

## Some results on packing in Orlicz sequence spaces

by

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**Abstract.** We present monotonicity theorems for index functions of  $N$ -functions, and obtain formulas for exact values of packing constants. In particular, we show that the Orlicz sequence space  $l^{(N)}$  generated by the  $N$ -function  $N(v) = (1 + |v|) \ln(1 + |v|) - |v|$  with Luxemburg norm has the Kottman constant  $K(l^{(N)}) = N^{-1}(1)/N^{-1}(1/2)$ , which answers M. M. Rao and Z. D. Ren's [8] problem.

Let  $N(v) = (1 + |v|) \ln(1 + |v|) - |v|$ , and let  $l^{(N)}$  and  $l^N$  be the corresponding Orlicz sequence spaces equipped with the Luxemburg norm and Orlicz norm, respectively. The following problem was raised in [8] (Problems 2.7 and 3.8): What are the packing constants of  $l^{(N)}$  and  $l^N$ ? We now present an answer. We first extend the theory of quantitative indices of Orlicz spaces.

1. Index functions of  $N$ -functions. Let

$$\Phi(u) = \int_0^{|u|} \phi(t) dt \quad \text{and} \quad \Psi(v) = \int_0^{|v|} \psi(s) ds$$

be a pair of complementary  $N$ -functions, Krasnosel'skiĭ and Rutickiĭ [5] studied the limit of the function  $t\phi(t)/\Phi(t)$  as  $t \rightarrow \infty$ . Later, Rao and Ren [7] and Lindenstrauss and Tzafriri [6] systematically investigated the following related values which Ren [8], [10], [11] calls *quantitative indices*:

$$\begin{aligned} (1) \quad A_\Phi &= \liminf_{t \rightarrow \infty} \frac{t\phi(t)}{\Phi(t)}, & B_\Phi &= \limsup_{t \rightarrow \infty} \frac{t\phi(t)}{\Phi(t)}, \\ (2) \quad A_\Phi^0 &= \liminf_{t \rightarrow 0} \frac{t\phi(t)}{\Phi(t)}, & B_\Phi^0 &= \limsup_{t \rightarrow 0} \frac{t\phi(t)}{\Phi(t)}, \\ (3) \quad \bar{A}_\Phi &= \inf_{t > 0} \frac{t\phi(t)}{\Phi(t)}, & \bar{B}_\Phi &= \sup_{t > 0} \frac{t\phi(t)}{\Phi(t)}. \end{aligned}$$

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The same six indices can be defined for  $\Psi(v)$ . The relationships among them are as follows (cf. [6], [7]):

PROPOSITION 1.1. *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions. Then*

$$(4) \quad \frac{1}{A_\Phi} + \frac{1}{B_\Psi} = 1 = \frac{1}{A_\Psi} + \frac{1}{B_\Phi},$$

$$(5) \quad \frac{1}{A_\Phi^0} + \frac{1}{B_\Psi^0} = 1 = \frac{1}{A_\Psi^0} + \frac{1}{B_\Phi^0},$$

$$(6) \quad \frac{1}{\overline{A}_\Phi} + \frac{1}{\overline{B}_\Psi} = 1 = \frac{1}{\overline{A}_\Psi} + \frac{1}{\overline{B}_\Phi}.$$

In 1985, on the basis of Cleaver [1], Ren studied the upper and lower limits of the function  $\Phi^{-1}(u)/\Phi^{-1}(2u)$  as  $u \rightarrow \infty$ . Later, Rao and Ren [7] (p. 23) and Ren and Chen [11] defined the following “new” quantitative indices:

$$(7) \quad \alpha_\Phi = \liminf_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_\Phi = \limsup_{u \rightarrow \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$

$$(8) \quad \alpha_\Phi^0 = \liminf_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_\Phi^0 = \limsup_{u \rightarrow 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$

$$(9) \quad \overline{\alpha}_\Phi = \inf_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \overline{\beta}_\Phi = \sup_{u > 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

The six indices of  $\Psi$  can be similarly defined. It is easy to check the following relations among the above indices:

THEOREM 1.2. *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions. Then*

$$(10) \quad 2\alpha_\Phi\beta_\Psi = 1 = 2\alpha_\Psi\beta_\Phi,$$

$$(11) \quad 2\alpha_\Phi^0\beta_\Psi^0 = 1 = 2\alpha_\Psi^0\beta_\Phi^0,$$

$$(12) \quad 2\overline{\alpha}_\Phi\overline{\beta}_\Psi = 1 = 2\overline{\alpha}_\Psi\overline{\beta}_\Phi.$$

For a further study of the geometric constants, we should directly investigate the functions which produce the quantitative indices.

THEOREM 1.3. *Let  $\Phi$  be an  $N$ -function,  $\phi(t)$  being its left derivative. Define*

$$(13) \quad F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}, \quad G_\Phi(c, u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)} \quad (c > 1).$$

*Then  $F_\Phi(t)$  is increasing (resp. decreasing) on  $(0, \Phi^{-1}(u_0)]$  if and only if  $G_\Phi(c, u)$  is increasing (resp. decreasing) on  $(0, u_0/c]$  for every  $c > 1$ .*

To prove Theorem 1.3, we first introduce two auxiliary lemmas.

LEMMA 1.4. *A continuous function  $f$  is increasing if and only if both its left derivative  $f'_-$  and its right derivative  $f'_+$  are nonnegative.*

LEMMA 1.5. *Let  $\phi_+(t)$  be the right derivative of  $\Phi$ . Write  $F_{\Phi_+}(t) = t\phi_+(t)/\Phi(t)$ . Then  $F_{\Phi}(t)$  is increasing if and only if  $F_{\Phi_+}(t)$  is increasing.*

*Proof of Theorem 1.3. Necessity.* In view of Lemma 1.4, it suffices to show that both the left derivative  $G'_{\Phi}(c, t)$  and the right derivative  $G'_{\Phi_+}(c, t)$  of  $G_{\Phi}(c, t)$  are nonnegative when  $F_{\Phi}(t)$  is increasing.

In fact, for every  $u \in (0, u_0/c]$ , let  $\Phi^{-1}(u) = t_1$  and  $\Phi^{-1}(cu) = t_2$ . Then  $0 < t_1 < t_2 \leq \Phi^{-1}(u_0)$  and

$$\begin{aligned} G'_{\Phi}(c, u) &= \frac{1}{[\Phi^{-1}(cu)]^2} \left\{ \frac{\Phi^{-1}(cu)}{\phi[\Phi^{-1}(u)]} - \frac{c\Phi^{-1}(u)}{\phi[\Phi^{-1}(cu)]} \right\} \\ &= \frac{1}{t_2^2\phi(t_1)\phi(t_2)} [t_2\phi(t_2) - ct_1\phi(t_1)] \\ &= \frac{cu}{t_2^2\phi(t_1)\phi(t_2)} \left[ \frac{t_2\phi(t_2)}{\Phi(t_2)} - \frac{t_1\phi(t_1)}{\Phi(t_1)} \right] \\ &= \frac{\Phi(t_2)}{t_2^2\phi(t_1)\phi(t_2)} [F_{\Phi}(t_2) - F_{\Phi}(t_1)]. \end{aligned}$$

Analogously we have

$$G'_{\Phi_+}(c, u) = \frac{\Phi(t_2)}{t_2^2\phi_+(t_1)\phi_+(t_2)} [F_{\Phi_+}(t_2) - F_{\Phi_+}(t_1)].$$

Since  $F_{\Phi}(t_2) - F_{\Phi}(t_1) \geq 0$  and  $F_{\Phi_+}(t_2) - F_{\Phi_+}(t_1) \geq 0$  for  $0 < t_1 < t_2 \leq \Phi^{-1}(u_0)$  when  $u \in (0, u_0/c]$  by Lemma 1.5, one has  $G'_{\Phi}(c, u) \geq 0$  and  $G'_{\Phi_+}(c, u) \geq 0$ .

*Sufficiency.* To show that  $F_{\Phi}(t)$  is increasing, for arbitrary  $0 < t_1 < t_2 \leq \Phi^{-1}(u_0)$ , choose  $c_1 = \Phi(t_2)/\Phi(t_1)$  and  $u_1 = \Phi(t_1) \leq \Phi(u_0)/c_1$ . Then  $G'_{\Phi}(c_1, u)|_{u=u_1} \geq 0$ . Thus,  $F_{\Phi}(t_2) - F_{\Phi}(t_1) \geq 0$ , which yields that  $F_{\Phi}(t)$  is increasing on  $(0, \Phi^{-1}(u_0)]$ . ■

THEOREM 1.6. *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions, let  $\phi$  and  $\psi$  be their left derivatives, respectively, and  $C > 0$ . Let  $F_{\Phi}(t)$  and  $F_{\Psi}(s)$  be defined as (13), i.e.,*

$$F_{\Phi}(t) = \frac{t\phi(t)}{\Phi(t)}, \quad F_{\Psi}(s) = \frac{s\psi(s)}{\Psi(s)}, \quad t, s > 0.$$

Then

(i)  $F_{\Phi}(t)$  is increasing (resp. decreasing) on  $(0, \psi(C)]$  if and only if  $F_{\Psi}(s)$  is decreasing (resp. increasing) on  $(0, C]$ .

(ii) Define

$$(14) \quad a_{\Phi}^* = \inf\{F_{\Phi}(t) : t \in (0, \psi(C)]\},$$

$$(15) \quad b_{\Psi}^* = \sup\{F_{\Psi}(s) : s \in (0, C]\}.$$

Then

$$(16) \quad \frac{1}{a_{\Phi}^*} + \frac{1}{b_{\Psi}^*} = 1.$$

*Proof.* (i) Since  $\psi(s)$  is always increasing, the set of discontinuity points of  $\psi(s)$  which are leap points is nondense and at most countable.

(A) Suppose  $\psi(s)$  is continuous and strictly increasing on some subinterval  $[s_1, s_2]$  of  $(0, C]$ . Then its inverse function  $\phi(t)$  is continuous and strictly increasing on  $[\psi(s_1), \psi(s_2)] \subset (0, \psi(C)]$ , and hence  $\phi(\psi(s)) = s$ . Therefore

$$\begin{aligned} F_{\Psi}(s) &= \frac{s\psi(s)}{\Psi(s)} = \frac{\phi(t)\psi(\phi(t))}{\Psi(\phi(t))} = \frac{\phi(t)t}{\Psi(\phi(t))} \\ &= \frac{t\phi(t)}{t\phi(t) - \Phi(t)} = \frac{t\phi(t)/\Phi(t)}{t\phi(t)/\Phi(t) - 1} = \frac{F_{\Phi}(t)}{F_{\Phi}(t) - 1}, \end{aligned}$$

i.e.,

$$(17) \quad \frac{1}{F_{\Psi}(s)} + \frac{1}{F_{\Phi}(\psi(s))} = 1, \quad s \in [s_1, s_2].$$

(B) Suppose  $\psi(s)$  is a constant  $t_0$  on an interval  $[s_1, s_2] \subset (0, C]$ . Then for any  $s'_1, s'_2$  satisfying  $s_1 \leq s'_1 < s'_2 \leq s_2$ , we have

$$\begin{aligned} F_{\Psi}(s'_1) - F_{\Psi}(s'_2) &= \frac{s'_1\psi(s'_1)}{\Psi(s'_1)} - \frac{s'_2\psi(s'_2)}{\Psi(s'_2)} \\ &= \left[ \frac{s'_1}{\Psi(s'_1)} - \frac{s'_2}{\Psi(s'_2)} \right] t_0 > 0. \end{aligned}$$

This means  $F_{\Psi}(s)$  is decreasing on  $[s_1, s_2]$ .

(C) Suppose  $\psi(s)$  leaps at the point  $s_0$ . Let  $\psi(s_0+0) - \psi(s_0-0) = k > 0$ . For any  $\varepsilon > 0$ , there are  $s'_1, s'_2$  satisfying  $s'_1 \leq s_0 \leq s'_2$  and  $0 < s'_2 - s'_1 < \varepsilon$ . We deduce that  $\psi(s'_2) - \psi(s'_1) \geq k$ , and

$$\begin{aligned} F_{\Psi}(s'_2) - F_{\Psi}(s'_1) &= \frac{s'_2\psi(s'_2)}{\Psi(s'_2)} - \frac{s'_1\psi(s'_1)}{\Psi(s'_1)} \\ &\geq \left[ \frac{s'_2}{\Psi(s'_2)} - \frac{s'_1}{\Psi(s'_1)} \right] \psi(s'_1) + \frac{s'_2}{\Psi(s'_2)} k. \end{aligned}$$

When  $s'_2 - s'_1 \rightarrow 0$ , we can see

$$\frac{s'_2}{\Psi(s'_2)} - \frac{s'_1}{\Psi(s'_1)} \rightarrow 0, \quad \frac{s'_2}{\Psi(s'_2)} k \rightarrow \frac{s_0}{\Psi(s_0)} k > 0,$$

since the function  $s/\Psi(s)$  is continuous. Thus,  $F_{\Psi}(s'_2) - F_{\Psi}(s'_1) > 0$ , which implies that  $F_{\Psi}(s)$  also leaps at the point  $s_0$ .

(D) Assume that  $\psi(s)$  is a constant  $t_0$  on some maximum interval  $[s_1, s_2]$  (i.e.  $\inf\{s : \psi(s) = t_0\} = s_1, \sup\{s : \psi(s) = t_0\} = s_2$ ). We know that  $F_{\Psi}(s)$

decreases from  $F_\Psi(s_1 - 0)$  to  $F_\Psi(s_2 + 0)$  by (B). At the same time,  $\phi(t)$  leaps at the point  $t_0 = \psi(s_1) = \psi(s_2)$ . By (C), we know  $F_\Phi(t)$  also leaps from  $F_\Phi(t_0 - 0)$  to  $F_\Phi(t_0 + 0)$ . Letting  $s \rightarrow s_1 - 0$  resp.  $s \rightarrow s_2 + 0$  in (17), we have

$$(18) \quad \frac{1}{F_\Psi(s_1 - 0)} + \frac{1}{F_\Phi(t_0 - 0)} = 1,$$

$$(19) \quad \frac{1}{F_\Psi(s_2 + 0)} + \frac{1}{F_\Phi(t_0 + 0)} = 1.$$

This makes it clear that when  $F_\Psi(s)$  continuously decreases from  $F_\Psi(s_1 - 0)$  to  $F_\Psi(s_2 + 0)$ ,  $F_\Phi(t)$  leaps up from  $F_\Phi(t_0 - 0)$  to  $F_\Phi(t_0 + 0)$ .

(E) When  $\psi(s)$  leaps at the point  $s_0$ ,  $\phi(t)$  is a constant on the maximum interval  $[\psi(s_0), \psi(s_0 + 0)]$ . Therefore  $F_\Phi(t)$  decreases on that interval by (B). As in (D), let  $s \rightarrow s_0 - 0$  resp.  $s \rightarrow s_0 + 0$  in (17) to find that

$$(20) \quad \frac{1}{F_\Psi(s_0 - 0)} + \frac{1}{F_\Phi(\psi(s_0 - 0))} = 1,$$

$$(21) \quad \frac{1}{F_\Psi(s_0 + 0)} + \frac{1}{F_\Phi(\psi(s_0 + 0))} = 1,$$

which implies that when  $F_\Psi(s)$  leaps up from  $F_\Psi(s_0 - 0)$  to  $F_\Psi(s_0 + 0)$ ,  $F_\Phi(t)$  continuously decreases from  $F_\Phi(\psi(s_0 - 0))$  to  $F_\Phi(\psi(s_0 + 0))$ .

Consequently, we find from the above cases that  $F_\Psi(s)$  increases (resp. decreases) if and only if  $F_\Phi(t)$  decreases (resp. increases) on the corresponding intervals. The proof of (i) is complete.

(ii) From the discussion of (i) we know that if  $[s_1, s_2]$  is a maximum interval of constancy of  $\psi(s)$  in  $(0, C]$  or an interval where  $\psi(s)$  is strictly increasing continuous, we always have

$$(22) \quad \frac{1}{\sup\{F_\Psi(s) : s \in [s_1 - 0, s_2 + 0]\}} + \frac{1}{\inf\{F_\Phi(t) : t \in [t_1 - 0, t_2 + 0]\}} = 1,$$

where  $t_1 - 0 = \psi(s_1 - 0)$ ,  $t_2 + 0 = \psi(s_2 + 0)$ . When  $s_0$  is a leap point of  $\psi(s)$ , we treat it as an interval  $[s_0, s_0]$ , so that (22) still holds. Combining the above intervals, we have

$$(23) \quad \frac{1}{\sup\{F_\Psi(s) : s \in (0, C]\}} + \frac{1}{\inf\{F_\Phi(t) : t \in (0, \psi(C))\}} = 1,$$

i.e., (16) holds as soon as  $C$  is a strictly increasing continuous point or the right end of a maximum constancy interval of  $\psi(s)$ . It suffices to show that (16) holds when  $C$  is an internal point of a maximum constancy interval of  $\psi(s)$ . In that case, let  $s_0$  be the left end of that interval. Therefore,  $F_\Psi(s)$  is decreasing on  $[s_0, C]$ , and hence

$$(24) \quad b_\Psi^* = \sup\{F_\Psi(s) : s \in (0, C]\} = \sup\{F_\Psi(s) : s \in (0, s_0 + 0]\}.$$

Meanwhile,  $F_\Phi(t)$  leaps up at the point  $\psi(C)$ , so that

$$(25) \quad a_\Phi^* = \inf\{F_\Phi(t) : t \in (0, \psi(C))\} = \inf\{F_\Phi(t) : t \in (0, \psi(s_0 + 0))\}.$$

It follows that

$$(26) \quad \frac{1}{\sup_{s \in (0, s_0 + 0]} F_\Psi(s)} + \frac{1}{\inf_{t \in (0, \psi(s_0 + 0))} F_\Phi(t)} = 1$$

since (22) holds. Consequently, (16) holds in any case by (24)–(26). ■

## 2. Packing in Orlicz sequence spaces with Luxemburg norm

DEFINITION 2.1 ([4], [8]). The *packing constant*  $P(X)$  of a Banach space  $X$  is

$$P(X) = \sup\{r > 0 : \text{infinitely many balls of radius } r \\ \text{can be packed into the unit ball of } X\}.$$

The *Kottman constant* of an infinite-dimensional Banach space  $X$  is defined as

$$K(X) = \sup\{\inf_{i \neq j} \|x_i - x_j\| : \{x_i\}_{i=1}^\infty \subset S(X)\},$$

where  $S(X)$  is the unit sphere of  $X$ .

Clearly,  $1 \leq K(X) \leq 2$ . The following relationship was proved by Kottman [4] (cf. Ye [14]):

PROPOSITION 2.2. *For an infinite-dimensional Banach space  $X$ , one has*

$$(27) \quad P(X) = \frac{K(X)}{2 + K(X)}.$$

Hudzik [3] verified that  $K(X) = 2$  if  $X$  is a nonreflexive Banach lattice, therefore  $P(X) = 1/2$ . Since  $l^{(\Phi)}$  and  $l^\Phi$  are Banach lattices, it suffices to consider the Kottman constant when  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ , i.e., when  $l^{(\Phi)}$  and  $l^\Phi$  are reflexive (we calculate the packing constants  $P(l^{(\Phi)})$  and  $P(l^\Phi)$  from (27)).

In this section, we investigate  $K(l^{(\Phi)})$  of an Orlicz sequence space with Luxemburg norm and answer Rao and Ren's [8] Problem 2.7.

LEMMA 2.3. *For an  $N$ -function  $\Phi$ ,  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$  implies*

$$(28) \quad \max\left(\frac{1}{\alpha_\Phi^0}, \frac{1}{\alpha'_\Phi}\right) \leq K(l^{(\Phi)}) \leq \frac{1}{\tilde{\alpha}_\Phi},$$

where

$$(29) \quad \alpha'_\Phi = \inf\left\{\frac{\Phi^{-1}\left(\frac{1}{2k}\right)}{\Phi^{-1}\left(\frac{1}{k}\right)} : k = 1, 2, \dots\right\}$$

$$(30) \quad \tilde{\alpha}_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \leq \frac{1}{2}\right\}.$$

*Proof.* The inequality

$$(31) \quad \frac{1}{\alpha_{\Phi}'} \leq K(l^{(\Phi)}) \leq \frac{1}{\widetilde{\alpha}_{\Phi}}$$

is due to Ye [14]. Rao and Ren [8, Theorem 2.3] proved

$$(32) \quad 1/\alpha_{\Phi}^0 \leq K(l^{(\Phi)}),$$

so (28) holds in its entirety. ■

REMARK 2.4. Let

$$(33) \quad \widetilde{A}_{\Phi} = \inf\{t\phi(t)/\Phi(t) : 0 < t \leq \Phi^{-1}(1)\}.$$

By using the right side of (32), it is proved in [8] (Theorem 2.3) that

$$(34) \quad 2^{1/B_{\Phi}^0} \leq K(l^{(\Phi)}) \leq 2^{1/\widetilde{A}_{\Phi}}.$$

If  $F_{\Phi}(t) = t\phi(t)/\Phi(t)$  is increasing on  $t \in (0, \infty)$ , then  $\widetilde{A}_{\Phi} = B_{\Phi}^0 = C_{\Phi}^0 = \lim_{t \rightarrow 0} t\phi(t)/\Phi(t)$  and by (31) we have

$$(35) \quad K(l^{(\Phi)}) = 2^{1/C_{\Phi}^0}.$$

For a pair of complementary  $N$ -functions

$$(36) \quad M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1 + |v|) \ln(1 + |v|) - |v|,$$

since  $F_M(t) = tM'(t)/M(t)$  is increasing on  $(0, \infty)$ , the authors of [8] computed that  $K(l^{(M)}) = \sqrt{2}$  by (32). However,  $F_N(t)$  is decreasing on  $(0, \infty)$ . They only estimated (cf. (24) of [8]):

$$1.41 \approx \sqrt{2} \leq K(l^{(N)}) \leq K(l^N) \leq 2^{1/(e-1)} \approx 1.50.$$

They asked (see [8, Problem 2.7]): What is the exact value of  $K(l^{(N)})$ ? For the solution of the problem, we give the following lemma which follows from Theorem 1.3 by taking  $c = 2$ .

LEMMA 2.5. *Let  $\Phi$  be an  $N$ -function,  $\phi(t)$  being its left derivative. Set*

$$(37) \quad F_{\Phi}(t) = \frac{t\phi(t)}{\Phi(t)}, \quad G_{\Phi}(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$$

for  $t, u > 0$  (see (13)). *If  $F_{\Phi}(t)$  is increasing (resp. decreasing) on  $(0, \Phi^{-1}(u_0)]$ , then  $G_{\Phi}(u)$  is increasing (resp. decreasing) on  $(0, u_0/2]$ .*

Now we establish the following

THEOREM 2.6. *For  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ , we have:*

(i) *If  $F_{\Phi}(t) = t\phi(t)/\Phi(t)$  is increasing on  $(0, \Phi^{-1}(1)]$ , then the Kottman constant of the Orlicz sequence space  $l^{(\Phi)}$  equipped with the Luxemburg norm is*

$$(38) \quad K(l^{(\Phi)}) = 2^{1/C_{\Phi}^0},$$

where  $C_{\Phi}^0 = \lim_{t \rightarrow 0} F_{\Phi}(t)$ .

(ii) If  $F_{\Phi}(t)$  is decreasing on  $(0, \Phi^{-1}(1)]$ , then

$$(39) \quad K(l^{(\Phi)}) = \frac{\Phi^{-1}(1)}{\Phi^{-1}(1/2)}.$$

*Proof.* The proof of (i) has been given in Remark 2.4.

(ii) If  $F_{\Phi}(t)$  is decreasing on  $(0, \Phi^{-1}(1)]$ , then  $G_{\Phi}(u)$  is also decreasing on  $(0, 1/2]$  by Lemma 2.5. Therefore

$$(40) \quad \alpha'_{\Phi} = \tilde{\alpha}_{\Phi} = [G_{\Phi}(u)]|_{u=1/2} = \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)}.$$

Note that  $\alpha_{\Phi}^0 \geq \alpha'_{\Phi}$ , and (39) follows from Lemma 2.3 and the above. ■

EXAMPLE 2.7. Let  $N(v) = (1 + |v|) \ln(1 + |v|) - |v|$ . Then we have the exact value:

$$(41) \quad K(l^{(N)}) = \frac{N^{-1}(1)}{N^{-1}(1/2)} \approx 1.48699, \quad P(l^{(N)}) \approx 0.42644.$$

Indeed, it is easy to see that

$$F_N(t) = \frac{tN'(t)}{N(t)} = \frac{t \ln(1+t)}{(1+t) \ln(1+t) - t}$$

is decreasing on  $(0, \infty)$  and  $C_N^0 = \lim_{t \rightarrow 0} F_N(t) = 2$ , i.e.,  $N \in \Delta_2(0) \cap \nabla_2(0)$ . Hence, (41) follows from Theorem 2.6(ii). Formula (41) solves Problem 2.7 of [8].

REMARK 2.8. The index function  $F_{\Phi}(t)$  of an  $N$ -function is generally easy to find, and is monotonic. So the exact value of the packing constant of most Orlicz sequence spaces with Luxemburg norm can be computed in this way. For instance, for the  $N$ -function of Krasnosel'skiĭ and Rutickiĭ [5, p. 30]

$$\Phi(u) = \frac{u^2}{\ln(e + |u|)},$$

we have

$$(42) \quad K(l^{(\Phi)}) = \frac{\Phi^{-1}(1)}{\Phi^{-1}(1/2)} \approx 1.46987 \quad \text{and} \quad P(l^{(\Phi)}) \approx 0.42361.$$

Indeed, since the index function

$$F_{\Phi}(t) = \frac{t\phi(t)}{\Phi(t)} = 2 - \frac{t}{(e+t) \ln(e+t)}$$

is decreasing on  $(0, \Phi^{-1}(1)]$  and  $C_{\Phi}^0 = 2$ , we have  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ . Thus (42) follows from Theorem 2.6(ii).

Another example is the  $N$ -function of Gallardo [2]:

$$\Phi_{p,r}(u) = |u|^p \ln^r(1 + |u|), \quad 1 \leq p < \infty, \quad 0 < r < \infty.$$

Then one has

$$(43) \quad K(l^{(\Phi_{p,r})}) = \frac{\Phi_{p,r}^{-1}(1)}{\Phi_{p,r}^{-1}(1/2)}.$$

In fact,

$$F_{\Phi_{p,r}}(t) = p + \frac{rt}{(1+t)\ln(1+t)}$$

for  $t > 0$ . Since  $C_{\Phi_{p,r}}^0 = \lim_{t \rightarrow 0^+} F_{\Phi_{p,r}}(t) = p + r > 1$  we deduce that  $\Phi_{p,r} \in \Delta_2(0) \cap \nabla_2(0)$ . On the other hand,

$$\frac{d}{dt} F_{\Phi_{p,r}}(t) = \frac{r[\ln(1+t) - t]}{(1+t)^2 \ln^2(1+t)} < 0$$

for  $t > 0$ , i.e.,  $F_{\Phi_{p,r}}(t)$  is decreasing on  $(0, \Phi_{p,r}^{-1}(1)]$ . Thus the exact value (43) of  $K(l^{(\Phi_{p,r})})$  results from Theorem 2.6(ii). In particular, if  $p = r = 1$ , i.e.,  $\Phi_{1,1}(u) = |u| \ln(1 + |u|)$ , then

$$(44) \quad K(l^{(\Phi_{1,1})}) = \frac{\Phi_{1,1}^{-1}(1)}{\Phi_{1,1}^{-1}(1/2)} \approx 1.18806 \quad \text{and} \quad P(l^{(\Phi_{1,1})}) \approx 0.37266.$$

We now present an interpolation theorem which supplements the corresponding results in [8].

**THEOREM 2.9.** *Let  $\Phi$  be an  $N$ -function,  $\Phi_0(u) = u^2$ ,  $0 < s \leq 1$ , and let  $\Phi_s$  be the inverse of*

$$(45) \quad \Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s} [\Phi_0^{-1}(u)]^s, \quad 0 < s \leq 1, \quad u \geq 0.$$

Then

$$(46) \quad \max \left( \frac{1}{\alpha_{\Phi_s}^0}, \frac{1}{\alpha'_{\Phi_s}} \right) \leq K(l^{(\Phi_s)}) \leq \frac{1}{\tilde{\alpha}_{\Phi_s}} \leq 2^{1-s/2}.$$

In particular, if  $\Phi \notin \nabla_2(0)$ , then

$$(47) \quad K(l^{(\Phi_s)}) = 2^{1-s/2}.$$

Therefore, the following assertions hold:

(i) *If  $F_{\Phi}(t) = t\phi(t)/\Phi(t)$  is increasing on  $(0, \Phi^{-1}(1)]$ , then*

$$(48) \quad K(l^{(\Phi_s)}) = 2^{(1-s)/C_{\Phi}^0 + s/2}.$$

(ii) *If  $F_{\Phi}(t)$  is decreasing on  $(0, \Phi^{-1}(1)]$ , then*

$$(49) \quad K(l^{(\Phi_s)}) = 2^{s/2} \left[ \frac{\Phi^{-1}(1)}{\Phi^{-1}(1/2)} \right]^{1-s}.$$

*Proof.* Ren [10] proved that  $\Phi_s \in \Delta_2(0) \cap \nabla_2(0)$  for  $0 < s \leq 1$ . To prove (46), by Lemma 2.3 we only need to show the rightmost inequality. Indeed,

$\Phi^{-1}(u)/\Phi^{-1}(2u) \geq 1/2$  for  $u > 0$  and hence,

$$(50) \quad G_{\Phi_s}(u) = \left[ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \right]^{1-s} \left[ \frac{\sqrt{u}}{\sqrt{2u}} \right]^s = 2^{-s/2} [G_{\Phi}(u)]^{1-s}, \quad u > 0,$$

which implies that

$$(51) \quad \tilde{\alpha}_{\Phi_s} \geq 2^{-s/2} \left( \frac{1}{2} \right)^{1-s} = 2^{s/2-1}.$$

If  $\Phi \in \nabla_2(0)$ , then  $\alpha_{\Phi}^0 = 1/2$ , so that

$$(52) \quad \alpha_{\Phi_s}^0 = \liminf_{u \rightarrow 0} G_{\Phi_s}(u) = (\alpha_{\Phi}^0)^{1-s} 2^{-s/2} = 2^{s/2-1}.$$

Thus, (47) follows from (52) and (46).

The hypothesis of (i) implies that  $G_{\Phi}(u)$  is increasing on  $(0, 1/2]$  (cf. Lemma 2.5). It follows from (46) that

$$(53) \quad \tilde{\alpha}_{\Phi_s} = \alpha_{\Phi_s}^0 = 2^{-s/2} [\lim_{u \rightarrow 0} G_{\Phi}(u)]^{1-s} = 2^{-s/2-(1-s)/C_{\Phi}^0}.$$

Hence, (48) follows from (46) and (53).

Under the condition of (ii),  $G_{\Phi}(u)$  is decreasing on  $(0, 1/2]$ , so that

$$(54) \quad \tilde{\alpha}_{\Phi_s} = \alpha'_{\Phi_s} = 2^{-s/2} [G_{\Phi}(u)]_{u=1/2} = 2^{-s/2} \left[ \frac{\Phi^{-1}(1/2)}{\Phi^{-1}(1)} \right]^{1-s}.$$

Finally, (49) holds by (46) and (54). ■

REMARK 2.10. By using an interpolation inequality, Rao and Ren [8, Theorem 4.3, 4.8] proved that

$$(55) \quad K(l^{(\Phi_s)}) \leq 2^{1-s/2}.$$

From the proof of Theorem 2.9 we see that (55) can be derived from (52) without the use of the interpolation inequality.

EXAMPLE 2.11. Let  $\Phi_r(u) = e^{|u|^r} - 1$  with  $1 < r < \infty$ . The inverse function of  $\Phi_s$  is

$$\Phi_s^{-1}(u) = [\Phi_r^{-1}(u)]^{1-s} [\Phi_0^{-1}(u)]^s = u^{s/2} [\ln(1+u)]^{(1-s)/r}, \quad u > 0, \quad 0 < s \leq 1.$$

Then the Kottman constant of  $l^{(\Phi_s)}$  is

$$(56) \quad K(l^{(\Phi_s)}) = 2^{(1-s)/r+s/2}.$$

Indeed,  $\Phi_r \in \nabla_2(0)$  since

$$C_{\Phi_r}^0 = \lim_{t \rightarrow 0} \frac{rt^r e^{t^r}}{e^{t^r} - 1} = r.$$

Note that

$$G_{\Phi_r}(u) = \left[ \frac{\ln(1+u)}{\ln(1+2u)} \right]^{1/r}$$

is increasing on  $(0, \infty) \supset (0, 1/2] \cup (0, (Q_{\Phi_s} - 1)/2]$ . From (51) and (46) we have

$$\tilde{\alpha}_{\Phi_s} = \alpha_{\Phi_s}^0 = 2^{-s/2}(\alpha_{\Phi_r}^0)^{1-s} = 2^{-s/2-(1-s)/r}$$

and

$$K(l^{(\Phi_s)}) = 2^{(1-s)/r+s/2}.$$

REMARK 2.12. Example 2.11 (56) strengthens the estimate

$$2^{(1-s)/r+s/2} \leq K(l^{(\Phi_s)}) \leq 2^{1-s/2}$$

of [8, Example 4.4] to an equality.

**3. Packing in Orlicz sequence spaces with Orlicz norm.** Let  $N(v) = (1 + |v|) \ln(1 + |v|) - |v|$ , and let  $l^N$  be the Orlicz sequence space generated by  $N(v)$  equipped with the Orlicz norm. Rao and Ren [8, Example 2.7 and Problem 3.8] estimated

$$(57) \quad 1.41 \leq K(l^N) \leq 1.56$$

and asked what is the exact value of  $K(l^N)$ . In this section, we present some supplementary results to [8] and improve the estimate (57).

LEMMA 3.1. *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions, with  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ . Then the Kottman constant  $K(l^\Phi)$  of the Orlicz sequence space  $l^\Phi$  equipped with the Orlicz norm can be estimated as follows:*

$$(58) \quad \max(2\beta_\Psi^0, 2\beta'_\Psi) \leq K(l^\Phi) \leq \frac{1}{\alpha_\Phi^*},$$

where

$$(59) \quad \beta'_\Psi = \sup \left\{ \frac{\Psi^{-1}(1/2k)}{\Psi^{-1}(1/k)} : k = 1, 2, \dots \right\},$$

$$(60) \quad \alpha_\Phi^* = \inf \left\{ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \leq \frac{1}{2}(Q_\Phi - 1) \right\},$$

with

$$(61) \quad Q_\Phi = \sup_{\|x\|_\Phi=1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x} [1 + \rho_\Phi(k_x x)] \right\}.$$

*Proof.* Using Wang's [12] result, Rao and Ren [8] proved that  $K(l^\Phi) \leq 1/\alpha_\Phi^*$  for  $\Phi \in \Delta_2(0)$  and  $2\beta'_\Psi \leq K(l^\Phi)$  for any  $N$ -function  $\Phi$ . It follows that if  $\Phi \notin \nabla_2(0)$  then  $\beta_\Psi^0 = 1$ ,  $K(l^\Phi) = 2$ . Thus, to show (58), it suffices to prove that

$$(62) \quad 2\beta'_\Psi \leq K(l^\Phi).$$

For any given  $k \geq 1$ , let

$$Z_k = (0, \dots, 0) \quad \text{and} \quad X_k = [k\Psi^{-1}(1/k)]^{-1}(1, \dots, 1),$$

with  $\dim Z_k = \dim X_k = k$ , and define

$$x_i = (Z_k, \dots, Z_k, X_k, Z_k, Z_k, \dots), \quad i \geq 1,$$

with  $X_k$  being at the  $i$ th position. Then  $\|x_i\|_{\Phi} = 1$ ,  $i \geq 1$ , and for  $i \neq j$ ,

$$\|x_i - x_j\|_{\Phi} = \frac{1}{k\Psi^{-1}(1/k)} \left[ 2k\Psi^{-1}\left(\frac{1}{2k}\right) \right] = \frac{2\Psi^{-1}(1/(2k))}{\Psi^{-1}(1/k)},$$

so that

$$K(l^{\Phi}) \geq \inf_{i \neq j} \|x_i - x_j\|_{\Phi} = \frac{2\Psi^{-1}(1/(2k))}{\Psi^{-1}(1/k)}.$$

Since  $k \geq 1$  is arbitrary, by (59), one has

$$K(l^{\Phi}) \geq \sup \left\{ \frac{2\Psi^{-1}(1/(2k))}{\Psi^{-1}(1/k)} : k = 1, 2, \dots \right\} = 2\beta'_{\Psi}. \quad \blacksquare$$

**COROLLARY 3.2** ([8]). *Let  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ , and  $\Psi$  be its complementary  $N$ -function. Then*

$$(63) \quad 1 < 2^{1/B_{\Phi}^0} \leq K(l^{\Phi}) \leq 2^{1/A_{\Phi}^*} < 2,$$

where

$$(64) \quad A_{\Phi}^* = \inf \left\{ \frac{t\phi(t)}{\Phi(t)} : 0 < t \leq \Phi^{-1}(Q_{\Phi} - 1) \right\}.$$

In particular, if  $F_{\Phi}(t) = t\phi(t)/\Phi(t)$  is increasing on  $(0, \Phi^{-1}(Q_{\Phi} - 1)]$ , then  $A_{\Phi}^* = B_{\Phi}^0 = C_{\Phi}^0 = \lim_{t \rightarrow 0} t\phi(t)/\Phi(t)$ , and hence

$$(65) \quad K(l^{\Phi}) = 2^{1/C_{\Phi}^0}.$$

For the constant  $Q_{\Phi}$  of the space  $l^{\Phi}$  appearing in (60) and (64), one has  $1 \leq Q_{\Phi} \leq \infty$ . Wang [12] and Wu [13] proved:  $\Phi \in \nabla_2(0)$  if and only if  $Q_{\Phi} < \infty$ . We give a better estimate of  $Q_{\Phi}$ .

**LEMMA 3.3.** *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions, with  $\Phi \in \nabla_2(0)$ . Then*

$$(66) \quad Q_{\Phi} \leq \frac{a_{\Phi}^*}{a_{\Phi}^* - 1},$$

where

$$(67) \quad a_{\Phi}^* = \inf \left\{ \frac{t\phi(t)}{\Phi(t)} : 0 < t \leq \psi[\Psi^{-1}(1)] \right\}.$$

*Proof.* Note that  $\Phi \in \nabla_2(0)$  if and only if  $A_{\Phi}^0 = \liminf_{t \rightarrow 0} t\phi(t)/\Phi(t) > 1$ . For every  $0 < t \leq \psi[\Psi^{-1}(1)]$  one has  $t\phi(t)/\Phi(t) > 1$ , so that  $a_{\Phi}^* > 1$  for  $\Phi \in \nabla_2(0)$ . Observe that  $a_{\Phi}^* \leq t_0\phi(t_0)/\Phi(t_0) < \infty$ , where  $t_0 = \psi[\Psi^{-1}(1)]$ . Now we show (66), which implies  $Q_{\Phi} < \infty$  for  $\Phi \in \nabla_2(0)$ .

For any given  $x = (x(i)) \in l^\Phi$  with  $\|x\|_\Phi = 1$ , since

$$k_x^{**} = \sup \left\{ k_x > 1 : \sum_{i=1}^{\infty} \Psi[\phi(k_x|x(i)|)] \leq 1 \right\},$$

one has

$$\sum_{i=1}^{\infty} \Psi[\phi((k_x^{**} - \varepsilon)|x(i)|)] \leq 1$$

for all  $\varepsilon > 0$ , so that  $(k_x^{**} - \varepsilon)|x(i)| \leq \psi[\Psi^{-1}(1)]$  for any  $i \geq 1$ . By definition of  $a_\Phi^*$ ,

$$a_\Phi^* \Phi[(k_x^{**} - \varepsilon)|x(i)|] \leq (k_x^{**} - \varepsilon)|x(i)|\phi[(k_x^{**} - \varepsilon)|x(i)|], \quad i \geq 1.$$

It follows that

$$\begin{aligned} 1 &\geq \sum_{i=1}^{\infty} \Psi\{\phi[(k_x^{**} - \varepsilon)|x(i)|]\} \\ &= \sum_{i=1}^{\infty} \{(k_x^{**} - \varepsilon)|x(i)|\phi[(k_x^{**} - \varepsilon)|x(i)|] - \Phi[(k_x^{**} - \varepsilon)|x(i)|]\} \\ &\geq (a_\Phi^* - 1) \sum_{i=1}^{\infty} \Phi[(k_x^{**} - \varepsilon)|x(i)|] \\ &\geq (a_\Phi^* - 1)[\|(k_x^{**} - \varepsilon)x\|_\Phi - 1] \\ &= (a_\Phi^* - 1)[(k_x^{**} - \varepsilon) - 1], \end{aligned}$$

i.e.,

$$k_x^{**} \leq \frac{a_\Phi^*}{a_\Phi^* - 1} + \varepsilon,$$

which proves (66) since  $Q_\Phi = \sup_{\|x\|_\Phi=1} k_x^{**}$  and  $\varepsilon$  is arbitrary. ■

REMARK 3.4. The constant  $a_\Phi^*$  was introduced by Rao and Ren (cf. (36) of [8]). The estimate (66) is an improvement of [8, Remark 3.4]. Wu, Chen and Wang [13, Lemma 2(b)] gave the following estimate of  $Q_\Phi$ :

$$Q_\Phi \leq \max \left( 2, \frac{2}{a_\Phi^* - 1} \right) = \begin{cases} 2, & a_\Phi^* \geq 2, \\ 2/(a_\Phi^* - 1), & 1 < a_\Phi^* < 2. \end{cases}$$

(66) also improves the above estimate. In fact, since  $a_\Phi^*/(a_\Phi^* - 1) < 2$  if and only if  $a_\Phi^* > 2$ , and  $a_\Phi^*/(a_\Phi^* - 1) < 2/(a_\Phi^* - 1)$  for  $1 < a_\Phi^* < 2$ , we see that  $a_\Phi^* \neq 2$  implies

$$\frac{a_\Phi^*}{a_\Phi^* - 1} < \max \left( 2, \frac{2}{a_\Phi^* - 1} \right).$$

COROLLARY 3.5. For any  $N$ -function  $\Phi$ , we have

$$(68) \quad Q_\Phi \leq b_\Psi^*,$$

where

$$(69) \quad b_{\Psi}^* = \sup \left\{ \frac{s\psi(s)}{\Psi(s)} : 0 < s \leq \Psi^{-1}(1) \right\}.$$

*Proof.* If  $\Phi \notin \nabla_2(0)$ ,  $\Psi \notin \Delta_2(0)$ , and  $b_{\Psi}^* \geq B_{\Psi}^0 = \limsup_{s \rightarrow 0} s\psi(s)/\Psi(s) = \infty$ , then (68) naturally holds. If  $\Phi \in \nabla_2(0)$ ,  $\Psi \in \Delta_2(0)$ , and  $b_{\Psi}^* < \infty$ ,

$$(70) \quad \frac{1}{a_{\Phi}^*} + \frac{1}{b_{\Psi}^*} = 1,$$

i.e.,  $a_{\Phi}^*/(a_{\Phi}^* - 1) = b_{\Psi}^*$  by Theorem 1.5(ii). Therefore (68) follows from (66). ■

**THEOREM 3.6.** *Let  $\Phi, \Psi$  be a pair of complementary  $N$ -functions with  $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ .*

(i) *If  $F_{\Phi}(t) = t\phi(t)/\Phi(t)$  is increasing on  $(0, \Phi^{-1}(b_{\Psi}^* - 1)]$ , then*

$$(71) \quad K(l^{\Phi}) = 2^{1/C_{\Phi}^0}, \quad C_{\Phi}^0 = \lim_{t \rightarrow 0^+} F_{\Phi}(t).$$

(ii) *If  $F_{\Phi}(t)$  is decreasing on  $(0, \psi[\Psi^{-1}(1)])$ , then*

$$(72) \quad \frac{2\Psi^{-1}(1/2)}{\Psi^{-1}(1)} \leq K(l^{\Phi}) \leq \frac{\psi[\Psi^{-1}(1)]}{\Phi^{-1}\{\frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])\}}.$$

*Proof.* (i) From Corollary 3.5,  $F_{\Phi}(t)$  is increasing on  $(0, \Phi^{-1}(Q_{\Phi} - 1)]$ , and (71) follows from Corollary 3.2 (61).

(ii) By Theorem 1.6(i),  $F_{\Psi}(s)$  is increasing on  $(0, \Psi^{-1}(1)]$ . It follows from Theorem 1.3 that  $G_{\Psi}(v)$  is increasing on  $(0, 1/2]$ , and  $G_{\Phi}(u)$  is decreasing on  $(0, \frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])]$  and hence

$$\begin{aligned} b_{\Psi}^* &= [F_{\Psi}(s)]_{s=\Psi^{-1}(1)} = \Psi^{-1}(1)\psi[\Psi^{-1}(1)] \\ &= \Phi\{\psi[\Psi^{-1}(1)]\} + \Psi[\Psi^{-1}(1)] = \Phi\{\psi[\Psi^{-1}(1)]\} + 1, \end{aligned}$$

i.e.,

$$(73) \quad b_{\Psi}^* - 1 = \Phi\{\psi[\Psi^{-1}(1)]\}.$$

It follows that

$$(74) \quad (0, \frac{1}{2}(Q_{\Phi} - 1)] \subset (0, \frac{1}{2}(b_{\Psi}^* - 1)] = (0, \frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])].$$

Since  $G_{\Phi}(u)$  is decreasing, one has

$$(75) \quad \alpha_{\Phi}^* = \frac{\Phi^{-1}\{\frac{1}{2}(Q_{\Phi} - 1)\}}{\Phi^{-1}(Q_{\Phi} - 1)} \geq \frac{\Phi^{-1}\{\frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])\}}{\psi[\Psi^{-1}(1)]}.$$

Then by Lemma 3.1,

$$(76) \quad K(l^{\Phi}) \leq \frac{1}{\alpha_{\Phi}^*} \leq \frac{\psi[\Psi^{-1}(1)]}{\Phi^{-1}\{\frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])\}}.$$

On the other hand, again by Theorem 1.3,  $G_\Psi(v)$  is increasing on  $(0, 1/2]$ , thus

$$(77) \quad K(l^\Phi) \geq \max(2\beta_\Psi^0, 2\beta_\Psi') = 2\beta_\Psi' = \frac{2\Psi^{-1}(1/2)}{\Psi^{-1}(1)}.$$

Finally, (72) follows from (76) and (77). ■

EXAMPLE 3.7 (Rao and Ren [8, Problem 3.8]). Let

$$M(u) = e^{|u|} - |u| - 1, \quad N(v) = (1 + |v|) \ln(1 + |v|) - |v|.$$

Then  $F_N(t) = tN'(t)/N(t)$  is decreasing on  $(0, \infty)$ . Now we estimate  $K(l^N)$  by Theorem 3.6.

Note that  $M'(u) = e^u - 1$ . Let  $v_0 = M^{-1}(1)$  and  $v_1 = M^{-1}(1/2)$ . Then  $v_0$  is a root of  $e^{v_0} - v_0 - 1 = 1$ , and  $v_1$  is a root of  $e^{v_1} - v_1 - 1 = 1/2$ . We have  $M'[M^{-1}(1)] = e^{v_0} - 1 = v_0 + 1$ , and hence by (72) one gets

$$(78) \quad \frac{2v_1}{v_0} \leq K(l^N) \leq \frac{v_0 + 1}{N^{-1}[\frac{1}{2}N(v_0 + 1)]}.$$

By a simple computation,  $v_0 = 1.14619$ ,  $v_1 = 0.85767$ , therefore,

$$(79) \quad 1.496 \leq K(l^N) \leq 1.498.$$

This improves the estimate of Rao and Ren [8, Example 3.7]:  $1.41 \leq K(l^N) \leq 1.56$ .

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