\tilde{I}_{2^n-1,1} + \ldots + \tilde{I}_{2^n-1,2^n}) \log 2 \]
\[ + (-1)(\tilde{I}_{2^n,1} + \ldots + \tilde{I}_{2^n,2^n}) \log 2. \]

**Proposition 9.** If \( 1 \leq j \leq 2^n \), then \( \tilde{I}_{j,1} + \ldots + \tilde{I}_{j,2^n} \sim n/2^n \).

**Proof.** Suppose \( 2 \leq j \leq 2^n \). Then

\[
\tilde{I}_{j,1} + \ldots + \tilde{I}_{j,2^n} = \int_{2^{-2^n+j-1}}^{2^{-2^n+j}} \int_{0}^{1} M_y h_{2^n}(x, z) \, dz \, dx
\]
\[ = 2^{-2^n+j-1} \int_{0}^{1} M_y h_{2^n}(2^{-2^n+j-1/2}, z) \, dz \]
\[ \sim 2^{-2^n+j-1} \cdot \alpha_j, \]

where \( \alpha_j \) satisfies (by the fact that \( \int_{0}^{1} M_y h_{2^n}(u, z) \, dz \sim \|h_{2^n}(u, \cdot)\|_{L \log L} \)) the equation

\[
\frac{1}{2^n} \left[ \frac{1}{\alpha_j} \log \left( 3 + \frac{1}{\alpha_j} \right) + \frac{2}{\alpha_j} \log \left( 3 + \frac{2}{\alpha_j} \right) \right. \\
+ \ldots + \left. \frac{2^{2^n-j}}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right) \right] = 1.
\]

Now,

\[
\frac{1}{\alpha_j} \log \left( 3 + \frac{1}{\alpha_j} \right) + \ldots + \frac{2^{2^n-j}}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right)
\]
\[ \leq \frac{1}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right) + \ldots + \frac{2^{2^n-j}}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right)
\]
\[ \leq \frac{2^{2^n-j+1}}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right)
\]
\[ \leq 2 \left[ \frac{1}{\alpha_j} \log \left( 3 + \frac{1}{\alpha_j} \right) + \ldots + \frac{2^{2^n-j}}{\alpha_j} \log \left( 3 + \frac{2^{2^n-j}}{\alpha_j} \right) \right].
\]

So \( \alpha_j \sim \tilde{\alpha}_j \), where \( \tilde{\alpha}_j \) is defined by

\[
\frac{2^{2^n-j+1}}{\tilde{\alpha}_j} \log \left( 3 + \frac{2^{2^n-j}}{\tilde{\alpha}_j} \right) = 2^n.
\]
Then for \(2 \leq j, k \leq 2^n\) we have

\[
I_{j,1} + \ldots + I_{j,2^n} \sim I_{k,1} + \ldots + I_{k,2^n},
\]

since

\[
I_{j,1} + \ldots + I_{j,2^n} \sim 2^{-2^n+j-1} \cdot \tilde{\alpha}_j = 2^{-2^n+j-1} \cdot 2^{k-j} \cdot \tilde{\alpha}_k \sim 2^{-2^n+k-1} \cdot \tilde{\alpha}_k \sim I_{k,1} + \ldots + I_{k,2^n}.
\]

Clearly \(\tilde{I}_{1,1} + \ldots + \tilde{I}_{1,2^n} \sim \tilde{I}_{2,1} + \ldots + \tilde{I}_{2,2^n}\), so (15) holds for \(1 \leq j, k \leq 2^n\).

As

\[
\tilde{I}_{2^n,1} + \ldots + \tilde{I}_{2^n,2^n} = \frac{1}{2} \left( 2^{-n} + \int \frac{2^{-n}}{x} \, dx \right) = 2^{-(n+1)}(1 + \log 2^n) \sim n/2^n,
\]

we get the assertion. ■

We now finish the proof of Lemma 8. The proposition and (10) imply

\[
\int_Q M_x M_y h_{2^n} \sim \frac{n}{2^n} \cdot 2^n + (2^n - 2 + 2^n - 3 + \ldots + 1) \cdot \frac{n}{2^n} \cdot \log 2
\]

\[
= n + \frac{1}{2} (2^n - 1)(2^n - 2) \cdot \frac{n}{2^n} \cdot \log 2
\]

\[
\sim n + \frac{1}{2} \cdot \log 2 \cdot n \cdot 2^n \sim n \cdot 2^n. \quad \blacksquare
\]

**Lemma 10.**

\[
\int_Q M_y M_x h_{2^n} \leq 10 \cdot 2^n.
\]

**Proof.** Clearly if \((u, v) \in Q\), then

\[
M_x h_{2^n}(u, v) \leq \begin{cases} 2^{2^n-1}, & u \leq 1/2^{2^n-1}, \\ 1/u, & u > 1/2^{2^n-1}. \end{cases}
\]

So

\[
M_y M_x h_{2^n}(u, v) \leq \begin{cases} 2^{2^n-1}, & u \leq 1/2^{2^n-1}, \\ 1/u, & u > 1/2^{2^n-1}. \end{cases}
\]

Then

\[
\int_Q M_y M_x h_{2^n} \leq 2^{2^n-1} \left( \frac{1}{2^{2^n-1}} \right) + \int_{1/2^{2^n-1}}^{1} \frac{1}{x} \, dx = 1 + \log(2^{2^n-1}) \leq 10 \cdot 2^n. \quad \blacksquare
\]

We now define the function \(h\) by

\[
h = \sum_{k=1}^{\infty} \frac{1}{2k+1} \cdot \frac{1}{2^{4k-1}} h_{2^{4k-1}}.
\]
Then

$$\int_Q M_y M_x h \leq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^k}} \int_Q M_y M_x h_{2^{4^k-1}}$$

$$\leq 10 \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^k}} \cdot 2^{4^k-1} = 5.$$  

but

$$\int_Q M_x M_y h \geq \lim_{k \to \infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^k}} \int_Q M_x M_y h_{2^{4^k-1}}$$

$$\sim \lim_{k \to \infty} \frac{1}{2^{k+1}} \cdot \frac{1}{2^{4^k-1}} \cdot 2^{4^k-1} = \lim_{k \to \infty} 2^{k-3} = \infty. \quad \blacksquare$$

We now show that if $f$ is a function such that $\int_Q M_y M_x f < \infty$ but $\int_Q M_x M_y f = \infty$, then there exists a set $A$ of finite measure in $\mathbb{R}^2$ such that $\int_A M_S f = \infty$. This result is particularly interesting in view of the fact that M. E. Gomez has constructed a function $g$ supported on $Q$ such that $\int_Q M_x M_y g$ and $\int_Q M_y M_x g$ are infinite, but $M_S g$ is integrable over every set of finite measure in $\mathbb{R}^2$ (see [5]). Such a construction is also implicit in the work of Bagby and Jawerth and Morrow ([1], [9]).

**Theorem 11.** Suppose that $f$ is a measurable function supported on $Q$, $\int_Q M_y M_x f = \infty$, and $\int_Q M_y M_x f < \infty$. Then there exists a set $A$ of finite measure in $\mathbb{R}^2$ such that $\int_A M_S f = \infty$.

**Proof.** We first recall a theorem due to Fava, Gatto, and Gutiérrez.

**Theorem 12 ([3]).** Suppose $f$ is a measurable function supported in $Q$. Then $M_x M_y f$ is integrable over every set of finite measure in $\mathbb{R}^2$ if and only if $\|f\|_{L(\log L)^2(Q)} < \infty$.

The maximal operators $M, \overline{M}$, defined as follows, will be very useful to us:

**Definition 13.** Let $f$ be a measurable function supported on $Q$. The associated maximal function $M f$ is defined on $Q$ by

$$M f(p_1, p_2) = \sup_{x_1 < p_1 < x_2} \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} |f(x, y)| \, dy \, dx.$$ 

The associated maximal function $\overline{M} f$ is defined on $Q$ by

$$\overline{M} f(p_1, p_2) = \sup_{y_1 < p_2 < y_2} \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} |f(x, y)| \, dy \, dx.$$
Now, we define the auxiliary functions $g$ and $h$ as follows. If $p = (p_1, p_2) \in \mathbb{R}^2$, let
\[
g(p) = \int_0^1 |f(p_1, y)| \, dy \cdot \chi_Q(p), \quad h(p) = \int_0^1 |f(x, p_2)| \, dx \cdot \chi_Q(p).
\]

Note that if $p \in \mathbb{R}^2$, then
\[
MSf(p) \geq \frac{1}{2} [MSg(p) + MS h(p)].
\]

Since $g$ and $h$ are tensors on $Q$, also note that
\[
MSg(p) = M_x M_y g(p) = M_y M_x g(p),
\]
\[
MS h(p) = M_x M_y h(p) = M_y M_x h(p).
\]

Now, by Theorem 12, $M_x M_y g$ is integrable over every set of finite measure in $\mathbb{R}^2$ if and only if $g \in L(\log L)^2(Q)$, and similarly for $h$ in place of $g$. If $\|g + h\|_{L(\log L)^2(Q)} = \infty$, equations (19)–(21) imply the existence of a set of finite measure in $\mathbb{R}^2$ over which $MSf$ is not integrable. So it suffices to show $\|g + h\|_{L(\log L)^2(Q)} = \infty$.

An application of the Fubini Theorem and Theorem 6 yields that
\[
\|g\|_{L(\log L)^2(Q)} \sim \int_Q MMf, \quad \|h\|_{L(\log L)^2(Q)} \sim \int_Q M\overline{M}f.
\]

So it is enough to show $\int_Q MMf + \int_Q M\overline{M}f = \infty$. As $\int_Q M_x M_y f \lesssim \|f\|_{L(\log L)^2(Q)}$, we see that the proof reduces to proving the following.

**Theorem 14.** Let $f$ be a measurable function supported on $Q$. Then
\[
\|f\|_{L(\log L)^2(Q)} \lesssim \int_Q MMf + \int_Q M_y M_x f + \int_Q M\overline{M}f.
\]

**Proof.** It will be technically convenient to work with the dyadic analogues of the maximal operators $M_{\text{HL}}$, $M$, $\overline{M}$, $M_x$, and $M_y$. Recall that a dyadic interval in $[0, 1]$ is an interval of the form $[k \cdot 2^j, (k+1) \cdot 2^j]$, where $j$ is a nonpositive integer and $k$ is a nonnegative integer such that $(k+1) \cdot 2^j \leq 1$. We denote the set of dyadic subintervals of $[0, 1]$ by $T^\Delta$. A dyadic square in $Q$ is a set of the form $I \times J$, where $I$ and $J$ are dyadic intervals in $[0, 1]$ of the same length. We denote the set of dyadic squares in $Q$ by $S^\Delta$. We formally define the dyadic maximal operators $M_{\text{HL}}^\Delta$, $M^\Delta$, $\overline{M}^\Delta$, $M_x^\Delta$, and $M_y^\Delta$ as follows.

**Definition 15.** Let $f$ be a measurable function supported on $Q$. The dyadic Hardy–Littlewood maximal function $M_{\text{HL}}^\Delta f$ is defined on $Q$ by
\[
M_{\text{HL}}^\Delta f(p) = \sup_{p \in S \in S^\Delta} \frac{1}{|S|} \int_S |f|.
\]
The maximal function $M^\Delta f$ is defined by

$$M^\Delta f(p_1, p_2) = \sup_{p_1 \in I \in \mathcal{I}^\Delta} \frac{1}{|I|} \int_I \int_0^1 |f(x, y)| \, dy \, dx.$$ 

$\overline{M}^\Delta f$ is defined by

$$\overline{M}^\Delta f(p_1, p_2) = \sup_{p_2 \in I \in \mathcal{I}^\Delta} \frac{1}{|I|} \int_I \int_0^1 |f(x, y)| \, dy \, dx.$$ 

$M_x^\Delta f$ is defined by

$$M_x^\Delta f(p_1, p_2) = \sup_{p_2 \in I \in \mathcal{I}^\Delta} \frac{1}{|I|} \int_I |f(x, p_2)| \, dx.$$ 

Similarly, $M_y^\Delta f$ is defined by

$$M_y^\Delta f(p_1, p_2) = \sup_{p_2 \in I \in \mathcal{I}^\Delta} \frac{1}{|I|} \int_I |f(p_1, y)| \, dy.$$ 

If $f$ is a measurable function supported on $[0, 1]$, then the dyadic Hardy–Littlewood maximal function $M_{\text{HL}}^\Delta f$ is defined on $[0, 1]$ by

$$M_{\text{HL}}^\Delta f(p) = \sup_{p \in I \in \mathcal{I}^\Delta} \frac{1}{|I|} \int_I |f|.$$ 

We will also need the following results found in [7].

**Lemma 16 ([7]).** Let $f$ be a measurable function supported on $Q$. Then

$$\int_Q M_{\text{HL}} f \sim \int_Q (M_x f + M_y f).$$ 

Moreover,

$$\int_Q M_{\text{HL}} f \sim \int_Q (M f + M_y f).$$ 

**Lemma 17 ([7]).** Let $f$ be a measurable function supported on $Q$. Then

$$\int_Q M^\Delta M_x^\Delta f \lesssim \int_Q M^\Delta M^\Delta f + \int_Q M_y^\Delta M_x^\Delta f.$$ 

Furthermore,

$$\int_Q M_x M_x f \lesssim \int_Q M f + \int_Q M_y M_x f.$$ 

**Lemma 18 ([7]).** Let $f$ be a nonnegative measurable function supported on $Q$. Let $\tilde{f}(x, y)$ be the function supported on $Q$ which is nonincreasing in $x$. 

(i.e. $\tilde{f}(x_1, y) \geq \tilde{f}(x_2, y)$ whenever $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y \leq 1$) and such that, for each $y \in [0, 1]$, $\tilde{f}(\cdot, y)$ and $f(\cdot, y)$ are equidistributed. Then

$$\int_Q M_{yM} \tilde{f} \lesssim \int_Q M_{yM} f.$$

**Lemma 19.** Suppose $f(x, y)$ is a nonnegative measurable function supported on $Q$ which is nonincreasing in $x$. Then

$$\int_Q M_x M_y f \lesssim \int_Q M_x M_y f + \int_Q M_y M_x f.$$

**Proof.** As $f$ is nonincreasing in $x$, $\int_Q M_x M_y f = \int_Q M_x M_y f$. So, letting $f'$ be a function supported on $Q$ such that $f'(x, \cdot)$ and $f(x, \cdot)$ are equidistributed for each $x \in [0, 1]$ and also such that $f'(x, y)$ is nonincreasing in $y$, we see that

$$\int_Q (M_x M_y f + M_y M_x f) = \int_Q (M_x M_y f + M_y M_x f)$$

$$\sim \int_Q M_M M_x f \quad \text{(by Lemma 16)}$$

$$\gtrsim \int_Q M_M M_x f' \quad \text{(by Lemma 18)}$$

$$\sim \int_Q M_x M_y f' + \int_Q M_y M_x f'$$

$$\gtrsim \int_Q M_y M_x f'$$

$$= \int_Q M_x M_y f' \quad \text{(since $f'$ is nonincreasing in both variables)}$$

$$\sim \int_Q M_x M_y f \quad \text{(by Theorem 6 and the Fubini Theorem).} \quad \blacksquare$$

**Lemma 20.** Let $B = [0, 2^{-n}] \times [0, 1]$ be a subset of $Q$. Suppose $f$ is a measurable function supported in $B$. Then

$$\int_B M_{yM} f \lesssim \int_B (M_x + M_y) f.$$

**Proof.** Let $g(x, y) = f(x \mod 2^{-n}, y)$. As

$$|\{p \in Q : M_{HL} g(p) > \alpha\}| \geq 2^n |\{p \in B : M_{HL} f(p) > \alpha\}|,$$
we see that \( \int_Q M_{HL}g \geq |B|^{-1} \int_B M_{HL}f \). As
\[
\int_Q M_{HL}g \sim \int_Q (M_x + M_y)g \sim \frac{1}{|B|} \int_B (M_x + M_y)f,
\]
we see that \( \int_B M_{HL}f \lessapprox \int_B (M_x + M_y)f \). ■

**Corollary 21.** Let \( R \) be a rectangle supported in \( Q \) of width or height one. Let \( f \) be a measurable function supported on \( R \). Then
\[
\frac{1}{|R|} \int_R M_{HL}f \lessapprox \frac{1}{|R|} \int_R (M_x + M_y)f.
\]

**Proof.** This follows from Lemma 20 by symmetry arguments. ■

**Lemma 22.** Suppose \( f \) is a nonnegative measurable function supported on \( Q \) which is nonincreasing in \( x \). Then
\[
\|f\|_{L(\log L)^2(Q)} \lessapprox \int_Q (MMf + MM\overline{M}f + M_yM_xf).
\]

**Proof.** We assume without loss of generality that \( f \in C^\infty(Q) \). Note that
\[
|\{p \in Q : M_{HL}(p) > \alpha\}| \leq 1000|\{p \in Q : M_{HL}^\Delta(p) > \alpha/1000\}|
\]
for all \( \alpha > 0 \). Then
\[
\int_Q M_{HL}M_{HL}f \sim \int_Q M_{HL}^\Delta M_{HL}^\Delta f.
\]
Hence
\[
\|f\|_{L(\log L)^2(Q)} \sim \int_Q M_{HL}^\Delta M_{HL}^\Delta f
\]
\[
\sim \int_Q \overline{M}^\Delta M_{HL}^\Delta f + \int_Q M_x^\Delta M_{HL}^\Delta f \quad \text{(by Lemma 16)}.
\]
It is then enough to show
\[
(i) \quad \int_Q \overline{M}^\Delta M_{HL}^\Delta f \lessapprox \int_Q (MMf + \overline{M}Mf + M_yM_xf),
\]
\[
(ii) \quad \int_Q M_x^\Delta M_{HL}^\Delta f \lessapprox \int_Q (MMf + \overline{M}Mf + M_yM_xf).
\]
To prove (i), let \( \ell \) be a unit horizontal line segment through \( Q \). Let \( p \in \ell \). Let \( B \) be a horizontal dyadic band through \( Q \) such that \( \overline{M}^\Delta M_{HL}^\Delta f(p) \sim |B|^{-1} \int_B M_{HL}^\Delta f \). Let now \( f_{\text{int}} = f \cdot \chi_B, f_{\text{ext}} = f \cdot \chi_{B^c} \). Now by Corollary 21,
\[
\frac{1}{|B|} \int_B M_{HL}^\Delta f_{\text{int}} \lessapprox \frac{1}{|B|} \int_B (M_x^\Delta + M_y^\Delta)f_{\text{int}} \lessapprox \overline{M}^\Delta (M_x^\Delta + M_y^\Delta)f(p).
\]
Also,
\[
\frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{ext}} \ll \frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{int}} \ll \frac{1}{|B|} \int_B M_y M_x f.
\]
(The equality above holds because \(M_{\text{HL}} f_{\text{ext}}\) is constant on vertical slices of \(B\).)
So
\[
\int_{\ell} M_{\text{HL}}^A f \ll \int_{\ell} M_{\text{HL}} (M_x^A + M_y^A) f + \int_{\ell} M_y M_x f.
\]
Hence
\[
\int_Q M_{\text{HL}}^A f \ll \int_Q M_{\text{HL}} (M_x^A + M_y^A) f + \int_Q M_y M_x f.
\]
Now \(\int_Q M_{\text{HL}} (M_x^A + M_y^A) f \ll \int_Q M_{\text{HL}} M_x^A f + \int_Q M_{\text{HL}} M_y^A f\). It is clear that
\(\int_Q M_{\text{HL}} M_x^A f \leq \int_Q M_y M_x f\). We also have
\[
\int_Q M_{\text{HL}} M_y^A f \ll \int_Q M M f + \int_Q M y M x f
\]
(by Lemma 17)
\[
\ll \int_Q M M f + \int_Q (M M f + M y M x f)
\]
(by Lemma 19).
Hence
\[
\int_Q M_{\text{HL}} (M_x^A + M_y^A) f \ll \int_Q (M M f + M M f + M y M x f),
\]
and thus
\[
\int_Q M_{\text{HL}}^A f \ll \int_Q (M M f + M M f + M y M x f).
\]
So (i) is proved.

We now prove (ii). Since \(f\) is nonincreasing in \(x\), we have \(\int_Q M_{\text{HL}}^A f \ll \int_Q M_{\text{HL}}^A f\). Let \(\ell\) now denote a vertical unit segment in \(Q\), and \(B\) the part of \(Q\) to the left of \(\ell\). We want to show
\[
\int_{\ell} M_{\text{HL}}^A f \ll \frac{1}{|B|} \int_B M_{\text{HL}}^A f \ll \frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{int}} + \frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{ext}}.
\]
Let \(f_{\text{int}} = f \cdot \chi_B\), \(f_{\text{ext}} = f \cdot \chi_{B^c}\). Now
\[
\frac{1}{|B|} \int_B M_{\text{HL}}^A f \ll \frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{int}} + \frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{ext}}.
\]
By Corollary 21 we have
\[
\frac{1}{|B|} \int_B M_{\text{HL}} f_{\text{int}} \ll \frac{1}{|B|} \int_B (M_x + M_y) f_{\text{int}} \ll M (M_x + M_y) f(p)
\]
for any \( p \in \ell \). So

\[
\frac{1}{|B|} \int_B M_{\text{HL}}^A f_{\text{int}} \lesssim \int_{\ell} M(M_x + M_y)f.
\]

Now if \( p \in B \), then \( M_{\text{HL}}^A f_{\text{ext}}(p) \leq M_y f(p) \) since \( f(x, y) \) is nonincreasing in \( x \) and \( f_{\text{ext}} \) is supported to the right of \( B \). Hence

\[
\frac{1}{|B|} \int_B M_{\text{HL}}^A f_{\text{ext}} \lesssim \frac{1}{|B|} \int_B M_y f \lesssim M(M_y f)(q)
\]

for any point \( q \in \ell \). So

\[
\frac{1}{|B|} \int_B M_{\text{HL}}^A f_{\text{ext}} \lesssim \int_{\ell} M M_y f.
\]

Hence \( \int_{\ell} M^A M_{\text{HL}}^A f \lesssim \int_{\ell} M(M_x + M_y)f \). This implies

\[
\int_Q M M_{\text{HL}}^A f \lesssim \int_Q M(M_x + M_y)f.
\]

As \( \int_Q M M_y f \sim \int_Q M M f \), since \( f \) is nonincreasing in \( x \), and

\[
\int_Q M M_y f \leq \int_Q M_x M_y f \lesssim \int_Q (M M f + M_y M_x f)
\]

by Lemma 19, we get the desired result. \( \blacksquare \)

We now complete the proof of Theorem 14, and hence of Theorem 11. \( f \) is a measurable function supported on \( Q \). Without loss of generality we assume \( f \) is a nonnegative function as well. Let \( \tilde{f}(x, y) \) be a function supported on \( Q \) which is nonincreasing in \( x \) and such that \( \tilde{f}(\cdot, y) \) and \( f(\cdot, y) \) are equidistributed for each \( y \in [0, 1] \). Now

\[
\|f\|_{L(\log L)^2(Q)} = \|\tilde{f}\|_{L(\log L)^2(Q)}
\]

\[
\lesssim \int_Q (M M \tilde{f} + M_y M_x \tilde{f} + \overline{M M} \tilde{f}) \quad \text{(Lemma 22)}
\]

\[
\sim \int_Q (M_x M \tilde{f} + M_y M_x \tilde{f}) + \int_Q \overline{M M} \tilde{f}
\]

\[
\sim \int_Q M_{\text{HL}} M_x \tilde{f} + \int_Q \overline{M M} \tilde{f} \quad \text{(Lemma 16)}
\]

\[
\sim \int_Q M_{\text{HL}} M_x f + \int_Q \overline{M M} f \quad \text{(as } \|M_x f\|_{L \log L(Q)} \sim \|M_x \tilde{f}\|_{L \log L(Q)})
\]}
\[ \sim \sum_{Q} M_xM_xf + \sum_{Q} M_yM_xf + \sum_{Q} \overline{M}Mf \quad \text{(Lemma 16)} \]

\[ \lesssim \sum_{Q} MMf + \sum_{Q} M_yM_xf + \sum_{Q} \overline{M}Mf \quad \text{(Lemma 17)} \]
as desired.

References


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