STUDIA MATHEMATICA 147 (1) (2001)

A transplantation theorem for ultraspherical polynomials at critical index

by

$\boxed{\text{J. J. Guadalupe}}$ (Logroño) and V. I. Kolyada (Odessa)

Abstract. We investigate the behaviour of Fourier coefficients with respect to the system of ultraspherical polynomials. This leads us to the study of the "boundary" Lorentz space $\mathcal{L}_\lambda$ corresponding to the left endpoint of the mean convergence interval. The ultraspherical coefficients $\{c_n^{(\lambda)}(f)\}$ of $\mathcal{L}_\lambda$-functions turn out to behave like the Fourier coefficients of functions in the real Hardy space $\operatorname{Re} H^1$. Namely, we prove that for any $f \in \mathcal{L}_\lambda$ the series $\sum_{n=1}^\infty c_n^{(\lambda)}(f)\cos n\theta$ is the Fourier series of some function $\varphi \in \operatorname{Re} H^1$ with $\|\varphi\|_{\operatorname{Re} H^1} \le c\|f\|_{\mathcal{L}_\lambda}$.

1. Introduction. Let $I = [-1,1]$, $0 < \lambda < \infty$ and $dm_\lambda(x) = (1-x^2)^{\lambda-1/2}\,dx$. Denote by $\{\varphi_n^{(\lambda)}\}_{n=0}^\infty$ the normalized system of ultraspherical polynomials, that is, the orthonormal system in $L^2(I, m_\lambda)$ obtained from $\{x^n\}_{n=0}^\infty$ by the Gram–Schmidt process. It is well known (see [18], [19]) that $\{\varphi_n^{(\lambda)}\}$ is a basis in $L^p(I, m_\lambda)$ if and only if $p_\lambda < p < p'_\lambda$, where $p_\lambda = (2\lambda+1)/(\lambda+1)$ (for any $p \in (1,\infty)$, by p' we denote its conjugate exponent, $p' = p/(p-1)$).

Let $f \in L^1(I, m_\lambda)$ and

$$a_n(f) = \int_I f(x)\varphi_n^{(\lambda)}(x)\,dm_\lambda(x). \tag{1.1}$$

Form a cosine series with these coefficients:

$$\sum_{n=1}^\infty a_n(f)\cos n\theta, \tag{1.2}$$

and consider the following problem. Suppose that f belongs to some Lorentz

2000 *Mathematics Subject Classification*: 42C10, 33C45, 42A50.

Key words and phrases: transplantation, Fourier coefficients, orthogonal polynomials, Hardy spaces, Lorentz spaces.

Professor José J. Guadalupe tragically died on April 1, 2000.

This paper was written while the second named author stayed at the University of La Rioja (Spain) as a visiting professor.

space $L^{p,r}(I, m_\lambda)$. What conditions imply that the series (1.2) is the Fourier series of a 2π-periodic integrable function φ and what class does φ belong to?

Observe that "transplantation" problems of such type and their applications have been studied in a lot of works (see [1–3], [8], [17]). Our interest in these problems was motivated by estimations of Fourier coefficients (1.1). It can be easily proved that if $f \in L^{p,r}(I, m_\lambda)$, $p_\lambda < p < 2$, then $\{a_n(f)\} \in l^{q',r}$, where $q = p/[(2-p)\lambda+1]$ (See Section 4 below). The related transplantation theorem follows immediately from the results of Askey and Wainger [2] (see also Lemma 2).

Theorem A. *Given* $f \in L^{p,r}(I, m_\lambda)$ $(\lambda > 0,\ p_\lambda < p < 2,\ r \geq 1)$, *the series* (1.2) *is the Fourier series of some function* $\varphi \in L^{q,r}[0, 2\pi]$, $q = p/[(2-p)\lambda + 1]$, *such that*

$$\|\varphi\|_{L^{q,r}} \leq c\|f\|_{L^{p,r}(m_\lambda)}.$$

It is well known that for any function $\varphi \in L^{q,r}[0, 2\pi]$ $(1 < q < 2,$ $r > 0)$ the sequence of its trigonometric Fourier coefficients belongs to $l^{q',r}$ (see [9], [21]).

The situation changes in the critical case $p = p_\lambda$. Using the asymptotic formula for ultraspherical polynomials (see [10, Proposition 2.3]), it is easy to deduce that for $1 \leq s \leq \infty$,

$$\|\varphi_n^{(\lambda)}\|_{L^{p'_\lambda, s}} \geq c(\ln n)^{1/s} \qquad (c > 0).$$

The uniform boundedness principle then implies that for any $r > 1$ there exists $f \in L^{p_\lambda, r}(I, m_\lambda)$ whose sequence of Fourier coefficients with respect to $\{\varphi_n^{(\lambda)}\}$ is unbounded (see also [5], [15], [18]). On the other hand, it follows from the asymptotic formula that for any function $f \in L^{p_\lambda,1}(I, m_\lambda)$,

$$|a_n(f)| \leq c\|f\|_{L^{p_\lambda,1}(m_\lambda)},$$

which in turn implies that $a_n(f) \to 0$. Moreover, we prove that for each $f \in L^{p_\lambda,1}(I, m_\lambda)$,

$$\sum_{n=0}^{\infty} \frac{|a_n(f)|}{n+1} \leq c\|f\|_{L^{p_\lambda,1}(m_\lambda)} \qquad (\lambda > 0). \tag{1.3}$$

The known transplantation theorems cannot be applied to the limiting case $p = p_\lambda$, $r = 1$. Formally, in this case Theorem A should give the corresponding function φ belonging to $L^1[0, 2\pi]$. But it is well known that for $\varphi \in L^1[0, 2\pi]$ the series

$$\sum_{n\in\mathbb{Z}} \frac{|\widehat{\varphi}(n)|}{|n|+1}, \qquad \widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)e^{-in\theta}\,d\theta, \tag{1.4}$$

may diverge. At the same time, by Hardy's theorem, the series (1.4) converges for any $\varphi \in \operatorname{Re} H^1$.

Recall that $\operatorname{Re} H^1$ is the real Hardy space of all 2π-periodic functions $\varphi \in L^1[0,2\pi]$ such that the conjugate function $\widetilde{\varphi}$ also belongs to $L^1[0,2\pi]$; the norm in $\operatorname{Re} H^1$ is defined by

$$\|\varphi\|_{\operatorname{Re} H^1} = \|\varphi\|_1 + \|\widetilde{\varphi}\|_1.$$

The main result of this paper is the following theorem.

THEOREM 1. *Let $f \in L^{p_\lambda,1}(I,m_\lambda)$, $0<\lambda<\infty$, and let $\{a_n(f)\}$ be the sequence of Fourier coefficients of f with respect to the system $\{\varphi_n^{(\lambda)}\}$. Then the series* (1.2) *is the Fourier series of some function $\varphi \in \operatorname{Re} H^1$ such that*

$$\|\varphi\|_{\operatorname{Re} H^1} \le c\|f\|_{L^{p_\lambda,1}(m_\lambda)}.$$

To prove this theorem we use Mehler's integral representation of ultraspherical polynomials and Weyl's fractional integrals.

Notice that the series

$$\sum_{n=1}^{\infty} \frac{a_n^*(f)}{n},$$

where $f \in L^{p_\lambda,1}(I,m_\lambda)$ and $\{a_n^*(f)\}$ is the non-increasing rearrangement of $\{a_n(f)\}$, may diverge (see Section 4). At the same time, applying L^1-estimates of exponential sums [12], [14] and Theorem 1, we immediately obtain the following analogue of Littlewood's conjecture.

COROLLARY. *Given $\lambda>0$, there exists a constant $A_\lambda>0$ such that for any set of positive integers $n_1<\ldots<n_N$,*

$$\Big\|\sum_{k=1}^{N} \varphi_{n_k}^{(\lambda)}\Big\|_{L^{p_\lambda,1}(m_\lambda)} \ge A_\lambda \log N.$$

2. Auxiliary propositions. First we recall the definition of the Lorentz space (see [4]). Let (R,μ) be a measure space with a finite measure μ. The non-increasing rearrangement of a μ-measurable function f defined on R will be denoted by f_μ^*. The *Lorentz space* $L^{p,r}(R,\mu)$ $(1\le p,r<\infty)$ consists of all f such that

$$\|f\|_{p,r} \equiv \Big(\int_0^{\mu(R)} [t^{1/p} f_\mu^*(t)]^r \,\frac{dt}{t}\Big)^{1/r} < \infty.$$

Let $f \in L^{p,r}(I,m_\lambda)$, $0<\lambda<\infty$. Consider the measure μ_λ in $[0,\pi]$ defined by $d\mu_\lambda(\theta) = (\sin\theta)^{2\lambda}\,d\theta$. Set $h(\theta) = f(\cos\theta)$, $\theta\in[0,\pi]$. It is easy to see that $f_{m_\lambda}^*(t) = h_{\mu_\lambda}^*(t)$. Thus, $\|f\|_{L^{p,r}(I,m_\lambda)} = \|h\|_{L^{p,r}([0,\pi],\mu_\lambda)}$. Furthermore, we have (see (1.1))

$$a_n(f) = \int_0^\pi h(\theta)\varphi_n^{(\lambda)}(\cos\theta)\,d\mu_\lambda(\theta).$$

It will be more convenient to consider the function $h(\theta)$ instead of $f(x)$. Also, we define

$$\mathcal{L}_\lambda = L^{p_\lambda,1}([0,\pi],\mu_\lambda) \qquad \left(0 \le \lambda < \infty,\ p_\lambda = \frac{2\lambda+1}{\lambda+1}\right).$$

Lemma 1. *Let $0 \le \alpha < \beta$. Then for any measurable set $E \subset [0,\pi]$,*

$$\mu_\alpha(E) \le 8[\mu_\beta(E)]^{(2\alpha+1)/(2\beta+1)}. \tag{2.1}$$

Proof. Let $\tau = \mu_0(E) \equiv |E|$. Then

$$\mu_\beta(E) = \int_E (\sin x)^{2\beta}\,dx \ge 2\int_0^{\tau/2} (\sin x)^{2\beta}\,dx \ge \frac{\tau^{2\beta+1}}{(2\beta+1)\pi^{2\beta}}.$$

Thus, $|E| \le 8[\mu_\beta(E)]^{1/(2\beta+1)}$. Using this estimate, for $0 < \alpha < \beta$ we have

$$\begin{aligned}\mu_\alpha(E) &= \int_E (\sin x)^{2\alpha}\,dx \le \left(\int_E (\sin x)^{2\beta}\,dx\right)^{\alpha/\beta} |E|^{(\beta-\alpha)/\beta}\\ &\le 8[\mu_\beta(E)]^{(2\alpha+1)/(2\beta+1)}.\end{aligned}$$

The lemma is proved.

In what follows we will write f_α^* instead of $f_{\mu_\alpha}^*$.

Lemma 2. *Let $\sigma > 0$, $0 \le \alpha < \beta$ and $p = 2(\beta-\alpha)/\sigma$. Suppose that f is a measurable function on $[0,\pi]$ and $\varphi(x) = f(x)(\sin x)^\sigma$. Then*

$$\varphi_\alpha^*(t) \le c'\left(\frac{1}{t}\int_{ct^\gamma}^\pi f_\beta^*(u)^p\,du\right)^{1/p}, \quad 0 < t \le \pi, \tag{2.2}$$

where $\gamma = (2\beta+1)/(2\alpha+1)$ and c, c' are some positive constants. Furthermore, if $f \in \mathcal{L}_\beta$ and $\sigma = \beta - \alpha$, then $\varphi \in \mathcal{L}_\alpha$ and

$$\|\varphi\|_{\mathcal{L}_\alpha} \le C\|f\|_{\mathcal{L}_\beta}. \tag{2.3}$$

Proof. Let $0 < t < 1/4$ and let $E_t \subset [0,\pi]$ be a measurable set such that $\mu_\beta(E_t) = t^\gamma$ and

$$\{x \in [0,\pi] : |f(x)| > f_\beta^*(t^\gamma)\} \subset E_t \subset \{x \in [0,\pi] : |f(x)| \ge f_\beta^*(t^\gamma)\}.$$

Set $f_t(x) = f(x)\chi_{[0,\pi]\setminus E_t}(x)$. Then (see [4], p. 49)

$$\int_0^\pi |f_t(x)|^p\,d\mu_\beta(x) = \int_{t^\gamma}^\pi f_\beta^*(u)^p\,du.$$

By (2.1), $\mu_\alpha(E_t) \le 8t$. Setting $\varphi_t(x) = f_t(x)(\sin x)^\sigma$, we have

$$\int_0^\pi |\varphi_t(x)|^p \, d\mu_\alpha(x) = \int_{[0,\pi]\setminus E_t} |\varphi(x)|^p \, d\mu_\alpha(x) \ge \int_{\mu_\alpha(E_t)}^\pi \varphi_\alpha^*(u)^p \, du \ge \int_{8t}^\pi \varphi_\alpha^*(u)^p \, du.$$

Taking into account that

$$\int_0^\pi |\varphi_t(x)|^p \, d\mu_\alpha(x) = \int_0^\pi |f_t(x)|^p \, d\mu_\beta(x),$$

we obtain

$$t\varphi_\alpha^*(9t)^p \le \int_{8t}^\pi \varphi_\alpha^*(u)^p \, du \le \int_{t^\gamma}^\pi f_\beta^*(u)^p \, du,$$

which implies (2.2).

Let now $\sigma = \beta - \alpha$. Using (2.2) and standard estimates (see [4], p. 217), we get

$$\varphi_\alpha^*(t) \le c' \left(\frac{1}{t} \int_{ct^\gamma}^\pi f_\beta^*(u)^2 \, du \right)^{1/2} \le c'' t^{-1/2} \int_{ct^\gamma/2}^\pi f_\beta^*(u) \frac{du}{\sqrt{u}}.$$

From this the inequality (2.3) follows immediately.

Now we will consider some results related to fractional integrals.

Let $\varphi \in L^1[0, 2\pi]$ and $0 < \alpha < 1$. The *Weyl fractional integrals* of order α of the function φ are defined by the equality (see [20], §19)

$$I_\pm^{(\alpha)} \varphi(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_\pm^{(\alpha)}(x-t)\varphi(t) \, dt,$$

where

$$\Psi_+^{(\alpha)}(t) = \psi_\alpha(t) + r_\alpha(t), \quad -2\pi < t \le 2\pi, \tag{2.4}$$

with $\psi_\alpha(t) = 2\pi|t|^{\alpha-1}\chi_{(0,\infty)}(t)/\Gamma(\alpha)$ $(t \in \mathbb{R})$ and $r_\alpha(t) \in C^\infty(-2\pi, 2\pi]$; furthermore,

$$\Psi_-^{(\alpha)}(t) = \Psi_+^{(\alpha)}(-t). \tag{2.5}$$

Lemma 3. *Let $g \in L^1[0,a]$ $(0 < a \le \pi)$, $0 < \lambda < 1$, $\alpha \in \mathbb{R}$,*

$$G(x) = e^{i\alpha x} \int_x^a (y-x)^{\lambda-1} g(y) \, dy, \quad 0 \le x \le a,$$

$G(x) = 0$ for $a < x < 2\pi$, and $G(x+2\pi) = G(x)$. Then there exists $\varphi \in L^1[0, 2\pi]$ such that

$$\|\varphi\|_1 \le c\|g\|_1 \tag{2.6}$$

and for almost all $x \in [0, 2\pi]$,

$$G(x) = A_0 + I_-^{(\lambda)}\varphi(x).$$

Proof. It is easy to see that

$$\int_0^a |G(x)|x^{-\lambda}\,dx \le c\|g\|_1. \tag{2.7}$$

Let

$$\begin{aligned} \Phi(x) \equiv I_-^{(1-\lambda)}G(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_-^{(1-\lambda)}(x-y)G(y)\,dy \\ &= \frac{1}{\Gamma(1-\lambda)}\int_x^{\pi} G(y)(y-x)^{-\lambda}\,dy + \frac{1}{2\pi}\int_0^a r_{1-\lambda}(y-x)G(y)\,dy \\ &\equiv \Phi_1(x) + \Phi_2(x), \qquad x \in [-\pi, \pi] \end{aligned}$$

(see (2.4), (2.5)). The function Φ is periodic with period 2π. The lemma will be proved if we show that Φ is absolutely continuous in $[-\pi, \pi]$ and $\varphi = -\Phi'$ satisfies (2.6) (see [20], p. 348).

Observe that Φ_2 is infinitely differentiable on $[-\pi, \pi]$ and

$$\|\Phi_2'\|_\infty \le c\|G\|_1 \le c'\|g\|_1. \tag{2.8}$$

Now consider the function Φ_1. For $x \in [-\pi, 0)$ we have

$$\Phi_1(x) = \frac{1}{\Gamma(1-\lambda)}\int_0^{\pi} G(y)(y-x)^{-\lambda}\,dy$$

and, by (2.7),

$$\int_{-\pi}^{0} |\Phi_1'(x)|\,dx \le \frac{1}{\Gamma(1-\lambda)}\int_0^{\pi} |G(x)|x^{-\lambda}\,dx \le c\|g\|_1. \tag{2.9}$$

Let now $x \in [0, a)$. Then

$$\Phi_1(x) = \frac{1}{\Gamma(1-\lambda)}\int_x^a g(y)s(x,y)\,dy,$$

where

$$s(x,y) = \int_0^1 \xi^{-\lambda}(1-\xi)^{\lambda-1}e^{i\alpha(x+\xi(y-x))}\,d\xi.$$

We have $|s'_x(x,y)| \le c$ and therefore

$$\Phi'_1(x) \le c\,(|g(x)| + \|g\|_1), \quad x \in [0,a),$$

and

$$\int_0^a |\Phi'_1(x)|\,dx \le c\|g\|_1. \tag{2.10}$$

Note also that $\Phi_1(x) = 0$ for $x \in [a,\pi]$ and by (2.10), Φ_1 is continuous at a. Thus, Φ_1 is absolutely continuous in $[-\pi,\pi]$. Moreover, it follows from (2.8)–(2.10) that $\|\Phi'\|_1 \le c\|g\|_1$. The lemma is proved.

Similarly, we have

LEMMA 4. *Let $g \in L^1[0,a]$ $(0 < a \le \pi)$, $0 < \lambda < 1$, $\alpha \in \mathbb{R}$,*

$$H(x) = e^{i\alpha x}\int_0^x (y-x)^{\lambda-1} g(y)\,dy, \quad 0 \le x \le a,$$

$H(x) = 0$ for $a < x < 2\pi$, and $H(x+2\pi) = H(x)$. Then there exists $\psi \in L^1[0,2\pi]$ such that

$$\|\psi\|_1 \le c\|g\|_1$$

and for almost all $x \in [0,2\pi]$,

$$H(x) = B_0 + I_+^{(\lambda)}\psi(x).$$

LEMMA 5. *Let $\lambda \ge 1$ and φ be a measurable function in an interval $[0,a]$, $(0 < a \le \pi)$ such that*

$$\int_0^a |\varphi(x)| x^{\lambda-1}\,dx < \infty. \tag{2.11}$$

Let

$$G_\lambda(x) = \int_x^a \varphi(y)(\cos x - \cos y)^{\lambda-1}\,dy, \quad 0 \le x \le a.$$

Then:

(i) *for any $0 \le k \le [\lambda]-1$, the derivative $G_\lambda^{(k)}$ is bounded in $(0,a]$;*
(ii) *if $\lambda \ge 2$, then for any odd $k \le [\lambda]-1$,*

$$\lim_{x\to+0} G_\lambda^{(k)}(x) = 0.$$

Proof. First suppose that φ is an arbitrary locally integrable function in $(0,a]$. We will show that for any integer $\nu \ge 2$ there exists a constant $C_\nu > 0$ such that for all $\lambda \in [\nu,\nu+1)$, $1 \le k \le \nu-1$ and $x \in (0,a)$,

$$|G_\lambda^{(k)}(x)| \le C_\nu x \int_x^a |\varphi(y)| y^{2\lambda-k-3}\,dy \quad \text{if } k \text{ is odd}, \tag{2.12}$$

$$|G_\lambda^{(k)}(x)| \le C_\nu \int_x^a |\varphi(y)| y^{2\lambda-k-2}\,dy, \qquad \text{if } k \text{ is even.} \tag{2.13}$$

We have

$$G'_\lambda(x) = (1-\lambda)\sin x \int_x^a \varphi(y)(\cos x - \cos y)^{\lambda-2}\,dy = (1-\lambda)G_{\lambda-1}(x)\sin x.$$

Hence

$$|G'_\lambda(x)| \le (\lambda-1)x \int_x^a |\varphi(y)|(1-\cos y)^{\lambda-2}\,dy \le (\lambda-1)x\int_x^a |\varphi(y)|y^{2\lambda-4}\,dy,$$

and therefore, for any $\nu \ge 2$ and $k=1$, the inequality (2.12) holds. If $\nu \ge 3$ and $k \ge 2$, then for any $\lambda \in [\nu, \nu+1)$,

$$G_\lambda^{(k)}(x) = (1-\lambda)\sum_{j=0}^{k-1}\binom{k-1}{j}\sin\left(x + (k-1-j)\frac{\pi}{2}\right)G_{\lambda-1}^{(j)}(x).$$

Applying induction, we easily derive inequalities (2.12) and (2.13).

Now, assume that the condition (2.11) holds. It implies that G_λ is continuous in $[0,a]$. Further, the statements (i) and (ii) follow immediately from (2.12) and (2.13). The proof is complete.

Lemma 6. *Let $\alpha \in \mathbb{R}$ and*

$$K_\alpha(x,y) = \left(\frac{\cos x - \cos y}{\sin y}\right)^\alpha, \qquad 0 \le x < y \le \pi/2.$$

Then there exist numbers $d_0 > 1$ and $c_0 > 0$ such that for $0 \le x < y \le \pi/2$,

$$K_\alpha(x,y) = (y-x)^\alpha\left[1 + \sum_{n=1}^\infty A_n(y)(y-x)^n\right], \tag{2.14}$$

where

$$|A_n(y)| \le c_0(d_0 y)^{-n}, \quad 0 < y \le \pi/2. \tag{2.15}$$

Proof. Let

$$\mu_\theta(z) = \frac{z - \sin z}{z} + \frac{1-\cos z}{z}\cot\theta \qquad (z \in \mathbb{C},\ z \ne 0).$$

Setting $|z| = \varrho$, we have (for $\varrho < 2$)

$$\left|\frac{z-\sin z}{z}\right| \le \frac{\varrho^2}{6}\sum_{n=0}^\infty\left(\frac{\varrho^2}{20}\right)^n = \frac{10\varrho^2}{3(20-\varrho^2)},$$

$$\left|\frac{1-\cos z}{z}\right| \le \frac{\varrho}{2}\sum_{n=0}^\infty\left(\frac{\varrho^2}{12}\right)^n = \frac{6\varrho}{12-\varrho^2}.$$

Thus, for $0 < \theta \le \pi/2$,

$$|\mu_\theta(z)| \le \frac{10\varrho^2}{3(20-\varrho^2)} + \frac{6\varrho}{12-\varrho^2}\cot\theta.$$

Setting $d_0 = \sqrt{10}/\pi$, we can easily see that

$$\sup_{0\le\theta\le\pi/2}\ \sup_{|z|=d_0\theta} |\mu_\theta(z)| < 1. \tag{2.16}$$

We have

$$\cos x - \cos y = z\sin y\,[1-\mu_y(z)], \quad z = y-x.$$

Set

$$\psi_y(z) = [1-\mu_y(z)]^\alpha, \quad |z| \le d_0 y,\ 0 \le y \le \pi/2.$$

This is an analytic function for $|z| \le d_0 y$ (see (2.16)). Thus,

$$\psi_y(z) = 1 + \sum_{n=1}^{\infty} A_n(y) z^n,$$

where

$$A_n(y) = \frac{1}{2\pi i}\int_{|z|=d_0 y} \frac{\psi_y(z)}{z^{n+1}}\,dz.$$

From this we immediately obtain the estimate (2.15). The equality (2.14) also holds, and the proof is complete.

3. Transplantation theorem. By H^1 we denote the space of all complex-valued 2π-periodic functions $f \in L^1[0,2\pi]$ such that

$$\widehat{f}(n) \equiv \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = 0 \quad \text{for all } n < 0.$$

The norm in H^1 is defined by

$$\|f\|_{H^1} = \int_0^{2\pi} |f(x)|\,dx.$$

It is well known that a complex-valued function $f \in L^1[0,2\pi]$ with $\int_0^{2\pi}\operatorname{Im} f(x)\,dx = 0$ belongs to H^1 if and only if $\widetilde{\operatorname{Re} f} = \operatorname{Im} f$; in this case both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $\operatorname{Re} H^1$.

The proof of Theorem 1 (for non-integer λ) is based on the following main lemma.

LEMMA 7. *Let $f \in \mathcal{L}_\lambda$ $(0<\lambda<1)$, $0<a<\pi$ and $\alpha\in\mathbb{R}$. Set*

$$F(x) = \int_x^a (y-x)^{\lambda-1} f(y)(\sin y)^\lambda\,dy, \quad 0 \le x \le a, \tag{3.1}$$

and

$$c_n = \frac{n^\lambda}{2\pi}\int_0^a F(x)e^{-i(n+\alpha)x}\,dx, \qquad n \in \mathbb{N}. \tag{3.2}$$

Then there exists a function $\Phi \in H^1$ *such that*

$$c_n = \widehat{\Phi}(n) \quad \textit{for all } n \in \mathbb{N} \tag{3.3}$$

and

$$\|\Phi\|_{H^1} \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.4}$$

Proof. Set $F(x) = 0$ for $a < x < 2\pi$ and extend F to the whole line with period 2π. Define $g(x) = f(x)(\sin x)^\lambda$, $x \in [0,\pi]$. By Lemma 2, $g \in L^1[0,\pi]$ and

$$\|g\|_1 \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.5}$$

Let $G(x) = e^{-i\alpha x}F(x)$. By Lemma 3, there exists $\varphi \in L^1[0,2\pi]$ such that

$$G(x) = A_0 + I_-^{(\lambda)}\varphi(x) \tag{3.6}$$

almost everywhere in $[0,2\pi]$ and (see (3.5))

$$\|\varphi\|_1 \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.7}$$

By (3.2), we have $c_n = n^\lambda \widehat{G}(n)$ $(n \in \mathbb{N})$. On the other hand, by (3.6) (see [20], p. 348),

$$\widehat{G}(n) = \widehat{\varphi}(n)|n|^{-\lambda}\exp\left(\frac{\lambda\pi i}{2}\operatorname{sign} n\right) \qquad (n \in \mathbb{Z},\ n \ne 0). \tag{3.8}$$

Therefore,

$$\widehat{\varphi}(n) = e^{-i\lambda\pi/2}c_n, \qquad n \in \mathbb{N}. \tag{3.9}$$

Now we will show that the function G can also be represented by a fractional integral of a "conjugate" type. We will use the following observation.

Consider a singular integral operator

$$(Sv)(x) = \frac{1}{\pi}\lim_{\varepsilon\to+0}\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^a\right)\frac{v(y)}{y-x}\,dy, \quad x \in (0,a),$$

where $v \in L^1[0,a]$. Further, define the linear operator

$$(Tu)(x) = x^{-2\lambda}(Sv)(x), \qquad v(x) = u(x)x^{2\lambda} \qquad (x \in (0,a))$$

for $u \in L^1([0,a],\mu_\lambda)$ $(0 < \lambda < 1)$. If $1 < p < (2\lambda+1)/(2\lambda)$, then $-1 < 2\lambda(1-p)$ and the operator S is bounded in $L^p([0,a], x^{2\lambda(1-p)}\,dx)$ (see [20], p. 200). Thus, for $1 < p < (2\lambda+1)/(2\lambda)$,

$$\left(\int_0^a |(Tu)(x)|^p\,d\mu_\lambda(x)\right)^{1/p} \le c_p\left(\int_0^a |u(x)|^p\,d\mu_\lambda(x)\right)^{1/p}.$$

By interpolation, T is bounded in $L^{p_\lambda,1}([0,a],\mu_\lambda) \equiv \overline{\mathcal{L}}_\lambda$,

$$\|Tu\|_{\overline{\mathcal{L}}_\lambda} \le c\|u\|_{\overline{\mathcal{L}}_\lambda}. \tag{3.10}$$

Now for any $h \in L^1[0,a]$ and $x \in [0,a]$ define

$$J_+^\lambda h(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-y)^{\lambda-1} h(y)\, dy,$$

$$J_-^\lambda h(x) = \frac{1}{\Gamma(\lambda)} \int_x^a (y-x)^{\lambda-1} h(y)\, dy.$$

We have $F(x) = \Gamma(\lambda) J_-^\lambda g(x)$. Note that it is sufficient to prove the lemma in the case when f is bounded (the general case then follows by standard arguments). Under this assumption we have the equality (see [20], p. 206)

$$J_-^\lambda g(x) = J_+^\lambda g(x) \cos\lambda\pi + J_+^\lambda \overline{g}(x) \sin\lambda\pi, \tag{3.11}$$

where $\overline{g}(x) = x^{-\lambda}(Sv)(x)$, $v(x) = x^\lambda g(x)$ $(x \in [0,a])$. Define $\bar{f}(x) = \overline{g}(x)(\sin x)^{-\lambda}$. By (3.10), $\|\bar{f}\|_{\overline{\mathcal{L}}_\lambda} \le c\|f\|_{\mathcal{L}_\lambda}$. Lemma 2 shows that $\overline{g} \in L^1[0,a]$ and

$$\|\overline{g}\|_1 \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.12}$$

Now, setting $h(x) = g(x)\cos\lambda\pi + \overline{g}(x)\sin\lambda\pi$ and using (3.5) and (3.12), we find that $h \in L^1[0,a]$ and

$$\|h\|_1 \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.13}$$

Furthermore, in view of (3.11),

$$F(x) = \int_0^x (x-y)^{\lambda-1} h(y)\, dy, \quad 0 \le x \le a.$$

Using Lemma 4, we obtain

$$G(x) = A_0 + I_+^{(\lambda)}\psi(x) \quad \text{a.e. in } [0,2\pi],$$

where $\psi \in L^1[0,2\pi]$ and

$$\|\psi\|_1 \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.14}$$

Thus,

$$\widehat{G}(n) = \widehat{\psi}(n)|n|^{-\lambda} \exp\left(-\frac{\lambda\pi i}{2}\operatorname{sign} n\right) \quad (n \in \mathbb{Z},\ n \ne 0). \tag{3.15}$$

and

$$\widehat{\psi}(n) = e^{\lambda\pi i/2} c_n, \quad n \in \mathbb{N}. \tag{3.16}$$

Set now

$$\Phi(x) = i(2\sin\lambda\pi)^{-1}[\varphi(x)e^{-\lambda\pi i/2} - \psi(x)e^{\lambda\pi i/2}].$$

Then $\varPhi \in L^1[0,2\pi]$ and, by (3.7) and (3.14),

$$\|\varPhi\|_1 \le c\|f\|_{\mathcal{L}_\lambda}.$$

Furthermore, applying (3.8), (3.9), (3.15) and (3.16), we have

$$\widehat{\varPhi}(n) = c_n \quad (n \in \mathbb{N}) \quad \text{and} \quad \widehat{\varPhi}(n) = 0 \quad (n \le 0).$$

Thus, $\varPhi \in H^1$ and $\varPhi$ satisfies the conditions (3.3) and (3.4). The proof is complete.

Now we will prove the main theorem. By Mehler's formula (see [6], p. 177),

$$\varphi_n^{(\lambda)}(\cos\theta) = t_n(\lambda)(\sin\theta)^{1-2\lambda}\int_0^\theta \frac{\cos(n+\lambda)\varphi}{(\cos\varphi-\cos\theta)^{1-\lambda}}\,d\varphi \tag{3.17}$$

for every $\theta \in [0,\pi]$ and $\lambda > 0$, where

$$\begin{aligned} t_n(\lambda) &= \frac{2^{2\lambda-1/2}\Gamma(\lambda+1/2)}{\pi\Gamma(2\lambda)}\left(\frac{(n+\lambda)\Gamma(n+2\lambda)}{\Gamma(n+1)}\right)^{1/2} \\ &= c_\lambda n^\lambda + O(n^{\lambda-1}). \end{aligned} \tag{3.18}$$

THEOREM 1. *Let $\lambda > 0$, $f \in \mathcal{L}_\lambda$ and*

$$a_n(f) = \int_0^\pi f(x)\varphi_n^{(\lambda)}(\cos x)\,d\mu_\lambda(x).$$

Then the series

$$\sum_{n=1}^\infty a_n \cos nx \tag{3.19}$$

is the Fourier series of some function $\varphi \in \operatorname{Re} H^1$ such that

$$\|\varphi\|_{\operatorname{Re} H^1} \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.20}$$

Proof. First we suppose that $f(x) = 0$ for $x \in (\pi/2,\pi]$. Set $g(x) = f(x)(\sin x)^\lambda$. By Lemma 2, $g \in L^1[0,\pi]$. Using (3.17), we get

$$\begin{aligned} a_n &= t_n(\lambda)\int_0^{\pi/2} f(x)\sin x\,dx\int_0^x \cos(n+\lambda)y\,(\cos y-\cos x)^{\lambda-1}\,dy \\ &= t_n(\lambda)\int_0^{\pi/2} F(x)\cos(n+\lambda)x\,dx, \end{aligned} \tag{3.21}$$

where

$$F(x) = \int_x^{\pi/2} g(y)K(x,y)\,dy, \quad K(x,y) = \left(\frac{\cos x-\cos y}{\sin y}\right)^{\lambda-1}$$

$(0 \le x < y \le \pi/2)$. By Lemma 6,

$$K(x,y) = (y-x)^{\lambda-1}\Big[1 + \sum_{n=1}^{\infty} A_n(y)(y-x)^n\Big],$$

where, for some $d_0 > 1$,

$$|A_n(y)| \le c_0(d_0 y)^{-n}, \quad 0 < y \le \pi/2. \tag{3.22}$$

Thus,

$$F(x) = \int_x^{\pi/2} g(y)(y-x)^{\lambda-1}\,dy + \sum_{n=1}^{\infty} Q_n(x),$$

where

$$Q_n(x) = \int_x^{\pi/2} g(y)A_n(y)(y-x)^{n+\lambda-1}\,dy.$$

Set

$$\xi_0(y) = g(y)A_n(y), \qquad \xi_{k+1}(y) = -\int_y^{\pi/2} \xi_k(y)\,dy \qquad (k = 0, 1, \ldots).$$

Applying $(n-1)$-fold integration by parts to the integral $Q_n(x)$, we get

$$Q_n(x) = (n+\lambda-1)\ldots\lambda \int_x^{\pi/2} (y-x)^{\lambda-1}\xi_{n-1}(y)\,dy.$$

Thus,

$$F(x) = \int_x^{\pi/2} (y-x)^{\lambda-1}h(y)\,dy, \tag{3.23}$$

where $h(y)$ is defined by

$$h(y) = g(y) + \sum_{n=1}^{\infty}(n+\lambda-1)\ldots\lambda\xi_{n-1}(y).$$

It remains to observe that

$$\xi_{n-1}(y) = \frac{1}{(n-1)!}\int_y^{\pi/2} (z-y)^{n-1}g(z)A_n(z)\,dz,$$

which we again obtain by integration by parts. Thus,

$$h(x) = g(x) + \sum_{k=1}^{\infty} \frac{(k+\lambda-1)\ldots\lambda}{(k-1)!}\int_x^{\pi/2} (y-x)^{k-1}g(y)A_k(y)\,dy. \tag{3.24}$$

Taking into account (3.22), we obtain

$$|h(x)| \le |g(x)| + c \int_x^{\pi/2} \frac{|g(y)|}{y}\, dy, \quad x \in [0, \pi/2]. \tag{3.25}$$

Set $f_0(x) = h(x)(\sin x)^{-\lambda}$, $x \in (0, \pi/2]$ ($f_0(x) = 0$ for $x \in (\pi/2, \pi]$). We will show that $f_0 \in \mathcal{L}_\lambda$. We have

$$|f_0(x)| \le |f(x)| + c\psi(x),$$

where

$$\psi(x) = x^{-\lambda} \int_x^{\pi/2} |f(y)| y^{\lambda-1}\, dy.$$

For any measurable set $E \subset [0, \pi/2]$ with $\mu_\lambda(E) = t$ we have $\sup E > t^{1/(2\lambda+1)}$. Thus,

$$\psi_\lambda^*(t) \le t^{-\lambda/(2\lambda+1)} \int_{t^{1/(2\lambda+1)}}^{\pi/2} |f(y)| y^{\lambda-1}\, dy.$$

It follows that

$$\|\psi\|_{\mathcal{L}_\lambda} = \int_0^{\pi/2} t^{1/p_\lambda} \psi_\lambda^*(t)\, dt/t \le c \int_0^{\pi/2} |f(y)| y^\lambda\, dy \le c' \|f\|_{\mathcal{L}_\lambda}.$$

Thus,

$$\|f_0\|_{\mathcal{L}_\lambda} \le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.26}$$

We now return to the equality (3.21). Observe that

$$F(x) = \int_x^{\pi/2} f(y) \sin y\, (\cos x - \cos y)^{\lambda-1}\, dy$$

and the function $\varphi(x) = f(x)\sin x$ satisfies the condition (2.11). Let $\lambda = \nu + \gamma$, where $\nu \ge 0$ is an integer and $0 < \gamma \le 1$. Applying ν-fold integration by parts to the integral on the right hand side of (3.21) and using Lemma 5, we obtain

$$a_n = (-1)^\nu \frac{t_n(\lambda)}{(n+\lambda)^\nu} \int_0^{\pi/2} F^{(\nu)}(x) \cos((n+\lambda)x - \nu\pi/2)\, dx. \tag{3.27}$$

On the other hand, from (3.23),

$$F^{(\nu)}(x) = (-1)^\nu (\lambda - 1) \dots (\lambda - \nu) \int_x^{\pi/2} (y - x)^{\gamma-1} h(y)\, dy.$$

Let $\bar f(x)=f_0(x)(\sin x)^\nu$. Then, by Lemma 2 and (3.26), $\bar f\in\mathcal{L}_\gamma$ and

$$\|\bar f\|_{\mathcal{L}_\gamma}\le c\|f\|_{\mathcal{L}_\lambda}. \tag{3.28}$$

Moreover, $h(x)=\bar f(x)(\sin x)^\gamma$.

Suppose that $0<\gamma<1$. Set

$$\bar F(x)=\int_x^{\pi/2}(y-x)^{\gamma-1}\bar f(y)(\sin y)^\gamma\,dy. \tag{3.29}$$

In view of (3.18), we have

$$a_n=n^\gamma\Big(c'_\lambda+O\Big(\frac{1}{n}\Big)\Big)\int_0^{\pi/2}\bar F(x)\cos((n+\lambda)x-\nu\pi/2)\,dx. \tag{3.30}$$

By Lemma 7, the sequence

$$a'_n\equiv n^\gamma\int_0^{\pi/2}\bar F(x)\cos((n+\lambda)x-\nu\pi/2)\,dx$$

is the sequence of Fourier coefficients of some $\sigma\in\operatorname{Re}H^1$ with

$$\|\sigma\|_{\operatorname{Re}H^1}\le c\|f\|_{\mathcal{L}_\lambda}.$$

Now suppose that $\alpha_n=O(1/n)$ and $\beta_n=a'_n\alpha_n$. Then

$$\Big(\sum_{n=1}^\infty\beta_n^2\Big)^{1/2}\le c\max_n|a'_n|\le c\|\sigma\|_1\le c'\|f\|_{\mathcal{L}_\lambda}.$$

Thus, $\{\beta_n\}$ is the sequence of Fourier coefficients of some function in L^2 whose L^2-norm does not exceed $C\|f\|_{\mathcal{L}_\lambda}$. It follows that (3.19) is the Fourier series of some $\varphi\in\operatorname{Re}H^1$ satisfying (3.20).

Now suppose that $\gamma=1$ and ν is an even number. Then (see (3.29) and (3.30))

$$a_n=(-1)^{\nu/2}\Big(c'_\lambda+O\Big(\frac{1}{n}\Big)\Big)\int_0^{\pi/2}\bar f(x)\sin x\sin(n+\lambda)x\,dx. \tag{3.31}$$

We will show that the function $h(x)=\bar f(|x|)\sin x$, $x\in[-\pi,\pi]$, belongs to $\operatorname{Re}H^1$.

Let $0<\delta\le\pi/4$, $Q_\delta=\{x:\delta\le|x|\le 2\delta\}$ and $h_\delta=h\chi_{Q_\delta}$. Choose some $1<r<3/2$. Since $\mu_1([\delta,2\delta])\le(2\delta)^3$, we have

$$\begin{aligned}\Big(\frac{1}{4\delta}\int_{-2\delta}^{2\delta}|h_\delta(x)|^r\,dx\Big)^{1/r}&\le c\delta^{1-3/r}\Big(\int_\delta^{2\delta}|\bar f(x)|^r\,d\mu_1(x)\Big)^{1/r}\\&\le c_1\delta^{1-3/r}\int_0^{\delta^3}\bar f_1^*(t)t^{1/r-1}\,dt\equiv\frac{c_1}{4\delta}\xi(\delta).\end{aligned} \tag{3.32}$$

Set $a_\delta(x) = h_\delta(x)/\xi(\delta)$. Since the function $h(\delta)$ is odd, it follows from (3.32) that a_δ is a $(1,r)$-atom supported in $[-2\delta, 2\delta]$ (see [7], p. 247). Set now $\delta_k = 2^{-k-1}\pi$ ($k \in \mathbb{N}$). Then

$$h(x) = \sum_{k=1}^{\infty} h_{\delta_k}(x) = \sum_{k=1}^{\infty} \xi(\delta_k) a_{\delta_k}(x). \tag{3.33}$$

Furthermore, using (3.28) we have

$$\begin{aligned}\sum_{k=1}^{\infty} \xi(\delta_k) &\le c \sum_{k=1}^{\infty} 2^{k(3/r-2)} \int_0^{2^{-3k}} \bar{f}_1^*(t) t^{1/r-1}\, dt \\ &\le c' \int_0^{\pi} t^{-1/3} \bar{f}_1^*(t)\, dt = c' \|\bar{f}\|_{\mathcal{L}_1} \le c'' \|f\|_{\mathcal{L}_\lambda}.\end{aligned}$$

Thus, (3.33) is a $(1,r)$-atomic decomposition of h ([7], p. 257). It follows that $h \in \operatorname{Re} H^1$ and

$$\|h\|_{\operatorname{Re} H^1} \le c \|f\|_{\mathcal{L}_\lambda}. \tag{3.34}$$

Let

$$b_n = \frac{2}{\pi} \int_0^{\pi/2} h(x) \sin(n+\lambda)x\, dx.$$

Then $\sum_{n=1}^{\infty} b_n \cos(n+\lambda)x$ is the Fourier series of some function in $\operatorname{Re} H^1$ whose $\operatorname{Re} H^1$-norm does not exceed $c\|f\|_{\mathcal{L}_\lambda}$ (see (3.34). It easily follows that the same is true for $\sum_{n=1}^{\infty} b_n \cos nx$, and therefore also for the series (3.19) (see (3.31)).

It remains to consider the case when λ is a positive even number. By Lemma 5, in this case $F^{(\lambda-1)}(0) = 0$. Thus, by (3.27), (3.23) and (3.18),

$$a_n = \left(c'_\lambda + O\left(\frac{1}{n}\right) \right) \int_0^{\pi/2} h(x) \cos(n+\lambda)x\, dx. \tag{3.35}$$

Recall that the function h is defined by (3.24) for $x \in [0,\pi]$. Set $h(x) = h(-x)$ for $x \in [-\pi, 0)$ and extend h by periodicity with period 2π. We will show that $h \in \operatorname{Re} H^1$.

For any $0 < \delta \le \pi/4$ set $f_\delta = f\chi_{[\delta,2\delta]}$, $g_\delta(x) = f_\delta(x)(\sin x)^\lambda$ and, as in (3.24),

$$h_\delta(x) = g_\delta(x) + \sum_{k=1}^{\infty} \frac{(k+\lambda-1)\dots\lambda}{(k-1)!} \int_x^{\pi/2} (y-x)^{k-1} g_\delta(y) A_k(y)\, dy \tag{3.36}$$

for $x \in [0,\pi]$ ($h_\delta(x) = h_\delta(-x)$ for $x \in [-\pi, 0)$). Let

$$F_\delta(x) = \int_x^{\pi/2} g_\delta(y) K(x,y)\,dy.$$

By Lemma 5, $F_\delta^{(\lambda-1)}(0) = 0$. On the other hand, similarly to (3.23),

$$F_\delta(x) = \int_x^{\pi/2} (y-x)^{\lambda-1} h_\delta(y)\,dy,$$

and therefore

$$\int_0^{\pi/2} h_\delta(y)\,dy = 0. \tag{3.37}$$

As above, the estimate (3.32) holds and, in view of (3.37), the function $a_\delta(x) = h_\delta(x)/\xi(\delta)$ is a $(1,r)$-atom supported in $[-2\delta, 2\delta]$. Furthermore, setting $\delta_k = 2^{-k-1}\pi$, we have

$$f(x) = \sum_{k=1}^{\infty} f_{\delta_k}(x), \quad x \in [0,\pi].$$

This equality, by (3.24) and (3.36), yields (3.33). As above, we have the estimate (3.34). Finally, by (3.35), we obtain the statement of the theorem.

Thus, the theorem is proved in the case when $\operatorname{supp} f \subset [0, \pi/2]$. Now suppose that $\operatorname{supp} f \subset [\pi/2, \pi]$. This case reduces at once to the preceding one. Indeed, set $f_1(x) = f(\pi - x)$, $x \in [0,\pi]$. Taking into account that

$$\varphi_n^{(\lambda)}(-z) = (-1)^n \varphi_n^{(\lambda)}(z)$$

(see [6], p. 175), we get $a_n(f) = (-1)^n a_n(f_1)$. As already proved, $\sum_{n=1}^{\infty} a_n(f_1)\cos nx$ is the Fourier series of some $\varphi_1 \in \operatorname{Re} H^1$ such that

$$\|\varphi_1\|_{\operatorname{Re} H^1} \le c\|f_1\|_{\mathcal{L}_\lambda} = c\|f\|_{\mathcal{L}_\lambda}. \tag{3.38}$$

Now, the series

$$\sum_{n=1}^{\infty} a_n(f)\cos nx = \sum_{n=1}^{\infty} (-1)^n a_n(f_1)\cos nx$$

is the Fourier series of $\varphi(x) \equiv \varphi_1(\pi - x)$. Furthermore, the conjugate function $\widetilde{\varphi}(x)$ is $-\widetilde{\varphi}_1(\pi - x)$. Thus, $\varphi \in \operatorname{Re} H^1$ and (see (3.38))

$$\|\varphi\|_{\operatorname{Re} H^1} \equiv \|\varphi\|_1 + \|\widetilde{\varphi}\|_1 = \|\varphi_1\|_{\operatorname{Re} H^1} \le c\|f\|_{\mathcal{L}_\lambda}.$$

The general case now follows immediately. The proof is complete.

4. Some remarks on Fourier coefficients. In this section we will consider some estimates of Fourier coefficients with respect to the system of ultraspherical polynomials.

We will use the following estimate ([23], Theorem 7.33.1 and formula (7.33.6)): for any $\lambda > 0$,

$$|\varphi_n^{(\lambda)}(\cos\theta)| \le C_\lambda \min(n^\lambda, (\sin\theta)^{-\lambda}), \quad \theta \in [0,\pi]. \tag{4.1}$$

Proposition 1. *Let* $f \in L^{p,r}([0,\pi],\mu_\lambda)$ $(1 < p < 2,\ 1 \le r \le 2)$ *and*

$$a_n(f) = \int_0^\pi f(x)\varphi_n^{(\lambda)}(\cos x)\,d\mu_\lambda(x). \tag{4.2}$$

Then

$$\Big(\sum_{n=1}^\infty |a_n(f)|^r n^{r\alpha-1}\Big)^{1/r} \le c\|f\|_{L^{p,r}(\mu_\lambda)}, \tag{4.3}$$

where $\alpha = (1+2\lambda)(1/p' - 1/p'_\lambda)$.

Proof. We apply some standard arguments (see [9], [16]). Let E_k $(k = 0,1,\ldots)$ be a measurable set with $\mu_\lambda(E_k) = 2^{-k(2\lambda+1)}$ such that

$$|f(x)| \begin{cases} \ge f_\lambda^*(2^{-k(2\lambda+1)}) & \text{for all } x \in E_k, \\ \le f_\lambda^*(2^{-k(2\lambda+1)}) & \text{for all } x \in [0,\pi]\setminus E_k. \end{cases}$$

Set

$$f_k(x) = f(x)\chi_{E_k}(x), \qquad g_k(x) = f(x) - f_k(x), \qquad x \in [0,\pi].$$

Then we have (setting $\delta_k = 2^{-k(2\lambda+1)}$)

$$\|f_k\|_{L^1(\mu_\lambda)} = \int_0^{\delta_k} f_\lambda^*(t)\,dt, \qquad \|g_k\|_{L^2(\mu_\lambda)} = \Big(\int_{\delta_k}^\pi f_\lambda^*(t)^2\,dt\Big)^{1/2}.$$

We also have (see (4.1) and (4.2))

$$|a_n(f_k)| \le cn^\lambda\|f_k\|_{L^1(\mu_\lambda)}, \qquad \sum_{n=0}^\infty a_n(g_k)^2 \le \|g_k\|^2_{L^2(\mu_\lambda)}.$$

Thus,

$$\sigma_k \equiv \Big(\frac{1}{2^k}\sum_{n=2^k}^{2^{k+1}-1} a_n(f)^2\Big)^{1/2} \le c2^{k\lambda}\int_0^{\delta_k} f_\lambda^*(t)\,dt + 2^{-k/2}\Big(\int_{\delta_k}^\pi f_\lambda^*(t)^2\,dt\Big)^{1/2}.$$

Using this estimate and applying Hardy type inequalities (see, for example, [11], Lemma 2), we easily get

$$\sum_{k=0}^\infty (2^{k\alpha}\sigma_k)^r \le \|f\|_{L^{p,r}(\mu_\lambda)}.$$

This implies (4.3).

Note that for $r = p$ the inequality (4.3) follows from the Marcinkiewicz–Zygmund inequality [13] (see also [11], [22]).

In the case $p = p_\lambda$, $r = 1$ we have the inequality (see (1.3))

$$\sum_{n=0}^{\infty} \frac{|a_n(f)|}{n+1} \le c\|f\|_{\mathcal{L}_\lambda}. \tag{4.4}$$

If $p_\lambda < p < 2$, $1 \le r \le 2$, then $0 < \alpha r < 1$. In this case the inequality (4.3) can be strengthened.

For any sequence $\{\alpha_n\}$ with $\alpha_n \to 0$ denote by $\{\alpha_n^*\}$ its non-increasing rearrangement. By the Hardy–Littlewood inequality [4, p. 44],

$$\sum_{n=1}^{\infty} |\alpha_n \beta_n| \le \sum_{n=1}^{\infty} \alpha_n^* \beta_n^*. \tag{4.5}$$

Proposition 2. *Let $f \in L^{p,r}([0,\pi], \mu_\lambda)$ ($\lambda > 0$, $p_\lambda < p < 2$, $r \ge 1$) and $\alpha = (1+2\lambda)(1/p' - 1/p'_\lambda)$. Then*

$$\Big(\sum_{n=1}^{\infty} (a_n^*(f))^r n^{r\alpha-1}\Big)^{1/r} \le c\|f\|_{L^{p,r}(\mu_\lambda)}. \tag{4.6}$$

Proof. Let $q = p/[(2-p)\lambda + 1]$. Then $1 < q < p$ and $\alpha = 1/q'$. Observe that the left hand side of (4.6) is the $l^{q',r}$-norm of the sequence $\{a_n(f)\}$.

Set $g(x) = f(x)(\sin x)^\lambda$. It follows from (2.2) that

$$g^*(t) \le c'\bigg(\frac{1}{t}\int_{ct^{2\lambda+1}}^{\pi} f_\lambda^*(u)^2\,du\bigg)^{1/2}.$$

Applying this inequality, we easily see that $g \in L^{q,r}[0,\pi]$ and

$$\|g\|_{L^{q,r}} \le c\|f\|_{L^{p,r}(\mu_\lambda)}. \tag{4.7}$$

Let now $u_n^{(\lambda)}(\theta) = \varphi_n^{(\lambda)}(\cos\theta)(\sin\theta)^\lambda$, $\theta \in [0,\pi]$. The system $\{u_n^{(\lambda)}\}$ is orthonormal in $L^2[0,\pi]$. Moreover, by (4.1), it is uniformly bounded. Also (see (4.2)),

$$a_n(f) = \int_0^\pi g(\theta)u_n^{(\lambda)}(\theta)\,d\theta.$$

Thus, using the generalized Paley inequality (see [9], [21]), we get

$$\|a_n(f)\|_{l^{q',r}} \le c\|g\|_{L^{q,r}}. \tag{4.8}$$

Applying (4.7), we obtain (4.6).

Actually, to prove (4.8) it would be sufficient to apply the same reasoning as in the proof of Proposition 1.

By (4.5), for $r < q'$ the inequality (4.6) is stronger than (4.3).

Now consider the case $p = p_\lambda$, $r = 1$. In this case we have the inequality (4.4). It is natural to ask whether it is possible to replace $a_n(f)$ by $a_n^*(f)$ in (4.4). The answer is negative.

Proposition 3. *For each $\lambda > 0$ there exists a function $f \in \mathcal{L}_\lambda$ such that the series*

$$\sum_{n=1}^{\infty} \frac{a_n^*(f)}{n} \tag{4.9}$$

diverges.

Proof. Let $\nu_j = 2^{2^j}$, $N_j = \nu_{j+1} - \nu_j$ $(j \in \mathbb{N})$. Set

$$b_n^{(j)} = \begin{cases} (n+\nu_j - N_j)^{-1} & \text{if } N_j \le n \le 2N_j, \\ 0 & \text{if } n \notin [N_j, 2N_j]. \end{cases}$$

We have

$$b_n^{(j)*} \le 1/n \quad \text{and} \quad \sum_{n=N_j}^{2N_j} b_n^{(j)} \ge 2^{j-1}. \tag{4.10}$$

Further, it follows from (3.17) that for $0 \le \theta \le \pi/[4(n+\lambda)]$,

$$\varphi_n^{(\lambda)}(\cos\theta) \ge c_\lambda n^\lambda, \tag{4.11}$$

where c_λ is a positive constant.

Let

$$g_j(\theta) = \sum_{n=N_j}^{2N_j} b_n^{(j)} \varphi_n^{(\lambda)}(\cos\theta).$$

Then, by (4.10) and (4.11),

$$g_j(\theta) \ge c_\lambda 2^{j-1} N_j^\lambda \quad \text{for } \theta \in \left[\frac{\pi}{16(N_j+\lambda)}, \frac{\pi}{8(N_j+\lambda)}\right] \equiv I_j.$$

Since $\mu_\lambda(I_j) \ge c'_\lambda N_j^{-(2\lambda+1)}$ $(c'_\lambda > 0)$, we have

$$\|g_j\|_{L^{p'_\lambda,\infty}(\mu_\lambda)} \ge A_\lambda 2^j \quad (A_\lambda > 0). \tag{4.12}$$

Now suppose that for any $f \in \mathcal{L}_\lambda$ the series (4.9) converges. Define a sequence of linear functionals on $\mathcal{L}_\lambda$ by

$$\Lambda_j(f) = \int_0^\pi f(\theta) g_j(\theta)\, d\mu_\lambda(\theta).$$

By (4.10) and (4.5), for each $f \in \mathcal{L}_\lambda$ we have

$$|\Lambda_j(f)| = \Big| \sum_{n=N_j}^{2N_j} a_n(f) b_n^{(j)} \Big| \le \sum_{n=1}^{\infty} \frac{a_n^*(f)}{n}.$$

By the uniform boundedness principle, $\{\|\Lambda_j\|\}$ is bounded. At the same time (see [4, p. 220] and (4.12)),

$$\|\Lambda_j\| \geq C\|g_j\|_{L^{p'_\lambda,\infty}(\mu_\lambda)} \geq C'2^j \quad (C' > 0).$$

This completes the proof.

References

[1] R. Askey, *A transplantation theorem for Jacobi series*, Illinois J. Math. 13 (1969), 583–590.

[2] R. Askey and S. Wainger, *A transplantation theorem between ultraspherical series*, ibid. 10 (1966), 322–344.

[3] —, —, *A transplantation theorem for ultraspherical coefficients*, Pacific J. Math. 16 (1966), 393–405.

[4] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[5] L. Colzani, *Expansions in Legendre polynomials and Lagrange interpolation*, Acta Math. Hungar. 61 (1993), 289–302.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, 1953.

[7] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North–Holland, Amsterdam 1985.

[8] J. E. Gilbert, *Maximal theorems for some orthogonal series I*, Trans. Amer. Math. Soc. 145 (1969), 495–515.

[9] W. B. Jurkat and G. Sampson, *On maximal rearrangement inequalities for the Fourier transform*, ibid. 282 (1984), 625–643.

[10] N. J. Kalton and L. Tzafriri, *The behaviour of Legendre and ultraspherical polynomials in L_p-spaces*, Canad. J. Math. 50 (1998), 1236–1252.

[11] V. I. Kolyada, *On some generalizations of the Hardy–Littlewood–Paley theorem*, Mat. Zametki 51 (1992), no. 3, 24–34 (in Russian); English transl.: Math. Notes 51 (1992), 235–244.

[12] S. V. Konyagin, *On a problem of Littlewood*, Izv. Akad. Nauk SSSR 45 (1981), 243–265 (in Russian); English transl.: Math. USSR-Izv. 18 (1982), 205–225.

[13] J. Marcinkiewicz and A. Zygmund, *Some theorems on orthogonal systems*, Fund. Math. 28 (1937), 309–335.

[14] O. C. McGehee, L. Pigno and B. Smith, *Hardy inequality and the L^1 norm of exponential sums*, Ann. of Math. 113 (1981), 613–618.

[15] C. Meaney, *Divergent Jacobi polynomial series*, Proc. Amer. Math. Soc. 87 (1983), 459–462.

[16] H. L. Montgomery, *A note on rearrangements of Fourier coefficients*, Ann. Inst. Fourier (Grenoble) 26 (1976), no. 2, 29–34.

[17] B. Muckenhoupt, *Transplantation theorems and multiplier theorems for Jacobi series*, Mem. Amer. Math. Soc. 356 (1986).

[18] J. Newman and W. Rudin, *Mean convergence of orthogonal series*, Proc. Amer. Math. Soc. 3 (1952), 219–222.

[19] H. Pollard, *The mean convergence of orthogonal series. II*, Trans. Amer. Math. Soc. 63 (1948), 355–367.

[20] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[21] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), 482–492.

[22] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, ibid. 87 (1958), 159–172.

[23] G. Szegő, *Orthogonal Polynomials*, 3rd ed., Amer. Math. Soc., Providence, RI, 1967.

Departamento de Matemáticas y Computación
Universidad de La Rioja
Edif. Vives, c. Luis de Ulloa
26004 Logroño, La Rioja, Spain

Department of Mathematics
Odessa National University
2 Dvoryanskaya st.
270000 Odessa, Ukraine
E-mail: kolyada@paco.net

Received July 7, 2000
Revised version February 12, 2001 (4565)