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Semicontinuity and continuous selections for the multivalued superposition operator without assuming growth-type conditions

by

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Abstract. Let Ω be a measure space, and E, F be separable Banach spaces. Given a multifunction $f : \Omega \times E \to 2^F$, denote by $N_f(x)$ the set of all measurable selections of the multifunction $f(\cdot, x(\cdot)) : \Omega \to 2^F$, $s \mapsto f(s, x(s))$, for a function $x : \Omega \to E$. First, we obtain new theorems on H-upper/H-lower/lower semicontinuity (without assuming any conditions on the growth of the generating multifunction $f(s, u)$ with respect to u) for the multivalued (Nemytskiĭ) superposition operator N_f mapping some open domain $G \subset X$ into 2^Y, where X and Y are Köthe–Bochner spaces (including Orlicz–Bochner spaces) of functions taking values in Banach spaces E and F respectively. Second, we obtain a new theorem on the existence of continuous selections for N_f taking nonconvex values in non-L_p-type spaces. Third, applying this selection theorem, we establish a new existence result for the Dirichlet elliptic inclusion in Orlicz spaces involving a vector Laplacian and a lower semicontinuous nonconvex-valued right-hand side, subject to Dirichlet boundary conditions on a domain $\Omega \subset \mathbb{R}^2$.

1. Introduction. Let Ω be a measure space, and E, F be separable Banach spaces. Given a multifunction $f : \Omega \times E \to 2^F$, denote by $N_f(x)$ the set of all measurable selections of the multifunction $f(\cdot, x(\cdot)) : \Omega \to 2^F$, $s \mapsto f(s, x(s))$, for a function $x : \Omega \to E$. In the present paper we consider semicontinuity properties and the existence of continuous selections for the so-called *multivalued* (*Nemytskiĭ*) *superposition operator* N_f acting in Köthe–Bochner spaces (Banach lattices) of measurable functions (concrete examples: Lebesgue/Bochner spaces, Lebesgue spaces with mixed norm, Orlicz–Bochner spaces, Lorentz spaces, Marcinkiewicz spaces, etc.).

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The multivalued operator N_f was studied recently by various mathematicians in connection with its natural applications to nonlinear elliptic/parabolic partial differential inclusions, abstract parabolic inclusions, integral inclusions, and equations with discontinuous right-hand side (see references since 1970 in [20] and [1, 2, 5, 7–16, 18, 29, 31, 32]). All semicontinuity results for N_f obtained in the above cited references (excepting the papers [5, 12, 31]) were established via different methods under certain conditions on the growth of the generating multifunction $f(s,u)$ with respect to u (e.g. $\|f(s,u)\|_F := \sup\{\|w\|_F : w \in f(s,u)\} \leq a(s) + b\|u\|_E$ for N_f mapping the Bochner–Lebesgue space $L_1(\Omega, E)$ into $2^{L_1(\Omega,F)}$). In the present paper we succeed in obtaining, via a new method, semicontinuity results for N_f without assuming any growth-type conditions on f.

First, we obtain a new theorem (Theorem 3.2) on H-upper/H-lower/lower semicontinuity for N_f mapping an open domain $G \subset X$ into 2^Y, where X and Y are Köthe–Bochner spaces of functions taking values in Banach spaces E and F respectively. Our proof of Theorem 3.2 relies crucially on the abstract semicontinuity Theorem 3.1 (whose proof relies on Lemmas 3.1–3.2) for a multivalued operator N satisfying the so-called local definedness property.

Second, using Lemma 3.2 and Theorem 3.1 (together with the Fryszkowski–Bressan–Colombo selection theorem [7, 16], [28, Corollary 2.1]) we obtain a first theorem (Theorem 3.3) on the existence of continuous selections for N_f taking nonconvex values in a non-L_p-type space Y.

Third, applying Theorem 3.3, we establish a new existence result (Theorem 7.1) for the Dirichlet elliptic inclusion in Orlicz spaces involving a vector Laplacian and a lower semicontinuous nonconvex-valued right-hand side, subject to Dirichlet boundary conditions on $\Omega \subset \mathbb{R}^2$ (in connection with Pokhozhaev–Trudinger's theorem on the exact noncompact embedding of the Sobolev space $H_0^1(\Omega)$ into some "exponential" Orlicz space [17, Theorem 7.15, Section 7.8]).

Let us point out the following comparisons. From Theorem 3.4 of Section 3 we can deduce by a different but more simple proof (see Section 5) the 1991 result of A. Cellina, A. Fryszkowski, and T. Rzeżuchowski [12] for N_f defined on a non-open set $G \subset X$. By the proofs of the present paper together with the recent theory of Banach function L_∞-modules (see references in [25, 27]) we can get analogs of our theorems of Section 3 for the operator N_f acting in these modules; in this way we refine the 1991 results of J. Appell, H. T. Nguyen, and P. P. Zabrejko [5, Theorem 4] who assume additionally that N_f maps an order-bounded set into an order-bounded set. By analogous arguments we can define the classes of "source spaces X" and "target spaces Y" for which our results in Section 3 remain valid; in this way in particular we can refine the 1994 results of S. Rolewicz and W. Song [31] for N_f in metric modular spaces.

In Section 4 we collect all proofs of the results of Section 3. Some standard facts about so-called Köthe–Bochner spaces (Banach lattices of measurable functions) are given in Section 2 as well as at the beginning of Section 4. In Section 5 we deduce (from Theorems 3.2, 3.4) Corollaries 5.1–5.2 on H-upper/H-lower/lower semicontinuity for N_f mapping a Bochner–Lebesgue space $L_p(\Omega, E)$ into $2^{L_q(\Omega,F)}$. In Section 6 we deduce (from Theorem 3.2) a new semicontinuity result (Corollary 6.1) for the multivalued superposition operator acting in Lebesgue spaces with mixed norm of functions defined on the product measure space $T \times \Omega$.

2. Some terminology and notation. First, we give some terminology and notation from set-valued analysis following, e.g., [10, 15]. Let (X, ϱ) be a metric (vector) space. For $x \in X$, $M \subset X$ and $\varepsilon > 0$ we define $d_X(x, M) = \inf\{\varrho(x, y) : y \in M\}$ and $U_\varepsilon(M) = \{y \in X : d_X(y, M) < \varepsilon\}$. For $A, B \subset X$ we define $h_X^+(A, B) := \sup\{d_X(a, B) : a \in A\}$. Denote by $B_X(r)$ the closed ball in X with center 0 and radius r. Given a multifunction $\Gamma : X \to 2^Y$, define $\operatorname{dom} \Gamma = \{x \in X : \Gamma(x) \neq \emptyset\}$.

Let X, Y be metric spaces and let $\Gamma : X \to 2^Y$ be a multifunction. Γ is called *lower semicontinuous* (or *l.s.c.*) *at* $x_0 \in X$ if for any open set $V \subset Y$ such that $\Gamma(x_0) \cap V \neq \emptyset$, there exists an open neighbourhood $U \subset X$ of x_0 such that $\Gamma(x) \cap V \neq \emptyset$ for all $x \in U$. We say that Γ is *H-upper semicontinuous* (or *H-u.s.c.*) *at* $x_0 \in X$ if for any $\varepsilon > 0$ one may find a $\delta > 0$ such that $\Gamma(B(x_0, \delta)) \subset U_\varepsilon(\Gamma(x_0))$. Γ is said to be *H-lower semicontinuous* (or *H-l.s.c.*) *at* $x_0 \in X$ if for any $\varepsilon > 0$ one may find a $\delta > 0$ such that $\Gamma(x_0) \subset U_\varepsilon(\Gamma(x))$ for all $x \in B(x_0, \delta)$. A multifunction Γ is called H-u.s.c. (resp., H-l.s.c., l.s.c.) if it is H-u.s.c. (resp., H-l.s.c., l.s.c.) at every $x \in X$.

Second, we give some terminology and notation from function space theory following, e.g., [33] and [6, 23]. From now on, unless stated to the contrary, E, F, etc. denote separable Banach spaces; $(\Omega, \mathfrak{A}, \mu)$ denotes a fixed measure space with a complete σ-finite σ-additive measure μ on a σ-algebra $\mathfrak{A}$ of subsets of Ω; μ_* *denotes any finite measure equivalent to* μ (i.e. $\mu_*(D) = 0 \Leftrightarrow \mu(D) = 0$). Further, $S(\Omega, E)$ denotes the complete metric vector space of all (equivalence classes of) measurable functions $x : \Omega \to E$, equipped with the metric topology of convergence in μ_* measure. Given a property P_s, we shall write $P_s \pmod 0$ if P_s is valid for almost all (a.a.) $s \in \Omega$.

A Banach space $\mathbb{X} \subset S(\Omega, \mathbb{R})$ with norm $\|\cdot\|_{\mathbb{X}}$ is called a *Köthe space* (also *Banach lattice*, *Banach ideal space*) if $x \in \mathbb{X}$ and $y \in S(\Omega, \mathbb{R})$ and $|y(s)| \le |x(s)|$ a.e. imply that $y \in \mathbb{X}$ and $\|y\|_{\mathbb{X}} \le \|x\|_{\mathbb{X}}$. Concrete examples of Köthe spaces are the Lebesgue spaces and many non-L_p-type spaces including general Orlicz/Lorentz/Marcinkiewicz spaces (see [6, 24, 30]).

Given a Köthe space $\mathbb{X} \subset S(\Omega, \mathbb{R})$, define the *Köthe–Bochner space* $X = \mathbb{X}[E] \subset S(\Omega, E)$ as the Banach space of all measurable functions $x : \Omega \to E$ such that $\|x(\cdot)\|_E \in \mathbb{X}$, with norm $\|x\|_X := \|\|x(\cdot)\|_E\|_{\mathbb{X}}$.

Given a Köthe–Bochner space $Y = \mathbb{Y}[F] \subset S(\Omega, F)$, define its *regular part* Y° as the Köthe–Bochner subspace of all measurable functions $y \in Y$ *with absolutely continuous norm* (or *having equi-continuous norm*, or *equi-integrable* in the case $Y = L_p$), i.e.

(1) $$Y^\circ = \{y \in Y : \lim_{\mu_*(D)\to 0} \|P_D y\|_Y = 0\},$$

where P_D denotes the multiplication operator by the characteristic function χ_D of a measurable set D. If $Y = Y^\circ$, then Y is called a *regular space.* The Bochner–Lebesgue spaces $\mathbb{L}_p[E]$ $(1 \le p < \infty)$ are regular. A set $M \subset Y$ is said to *have uniformly absolutely continuous norms of* Y, or briefly, *u.a.c. norms of* Y (or *equicontinuous norms of* Y, or to be *absolutely bounded*, or *equi-integrable* in the case $Y = L_p$) if

(2) $$\lim_{\mu_*(D)\to 0} \sup_{y\in M} \|P_D y\|_Y = 0.$$

Third, we give some notation specific for the present paper. Denote by $\operatorname{Sel} g$ the set of all measurable selectors of a multifunction $g : \Omega \to 2^F$, i.e.

(3) $$\operatorname{Sel} g = \{y \in S(\Omega, F) : y(s) \in g(s) \text{ a.e.}\}.$$

A multifunction $f : \Omega \times E \to 2^F$ is called [5] *superpositionally measurable*, or briefly *sup-measurable*, if for every single-valued measurable function $x : \Omega \to E$ the multifunction $\Gamma = f(\cdot, x(\cdot)) : \Omega \to 2^F$, $s \mapsto f(s, x(s))$, is a measurable multifunction with $\operatorname{dom} \Gamma = \Omega \pmod 0$. The measurability of Γ means that $\{s \in \Omega : \Gamma(s) \cap U \neq \emptyset\}$ is measurable for every open subset U of F. If $\Gamma : \Omega \to 2^F$ is measurable such that $\Gamma(s)$ is nonempty closed for a.a. $s \in \Omega$ then $\operatorname{Sel} \Gamma \neq \emptyset$ and moreover there exists a so-called Castaing representation $\{y_k : k \in \mathbb{N}\}$ for Γ (see [10]). Sufficient conditions for sup-measurability are given e.g. in [1].

We denote [1, 5] by $\mathcal{P}(F)$ (resp., $\mathrm{Cl}(F)$, $\mathrm{Bd}(F)$, $\mathrm{Cp}(F)$, $\mathrm{BdCl}(F)$, etc.) the family of all nonempty (resp., nonempty closed, nonempty bounded, nonempty compact, nonempty bounded closed, etc.) subsets of F.

3. Semicontinuity theorems in Köthe–Bochner spaces. A multi-valued operator $N : G \to \mathcal{P}(S(\Omega, F))$ with $\emptyset \neq G \subset S(\Omega, E)$ is called *locally defined* (or *locally determined*) provided the following condition holds:

(LD1) if $P_D x = P_D y$ with $x, y \in G$, $D \in \mathfrak{A}$, then $P_D N(x) = P_D N(y)$;

in the case $P_D x \in G$ $(D \in \mathfrak{A})$ this is equivalent to

(LD2) $P_D N(x) = P_D N(P_D x)$ $(D \in \mathfrak{A}$, $x \in G)$.

It is surprising that being locally defined is sufficient to get general theorems for N. Given a Köthe–Bochner space $Y = \mathbb{Y}[F]$, it is easily verified that the multivalued superposition operator with values in Y

$$N = N_f^Y, \qquad N_f^Y(x) := Y \cap \operatorname{Sel} f(\cdot, x(\cdot)), \tag{4}$$

is locally defined on its domain $\operatorname{dom} N_f^Y$.

In Lemma 3.1 we find a new property of a set $\mathcal{N}$ which does *not* satisfy (2) (i.e. it is not equi-integrable in the case $\mathcal{N} \subset Y = L_1$).

LEMMA 3.1. *Let $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space (i.e. $Y = Y^\circ$) and let $\mathcal{N} \subset Y$ be a set which fails* (2)*. Then there exist a positive number $\varepsilon > 0$, a sequence $y_n \subset \mathcal{N}$, and a sequence of mutually disjoint measurable sets Ω_n such that*

$$\mu_*(\Omega_n) \to 0 \quad (n \to \infty), \qquad \|P_{\Omega_n} y_n\|_Y > \varepsilon > 0 \quad (\forall n). \tag{5}$$

In Lemma 3.2 (whose proof crucially relies on Lemma 3.1) we obtain a new property for a locally defined multivalued operator N.

LEMMA 3.2. *Let $X = \mathbb{X}[E] \subset S(\Omega, E)$ be a Köthe–Bochner space, $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space, $r \in (0, \infty)$, and $N : B_X(r) \to \mathcal{P}(Y)$ be a locally defined operator such that each $N(x)$ has u.a.c. norms of Y. Suppose that $\{x_n : n = 1, 2, \ldots\} \subset B_X(r)$ satisfies*

$$\sum_{n=1}^{\infty} \|x_n\|_X \le r. \tag{6}$$

Then the set $M := \bigcup_{n=1}^{\infty} N(x_n)$ has u.a.c. norms of Y.

We shall use one of the following abstract conditions:

(LS) if $\|x_n - x_0\|_X \to 0$, $\|x_n - x_0\|_X \le R(x_0) < \infty$, $x_n \to x_0$ a.e. and $y_0 \in N(x_0)$, then there exist a subsequence n_k and $y_{n_k} \in N(x_{n_k})$ such that $y_{n_k} \to y_0$ in $S(\Omega, F)$;

(HL) if $\|x_n - x_0\|_X \to 0$, $\|x_n - x_0\|_X \le R(x_0) < \infty$, $x_n \to x_0$ a.e. and $y_n \in N(x_0)$, then there exist a subsequence n_k and $z_{n_k} \in N(x_{n_k})$ such that $y_{n_k} - z_{n_k} \to 0$ in $S(\Omega, F)$;

(HU) if $\|x_n - x_0\|_X \to 0$, $\|x_n - x_0\|_X \le R(x_0) < \infty$, $x_n \to x_0$ a.e. and $y_n \in N(x_n)$, then there exist a subsequence n_k and $z_{n_k} \in N(x_0)$ such that $y_{n_k} - z_{n_k} \to 0$ in $S(\Omega, F)$.

THEOREM 3.1. *Let G be a nonempty open subset of a Köthe–Bochner space $X = \mathbb{X}[E] \subset S(\Omega, E)$, $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space and $N : G \to \mathcal{P}(Y)$ be a locally defined operator such that each $N(x)$ has u.a.c. norms of Y. Suppose that N satisfies the condition* (LS) (*respectively*, (HL), (HU))*. Then N is l.s.c.* (*respectively*, *H-l.s.c*, *H-u.s.c.*)*.*

The next Theorem 3.2 is a first concrete realization of the abstract Theorem 3.1 for the multivalued superposition operator $N = N_f^Y$. The following simple Proposition 3.1 and Lemma 3.3 are first steps of this realization.

PROPOSITION 3.1. *Let $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a Köthe–Bochner space, and $H : \Omega \to \mathrm{Cl}\,(F) \cup \{\emptyset\}$ be an arbitrary multifunction. Then $Y \cap \mathrm{Sel}\, H$ is a closed subset of Y.*

LEMMA 3.3. *Let $f : \Omega \to \mathrm{Cl}(E)$ and $g : \Omega \to \mathrm{Cl}(E)$ be measurable multifunctions. Then the function $h_E^+(f(\cdot), g(\cdot))$ is measurable, and for every fixed $\varepsilon > 0$ and fixed selector $x \in \mathrm{Sel}\, f$ there exists a selector $y \in \mathrm{Sel}\, g$ such that for all $s \in \Omega$,*

$$d_E(x(s), y(s)) \le (1+\varepsilon) d_E(x(s), g(s)) \le (1+\varepsilon) h_E^+(f(s), g(s)).$$

THEOREM 3.2. *Let G be a nonempty open subset of a Köthe–Bochner space $X = \mathbb{X}[E] \subset S(\Omega, E)$, and $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space. Suppose that $f : \Omega \times E \to \mathrm{Cl}\,(F) \cup \{\emptyset\}$ is a sup-measurable multifunction such that $N(x) = N_f(x) := \mathrm{Sel}\, f(\cdot, x(\cdot)) \subset Y$ $(\forall x \in G)$. If $f(s, \cdot) : E \to \mathrm{Cl}\,(F) \cup \{\emptyset\}$ is l.s.c. (respectively, H-l.s.c., H-u.s.c.) for a.a. $s \in \Omega$, then the operator $N = N_f : G \to \mathrm{BdCl}(Y)$ is l.s.c. (respectively, H-l.s.c., H-u.s.c.).*

THEOREM 3.3. *Let μ be a complete σ-finite σ-additive continuous (non-atomic) measure, G be a nonempty open subset of a Köthe–Bochner space $X = \mathbb{X}[E] \subset S(\Omega, E)$, and $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space. Suppose that $N : G \to \mathcal{P}(Y)$ is a locally defined operator such that each $N(x)$ has u.a.c. norms of Y,*

$$N(x) = Y \cap \mathrm{Sel}\, H(x), \qquad H : G \to \Omega^{\mathrm{Cl}\,(F)} \ (x \in G), \tag{7}$$

and N satisfies the condition (LS) [*in particular, suppose that $f : \Omega \times E \to \mathrm{Cl}\,(F) \cup \{\emptyset\}$ is a sup-measurable multifunction such that $N(x) = N_f(x) := \mathrm{Sel}\, f(\cdot, x(\cdot)) \subset Y$ $(\forall x \in G)$, $f(s, \cdot) : E \to \mathrm{Cl}\,(F) \cup \{\emptyset\}$ is l.s.c. for a.a. $s \in \Omega$*]. *Then for every separable subset $K \subset G$ (and every compact subset $K \subset G$) the restriction $N : K \to \mathrm{Cl}(Y)$ has a continuous selection $u : K \to Y$, i.e. $u(x) \in N(x)$ $(\forall x \in K)$.*

If G is assumed to be a nonopen set, then we cannot apply the crucial Lemma 3.2, but modifying slightly the arguments in the proof of Theorem 3.1, we get the following abstract Theorem 3.4 assuming additionally some local property on values $N(x)$. Analogs of Theorems 3.2–3.3 for the case of G nonopen with this additional assumption can be easily formulated and proved.

THEOREM 3.4. *Let G be a nonempty subset of a metric space X, $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space, and $N : G \to \mathcal{P}(Y)$ be an operator such that each $N(x)$ has u.a.c. norms of Y. Suppose that N*

satisfies the condition (LS) (*respectively*, (HL), (HU)). *Assume additionally that there exists a subsequence* x_{n_k} *as in* (LS) (*respectively*, (HL), (HU)) *such that the set* $\bigcup_{k=1}^{\infty} N(x_{n_k})$ *has u.a.c. norms of* Y. *Then* N *is l.s.c.* (*respectively*, H*-l.s.c*, H*-u.s.c.*).

4. Proofs of results of Section 3. We point out that in the following proofs we shall constantly use the following well-known Propositions 4.A–4.C (see, e.g., [33, 6, 23]) for Köthe–Bochner spaces $X = \mathbb{X}[E] \subset S(\Omega, E)$ and $Y = \mathbb{Y}[F] \subset S(\Omega, F)$, as well as the Riesz Theorem [i.e. if $x_n \to x_0$ in $S(\Omega, E)$ (in measure), then there exists a subsequence n_k such that $x_{n_k} \to x_0$ a.e.]. Note that in the simplest case of Lebesgue spaces $X = Y = L_p(\Omega, \mathbb{R})$, $1 \le p < \infty$, Proposition 4.A(i), (iii) is easily checked, Proposition 4.A(ii) is a reformulation of the Lebesgue dominated convergence theorem, Proposition 4.B is the *Vitali–Krasnosel'skiĭ convergence criterion* (see e.g. [24, Lemma 9.2]), and Proposition 4.C is a simple consequence of the Hölder integral inequality.

A normed space $X \subset S(\Omega, E)$ is said to have the *Riesz–Fischer property* provided the inequality

$$\sum_{n=1}^{\infty} \|x_n\|_X < \infty$$

implies that the series $x_\infty := \sum_{n=1}^{\infty} x_n$ converges in $S(\Omega, E)$,

$$\|x_\infty\|_X \le \sum_{n=1}^{\infty} \|x_n\|_X < \infty \tag{8}$$

and $\sum_{j=1}^{n} x_j \to x_\infty$ in the norm of X.

PROPOSITION 4.A. *Let* $X = \mathbb{X}[E] \subset S(\Omega, E)$ *be a Köthe–Bochner space. Then*

(i) X *is continuously embedded in* $S(\Omega, E)$;
(ii) X *has the Riesz–Fischer property*;
(iii) X *has* P_D*-monotone norm*, *i.e.*

$$\|P_D x\|_X \le \|x\|_X \quad (x \in X,\ D \in \mathfrak{A}), \tag{9}$$

and the following inequality is true:

$$\|P_D x\|_X \le \sum_{n=1}^{\infty} \|P_{D_n} x\|_X \quad (x \in X,\ D \subset \textstyle\bigcup_{n=1}^{\infty} D_n;\ D, D_n \in \mathfrak{A}). \tag{10}$$

A regular space $Y \subset S(\Omega, F)$ is said to have the *Vitali property* if the fact that a sequence $\{y_n : n = 1, 2, \ldots\} \subset Y$ has u.a.c. norms of Y, i.e.

$$\lim_{\mu_*(D) \to 0} \sup_{n \in \mathbb{N}} \|P_D y_n\|_Y = 0,$$

and that y_n converges a.e. to $y \in S(\Omega, F)$, implies that y_n converges in the norm of Y to $y \in Y$.

PROPOSITION 4.B. *Let $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a regular Köthe–Bochner space. Then Y has the Vitali property.*

Let $\eta_0 \in S(\Omega, \mathbb{R})$ be a nonnegative function. We define the Bochner–Lebesgue space $L(\eta_0) = \mathbb{L}(\eta_0)[F]$ with the weight η_0 as follows:

$$L(\eta_0) = \mathbb{L}(\eta_0)[F] := \Big\{ y \in S(\Omega, F) : \|y\|_{L(\eta_0)} = \int_{\Omega} \eta_0(s)\|y(s)\|_F \, d\mu(s) < \infty \Big\}. \tag{11}$$

PROPOSITION 4.C. *Let $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ be a Köthe–Bochner space. Then Y is continuously embedded in the Bochner–Lebesgue space $L(\eta_0) = \mathbb{L}(\eta_0)[F]$ for some weight η_0 with* $\operatorname{supp} \eta_0 = \operatorname{supp} Y$.

We now prove the results of Section 3.

Proof of Lemma 3.1. Assume that $\mathcal{N} \subset Y$ fails (2), where $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ is a regular Köthe–Bochner space. Then there exists $\varepsilon > 0$ such that

$$\lim_{\mu_*(D)\to 0} \sup_{y\in\mathcal{N}} \|P_D y\|_Y > 2\varepsilon > 0. \tag{12}$$

So there exist $D_1 \in \mathfrak{A}$ and $y_1 \in \mathcal{N}$ such that $\|P_{D_1} y_1\|_Y > 2\varepsilon$. Since $y_1 \in Y^\circ$ has u.a.c. norm of Y, it follows from (12) that there exist $D_2 \in \mathfrak{A}$, $y_2 \in \mathcal{N}$ such that $\mu_*(D_2) < 2^{-2}\mu_*(D_1)$, $\|P_{D_2} y_2\|_Y > 2\varepsilon > 0$, $\|P_{D_2} y_1\|_Y < 2^{-2}\varepsilon$. Continuing by induction, we construct sequences D_k and $y_k \in \mathcal{N}$ such that

$$\mu_*(D_k) < 2^{-k}\mu_*(D_1), \quad \|P_{D_k} y_k\|_Y > 2\varepsilon > 0, \quad \|P_{D_k} y_i\|_Y < 2^{-k}\varepsilon \quad (i < k).$$

Put $\Omega_n = D_n \setminus \bigcup_{k>n} D_k$. Then by (10) we have

$$\|P_{\Omega_n} y_n\|_Y \ge \|P_{D_n} y_n\|_Y - \sum_{k>n} \|P_{D_k} y_n\|_Y \ge 2\varepsilon - \sum_{k>n} 2^{-k}\varepsilon > \varepsilon > 0,$$
$$\mu_*(\Omega_n) \le \mu_*(D_n) < 2^{-n}\mu_*(D_1) \to 0 \quad (n \to \infty).$$

Clearly, Ω_n are mutually disjoint, and hence the lemma follows. ■

Proof of Lemma 3.2. Since $X = \mathbb{X}[E] \subset S(\Omega, E)$ is a Köthe–Bochner space with P_D-monotone norm, $P_D(B_X(r)) \subset B_X(r)\,(D \in \mathfrak{A})$, and so N has property (LD2).

Suppose to the contrary that the set $M = \bigcup_{n=1}^\infty N(x_n)$ fails (2), although the sequence $\{x_n : n = 1, 2, \ldots\}$ satisfies (6):

$$\sum_{n=1}^{\infty} \|x_n\|_X \le r < \infty.$$

Since each value $N(x_n)$ is assumed to have u.a.c. norms of Y, where $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ is a regular Köthe–Bochner space, by applying Lemma 3.1 for the set M we can find $\varepsilon > 0$, subsequences n_j, y_{n_j} and a sequence of mutually disjoint sets $D_j \in \mathfrak{A}$ such that

$$\mu_*(D_j) \to 0 \quad (j \to \infty), \qquad y_{n_j} \in N(x_{n_j}), \qquad \|P_{D_j} y_{n_j}\|_Y > \varepsilon > 0. \tag{13}$$

Since X has the Riesz–Fischer property and P_D-monotone norm (see Proposition 4.A), there exists $x_* := \sum_{j=1}^{\infty} P_{D_j} x_{n_j}$ satisfying (cf. (8))

$$\|x_*\|_X \le \sum_{j=1}^{\infty} \|P_{D_j} x_{n_j}\|_X \le \sum_{j=1}^{\infty} \|x_{n_j}\|_X \le r < \infty.$$

So $x_* \in B_X(r)$, and hence by assumption,

$$\lim_{\mu_*(D)\to 0} \sup_{y \in N(x_*)} \|P_D y\|_Y = 0. \tag{14a}$$

By (13) together with the property (LD2) of the locally defined operator N, we get

$$\begin{aligned} P_{D_j} y_{n_j} \in P_{D_j} N(x_{n_j}) &= P_{D_j} N(P_{D_j} x_{n_j}) \\ &= P_{D_j} N(P_{D_j} x_*) = P_{D_j} N(x_*). \end{aligned} \tag{14b}$$

Since $\mu_*(D_j) \to 0$ $(j \to \infty)$ (see (13)), from (14a)–(14b) we obtain $\lim_{j\to\infty} \|P_{D_j} y_{n_j}\|_Y = 0$, which contradicts the last inequality in (13). ■

Proof of Theorem 3.1. Because H defined by $H(x) := N(x - x_0)$ for a fixed $x_0 \in G$ is again a locally defined operator on the nonempty open subset $G(x_0) = G - x_0$ containing 0, which satisfies all the assumptions of the theorem, it suffices to prove the semicontinuity of N at 0 for $0 \in G$. In this case, Lemma 3.2 is applicable for N considered on any $B_X(r) \subset G$.

First, suppose to the contrary that (LS) holds but N is not l.s.c. at $0 \in G$. Then there exist an open set O and a sequence x_n with $\|x_n\| \le 2^{-n} r$ such that $N(0) \cap O \neq \emptyset$, $0 < r < R(0)$, $B_X(r) \subset G$, and $N(x_n) \cap O = \emptyset$ for all n. Since $x_n \to 0$ in X and X is continuously embedded in $S(\Omega, E)$, by Riesz's theorem we can assume without loss of generality that $x_n \to 0$ a.e. By (LS), for a fixed $y_0 \in N(0) \cap O$ there exist n_k and $y_{n_k} \in N(x_{n_k})$ such that $y_{n_k} \to y_0$ a.e. By Lemma 3.2 for x_{n_k} satisfying

$$\sum_{k=1}^{\infty} \|x_{n_k}\|_X \le \sum_{k=1}^{\infty} 2^{-k} r = r < \infty,$$

the sequence $\{y_{n_k} : k = 1, 2, \ldots\} \subset \bigcup_{k=1}^{\infty} N(x_{n_k})$ has u.a.c. norms of Y, so by the Vitali property of the regular Köthe–Bochner space Y (see Proposition 4.B), $y_{n_k} \to y_0$ in the norm of Y. Since O is open and $y_0 \in N(0) \cap O$, it follows that $y_{n_k} \in O$, $y_{n_k} \in N(x_{n_k})$ for sufficiently large k, which contradicts the condition $N(x_{n_k}) \cap O = \emptyset$ for all n_k.

Second, suppose that (HL) holds but N is not H-l.s.c. at $0 \in G$. Then there exist x_n with $\|x_n\| \le 2^{-n}r$ such that $h^+(N(0), N(x_n)) > \varepsilon > 0$ and $0 < r < R(0)$ and $B_X(r) \subset G$. As before, again we can assume that $x_n \to 0$ a.e. Since $h^+(N(0), N(x_n)) = \sup_{y \in N(0)} \operatorname{dist}(y, N(x_n))$, there exists a sequence $y_n \in N(0)$ such that $\operatorname{dist}(y_n, N(x_n)) > \varepsilon > 0$. By (HL) there exist n_k and $z_{n_k} \in N(x_{n_k})$ such that $y_{n_k} - z_{n_k} \to 0$ a.e. Since $\{y_{n_k} : k = 1, 2, \ldots\} \subset N(0)$ has u.a.c. norms of Y by assumption, and again by Lemma 3.2 the set $\bigcup_{k=1}^{\infty} N(x_{n_k})$ has u.a.c. norms of Y, it follows that the sequence $\{y_{n_k} - z_{n_k} : k = 1, 2, \ldots\}$ has u.a.c. norms of Y. By the Vitali property, hence $y_{n_k} - z_{n_k} \to 0$ in the norm of Y, which contradicts the conditions $z_{n_k} \in N(x_{n_k})$ with $\operatorname{dist}(y_{n_k}, N(x_{n_k})) > \varepsilon > 0$ for all n_k.

Third, suppose that (HU) holds but N is not H-u.s.c. at $0 \in G$. Then there exist x_n with $\|x_n\| \le 2^{-n}r$ such that $h^+(N(x_n), N(0)) > \varepsilon > 0$ and $0 < r < R(0)$ and $B_X(r) \subset G$. As before, we can assume that $x_n \to 0$ a.e. Since $h^+(N(x_n), N(0)) = \sup_{y \in N(x_n)} \operatorname{dist}(y, N(0))$, there exists a sequence $y_n \in N(x_n)$ such that $\operatorname{dist}(y_n, N(0)) > \varepsilon > 0$. By (HU) there exist n_k and $z_{n_k} \in N(0)$ such that $y_{n_k} - z_{n_k} \to 0$ a.e. Since $\{z_{n_k} : k = 1, 2, \ldots\} \subset N(0)$ has u.a.c. norms of Y by assumption, and again by Lemma 3.2 the set $\bigcup_{k=1}^{\infty} N(x_{n_k})$ has u.a.c. norms of Y, it follows that the sequence $\{y_{n_k} - z_{n_k} : k = 1, 2, \ldots\}$ has u.a.c. norms of Y. By the Vitali property, hence $y_{n_k} - z_{n_k} \to 0$ in the norm of Y, which contradicts the conditions $z_{n_k} \in N(0)$ with $\operatorname{dist}(y_{n_k}, N(0)) > \varepsilon > 0$ for all n_k. ∎

Proof of Proposition 3.1. This is standard by the above mentioned Riesz theorem. ∎

Proof of Lemma 3.3. This is standard via the known selection theorems [10] and Castaing representations (see the same argument, e.g., in [5]). ∎

Proof of Theorem 3.2. We shall show that Theorem 3.2 is a consequence of Theorem 3.1 for $N(x) = N_f^Y(x) = \operatorname{Sel} f(\cdot, x(\cdot))$.

By the assumptions of Theorem 3.2, given a fixed $x \in G$, the multifunction $\Gamma_x = f(\cdot, x(\cdot)) : \Omega \to \operatorname{Bd}\operatorname{Cl}(F) \cup \{\emptyset\}$ is measurable with $\operatorname{dom} f(\cdot, x(\cdot)) = \Omega \pmod 0$ and $\operatorname{Sel}\Gamma_x \subset Y = \mathbb{Y}[F]$. Then by the Kuratowski–Ryll-Nardzewski measurable selection theorem [10] it is easy to construct a nonnegative function $\beta_x \in \mathbb{Y}$ such that $\|y(s)\|_F \le \beta_x(s)$ a.e. for every $y \in \operatorname{Sel}\Gamma_x$, and so $\|P_D y\|_Y \le \|P_D \beta_x\|_{\mathbb{Y}}$ for every $y \in \operatorname{Sel}\Gamma_x$, $D \in \mathfrak{A}$. Since $\mathbb{Y}$ is regular, i.e. $\mathbb{Y} = \mathbb{Y}^\circ$, the function $\beta_x \in \mathbb{Y}$ satisfies the equality (1) in the norm of $\mathbb{Y}$. Hence, each $N_f(x) = \operatorname{Sel}\Gamma_x$ has u.a.c. norms of Y.

So, it remains to verify the condition (LS) (respectively, (HL), (HU)) of Theorem 3.1 under the assumptions of Theorem 3.2.

First, suppose that $f(s, \cdot) : E \to \operatorname{Cl}(F) \cup \{\emptyset\}$ is l.s.c. for a.a. $s \in \Omega$. Let $x_n \to x_0$ in X, $x_n \to x_0$ a.e., $x_n \in B_X(x_0, R(x_0)) \subset G$ (G is open and $x_0 \in G$, so we can choose such $R(x_0) < \infty$). By our assumptions f is sup-measurable, so $f(\cdot,x_n(\cdot)) : \Omega \to \mathrm{Cl}\,(F)\cup\{\emptyset\}$ $(n=0,1,\ldots)$ are measurable with $\mathrm{dom}\,f(\cdot,x_n(\cdot)) = \Omega$ (mod 0). Fix $y_0 \in N(x_0) := \mathrm{Sel}\,f(\cdot,x_0(\cdot))$. Then, $d(y_0(\cdot), f(s,x_n(\cdot))) \to 0$ a.e., since $f(s,\cdot)$ is l.s.c. By our assumption $\mathrm{Sel}\,f(\cdot,x_n(\cdot)) \subset Y$ $(n=0,1,\ldots)$, hence by Lemma 3.3 for each $y_0 \in N(x_0)$ there exists a sequence $y_n \in N(x_n) := \mathrm{Sel}\,f(\cdot,x_n(\cdot)) \subset Y$ such that

$$d(y_0(s),y_n(s)) \le 2d(y_0(s), f(s,x_n(s)))\ (\mathrm{mod}\,0).$$

So, $y_n \to y_0$ a.e., which proves (LS).

Second, suppose that $f(s,\cdot) : E \to \mathrm{Cl}\,(F)\cup\{\emptyset\}$ is H-l.s.c. for a.a. $s \in \Omega$. Let $x_n \in X$ be as above. Then $h^+(f(\cdot,x_0(\cdot)), f(s,x_n(\cdot))) \to 0$ a.e. By our assumption $\mathrm{Sel}\,f(\cdot,x_n(\cdot)) \subset Y$ $(n=0,1,\ldots)$, hence by Lemma 3.3 for each $y_n \in N(x_0)$ there exists $z_n \in N(x_n)$ such that

$$d(y_n(s),z_n(s)) \le 2h^+(f(s,x_0(s)), f(s,x_n(s)))\ (\mathrm{mod}\,0).$$

So, $y_n - z_n \to 0$ a.e., which proves (HL).

Third, suppose that $f(s,\cdot) : E \to \mathrm{Cl}\,(F)\cup\{\emptyset\}$ is H-u.s.c. for a.a. $s \in \Omega$. As before, for each sequence $y_n \in N(x_n)$ we get some sequence $z_n \in N(x_0)$ such that $d(y_n(s),z_n(s)) \le 2h^+(f(s,x_n(s)), f(s,x_0(s)))$ (mod 0). Again we get $y_n - z_n \to 0$ a.e., which proves (HU). ■

Proof of Theorem 3.3. First, by Proposition 4.C we can choose some $\eta_0 \in S(\Omega,\mathbb{R})$ such that Y is continuously embedded in $L(\eta_0) = \mathbb{L}(\eta_0)[F]$. Then every set with u.a.c. norms of Y has u.a.c. norms of $L(\eta_0)$. So, we can apply Theorem 3.1 for $N : G \to \mathcal{P}(L(\eta_0))$ to deduce that $N : G \to \mathrm{Cl}(L(\eta_0))$ is lower semicontinuous.

We define $N' := T\circ N$ with $Ty(s) = \eta_0(s)y(s)$ a.e. So, $N' := T\circ N : G \to \mathrm{Cl}(L_1[F])$. By our construction, N' is clearly locally defined and lower semicontinuous on G. We easily verify by (7) that $N'(x)$ is a decomposable set in $L_1[F]$, i.e. if $a,b \in N'(x)$ and $D \in \mathfrak{A}$ then $y = P_D a + P_{\Omega\setminus D} b \in N'(x)$. Now we can apply the well-known selection theorem of Bressan–Colombo [7] or Fryszkowski [16] (see a minor modification for an unbounded σ-additive measure μ in [28, Corollary 2.1]) for the lower semicontinuous operator $N' : K \to \mathrm{Cl}(L_1[F])$ with decomposable values on a separable metric space $K \subset G$ or a compact metric space $K \subset G$, respectively. Hence we can find a continuous selection $u' : K \to \mathrm{Cl}(L_1[F])$ of N'. Then $u = T^{-1}u' : K \to L(\eta_0)$ is a continuous selection of $N : K \to \mathrm{Cl}(L(\eta_0))$.

Clearly, $u : K \to Y$, since $u(x) \in N(x) \subset Y$. Note that u is generally not locally defined, but we shall establish that $u : K \to Y$ is continuous.

Assume to the contrary that $u : K \to Y$ is not continuous at some $x_0 \in K$. Then we can find some sequence $x_n \in K$ such that $x_n \to x_0$ in norm and

$$\|u(x_n) - u(x_0)\|_Y \text{ does not tend to } 0\ (n\to\infty). \tag{15}$$

By the same arguments as in the proof of Theorem 3.1, by using Lemma 3.2 we get a subsequence $x_{n_k} \subset K \subset G$ such that the set $M = \bigcup_{k=1}^{\infty} N(x_{n_k})$ has u.a.c. norms of Y. Since $u(x) \in N(x) \subset Y$ $(\forall x \in K)$, the sequence $\{u(x_{n_k}) : k = 1, 2, \ldots\}$ has u.a.c. norms of Y. Since $u : K \to L(\eta_0)$ is continuous and $L(\eta_0)$ is continuously embedded in $S(\Omega, F)$, $u(x_{n_k})$ converges to $u(x_0)$ in $S(\Omega, F)$ and so by Riesz's theorem we can assume that $u(x_{n_k})$ converges to $u(x_0)$ a.e. By the Vitali property of Y (see Proposition 4.B), $u(x_{n_k})$ converges to $u(x_0)$ in the norm of Y, which contradicts (15). ∎

Proof of Theorem 3.4. Since in Theorem 3.4 we assume additionally that there exists a subsequence x_{n_k} as in (LS) (respectively, (HL), (HU)) such that $\bigcup_{k=1}^{\infty} N(x_{n_k})$ has u.a.c. norms of Y, the arguments in the proof of Theorem 3.1 yield the assertion of Theorem 3.4 without using Lemma 3.2. ∎

5. Semicontinuity results in Bochner–Lebesgue spaces

COROLLARY 5.1. *Let* $1 \le p \le \infty$, $1 \le q < \infty$, $f : \Omega \times E \to \mathrm{BdCl}\,(F) \cup \{\emptyset\}$, *and* G *be a nonempty open subset of* $\mathbb{L}_p[E]$. *Suppose that* $N_f(x) := \mathrm{Sel}\, f(\cdot, x(\cdot)) \subset \mathbb{L}_q[F]$ $(\forall x \in G)$. *If* f *is sup-measurable and* $f(s, \cdot) : E \to \mathrm{BdCl}\,(F) \cup \{\emptyset\}$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*) *for a.a.* $s \in \Omega$, *then* $N_f : G \to \mathrm{BdCl}(\mathbb{L}_q[F])$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*).

Proof. Since $\mathbb{L}_q[F]$ $(1 \le q < \infty)$ is a regular Köthe–Bochner space, the corollary is a particular case of Theorem 3.2. ∎

In the following Corollary 5.2 we do not assume that G is open but we must assume some additional local condition on the values of N_f. Note that the following condition (CFR) is taken from Cellina–Fryszkowski–Rzeżuchowski [12] whose H-u.s.c. result for N_f acting in L_p-type spaces was proved by a different, complicated proof under the additional assumption of the graph measurability of f. In Corollary 5.2 we drop this assumption, and get also the l.s.c. and H-l.s.c. results under the same condition (CFR).

COROLLARY 5.2. *Let* $f : \Omega \times E \to \mathrm{BdCl}\,(F) \cup \{\emptyset\}$ *and* G *be a nonempty subset of* $\mathbb{L}_p[E]$. *Suppose that* $N_f(x) := \mathrm{Sel}\, f(\cdot, x(\cdot)) \subset \mathbb{L}_q[F]$ $(\forall x \in G)$, *and assume the following local condition*:

(CFR) $1 \le p, q < \infty$, *given* $x_0 \in G$ *there exist* $R(x_0) > 0$, $a \in \mathbb{L}_1(\Omega, \mathbb{R})$, $b \ge 0$ *such that*

$$\|y_0(s) - y(s)\|^q \le a(s) + b\|x_0(s) - x(s)\|^p \quad (\text{for a.a. } s \in \Omega)$$

for every $y_0 \in N_f(x_0)$, $y \in N_f(x)$, *and* $x \in G$ *with* $\|x_0 - x\| \le R(x_0)$.

If f *is sup-measurable and* $f(s, \cdot) : E \to \mathrm{BdCl}\,(F) \cup \{\emptyset\}$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*) *for a.a.* $s \in \Omega$, *then* $N_f : G \to \mathrm{BdCl}(\mathbb{L}_q[F])$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*).

Proof. We shall show that the corollary is a consequence of Theorem 3.4 together with the verification given in the proof of Theorem 3.2. Without loss of generality we may consider only the case $0 \in G$ and prove semicontinuity of N_f at 0, since otherwise we can pass to $M(x) := \operatorname{Sel} f(\cdot, x(\cdot) - x_0(\cdot))$. Since $\mathbb{L}_q[F]$ $(1 \le q < \infty)$ is a regular Köthe–Bochner space, it suffices to show that for any sequence $x_n \in G \cap B_{L_p}(R(0))$ such that $x_n \to 0$ in norm, there exists a subsequence n_k such that $\bigcup_{k=1}^{\infty} N_f(x_{n_k})$ has u.a.c. norms. To this end we take n_k such that $\|x_{n_k}\|_p \le 2^{-k-1}R(0)$. By Lebesgue's dominated convergence theorem we get $\alpha(\cdot) := \sum_k \|x_{n_k}(\cdot)\|_E \in \mathbb{L}_p$. From (CFR) and the equi-integrability of Lebesgue integrable functions we obtain

$$\begin{aligned}\|P_D y\|_{L_q}^q &\le 2^{q-1}(\|P_D y_0\|_{L_q}^q + \|P_D(y - y_0)\|_{L_q}^q) \\ &\le 2^{q-1}(\|P_D y_0\|_{L_q}^q + \|P_D a\|_{\mathbb{L}_1} + b\|P_D \alpha\|_{\mathbb{L}_p}^p) \to 0 \\ &\qquad (\mu_*(D) \to 0, \text{ uniformly for } y \in \textstyle\bigcup_{k=1}^{\infty} N_f(x_{n_k}))\end{aligned}$$

for a fixed $y_0 \in N_f(0)$. Therefore, $\bigcup_{k=1}^{\infty} N_f(x_{n_k})$ has u.a.c. norms. ■

6. Semicontinuity results in Lebesgue spaces with mixed norm. We now collect some known auxiliary information. Let T and Ω be two measure spaces, and $T \times \Omega$ be the product measure space. Let $E \subset S(\Omega, \mathbb{R}^m)$, $F \subset S(T, \mathbb{R})$ be two Lebesgue spaces. Define $F(E)$ to be the space of all $z \in S(T \times \Omega, \mathbb{R}^m)$ such that $z(t, \cdot) \in E$ for a.a. $t \in T$ and the function h_z, $h_z(t) := \|z(t, \cdot)\|_E$, belongs to F. The space $F(E)$ with the norm $\|h_z(\cdot)\|_F$, where $h_z(t) = \|z(t, \cdot)\|_E$ $(z \in F(E))$, is called a *Lebesgue space with mixed norm* on $T \times \Omega$. From e.g. [19, Theorem XI.1.2] one can get the following:

LEMMA 6.A. *Let $E \subset S(\Omega, \mathbb{R}^m)$ be a Lebesgue space and let $z \in S(T \times \Omega, \mathbb{R}^m)$ be such that $z(t, \cdot) \in E$ for a.a. $t \in T$. Then the function h_z, $h_z(t) := \|z(t, \cdot)\|_E$, is measurable.*

By Lemma 6.A one can easily check (see, e.g., [19]) that $F(E)$ is a Köthe–Bochner space in $S(T \times \Omega, \mathbb{R}^m)$, and $F(E)$ is regular if both F and E are regular.

Hence we may apply all results of Section 3 to the multivalued superposition operator N_g generated by a multivalued function $g : (T \times \Omega) \times \mathbb{R}^m \to 2^{\mathbb{R}^n}$ and acting in two Lebesgue spaces with mixed norm, i.e. $N_g : F(E) \to 2^{F_0(E_0)}$. So from Theorem 3.2 we deduce immediately Corollary 6.1 below, since the Köthe–Bochner space $F_0(E_0)$ is regular.

COROLLARY 6.1. *Let G be a nonempty open subset of a Lebesgue space with mixed norm $X = F(E) \subset S(T \times \Omega, \mathbb{R}^m)$, and let $Y = F_0(E_0) \subset S(T \times \Omega, \mathbb{R}^n)$ be another Lebesgue space with mixed norm such that both $F_0 \subset S(T, \mathbb{R})$ and $E_0 \subset S(\Omega, \mathbb{R}^n)$ are regular spaces. Suppose that $g : (T \times \Omega) \times \mathbb{R}^m \to \operatorname{Cp}(\mathbb{R}^n) \cup \{\emptyset\}$ is a sup-measurable multifunction such that $N_g(x) :=$*

$\operatorname{Sel} g(\cdot,\cdot,x(\cdot,\cdot)) \subset Y$ $(\forall x \in G)$. *If* $g(t,\omega,\cdot) : (T \times \Omega) \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^n) \cup \{\emptyset\}$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*) *for a.a.* $(t,\omega) \in T \times \Omega$, *then the operator* $N_g : G \to \mathrm{BdCl}(Y)$ *is l.s.c.* (*respectively*, H*-l.s.c.*, H*-u.s.c.*).

7. The Dirichlet problem for quasilinear elliptic differential systems with lower semicontinuous nonconvex-valued right-hand side. Let Ω be a bounded domain in $\mathbb{R}^2$, and $f : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}$ be a multifunction of two variables $(s,u) \in \Omega \times \mathbb{R}^m$. We shall consider the problem

$$\begin{cases} -\Delta_m x(s) \in f(s,x(s)) & \text{for a.a. } s \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \tag{16}$$

where $\Delta_m = (\Delta,\dots,\Delta)$ is the m-vector Laplacian. In what follows, we denote the scalar product and norm in the Euclidean space $\mathbb{R}^m$ by $(\cdot,\cdot)$ and $|\cdot|$, respectively, and the scalar product and norm in the Lebesgue space $L_2 = L_2(\Omega,\mathbb{R}^m)$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. As usual, $H^1 = H^1(\Omega,\mathbb{R}^m)$ is the Sobolev space defined by the norm $\|x\|_1 = \|x\| + \|\nabla_m x\|$, while $H_0^1 = H_0^1(\Omega,\mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^m)$ with respect to this norm. Denote by H^{-1} the dual space to H_0^1 with respect to the L_2-pairing $\langle\cdot,\cdot\rangle$. Given a Young function $M : \mathbb{R} \to [0,\infty)$, the term *Orlicz space* (e.g. [21, 24]) will refer to the space $L_M = L_M(\Omega,\mathbb{R}^m)$ (of equivalence classes) of measurable functions u on Ω taking values in $\mathbb{R}^m$, which is equipped with the Luxemburg norm $\|x\|_M = \inf\{k > 0 : \int_\Omega M(k^{-1}\|x(s)\|_{\mathbb{R}^m})\,ds \le 1\}$; in this case the regular part L_M° coincides with the closure E_M in L_M-norm of the set of all continuous functions (see e.g. [21, Subsection 1.8], [30]).

Throughout this section, we denote by Z and X the "exponential" Orlicz spaces:

$$\begin{aligned} &Z = L_\Phi, \qquad X = L_\Psi; \\ &\Phi(\alpha) = e^{|\alpha|^2} - 1, \qquad \Psi(\alpha) = e^{|\alpha|^\varrho} - 1, \qquad \alpha \in \mathbb{R},\ 1 < \varrho < 2. \end{aligned} \tag{17}$$

By Pokhozhaev–Trudinger's exact embedding theorem (see e.g. [17, Theorem 7.15, Section 7.8]), the Sobolev space H_0^1 (on $\Omega \subset \mathbb{R}^2$) is always continuously non-compactly embedded in Z. By e.g. [3, Lemma 1], H_0^1 is compactly embedded in X, since $Z \subset X$ and the unit ball of Z has u.a.c. norms of X (see e.g. [21, Lemma 2.5], [30]).

Note that

$$H_0^1 \subset Z \subset X \subset L_2 \subset X' \subset Z' \subset H^{-1} \tag{18}$$

continuously. Here, Z' denotes the *Köthe associate space* of all integral linear functionals on Z: $Z' := \{z' : z' : \Omega \to \mathbb{R}^m$ is measurable, $\langle z,z'\rangle < \infty$ $(\forall z \in Z)\}$ with the norm

$$\|z'\|_{Z'} = \sup\Big\{\langle z,z'\rangle := \int_\Omega (z(s),z'(s))\,ds : \|z\|_Z \le 1\Big\}.$$

Later on, we denote by μ_Δ the *first Dirichlet eigenvalue* of the Laplacian $-\Delta$.

The main result of this section is

THEOREM 7.1. *Let Z, X be the Orlicz spaces in* (17). *Suppose that the following conditions are satisfied*:

(E1) *The multivalued superposition operator N_f maps X into $2^{Z'}$, where $Z' = L_{\Phi^*}$ with Φ^* the dual to the Young function Φ.*

(E2) *The multifunction $f : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^m) \cup \{\emptyset\}$ is sup-measurable such that $f(s,\cdot) : \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^m) \cup \{\emptyset\}$ is lower semicontinuous for a.a. $s \in \Omega$.*

(E3) *The following one-sided inequality holds*:

$$(u,w) \le \gamma(u,u) + \delta(s) \quad (\textit{for a.a. } s \in \Omega \textit{ and all } w \in F(s,u)), \tag{19}$$

where $0 \le \gamma < \mu_\Delta$, $\delta \in L_1(\Omega,\mathbb{R})$ is non-negative.

Then the problem (16) *has at least one solution $x_* \in H_0^1 \subset Z$.*

REMARKS 7.1. (i) In [1–2] the problem (16) was treated only in the case of $f(s,\cdot)$ upper semicontinuous. In [9, 18] (and references cited therein) the problem (16) for $f(s,\cdot)$ lower semicontinuous was treated only if N_f maps L_q into 2^{L_p} ($2 \le q < \infty$, $1 < p \le 2$) (via the Fryszkowski–Bressan–Colombo selection theorem [7, 16]). Since X is contained *properly* in $\bigcup_{2\le q<\infty} L_q$, and $\bigcup_{1<p\le 2} L_p$ is contained *properly* in Z' by our choice (17), no existence results from [1–2, 9, 18] can be applied in the situation of Theorem 7.1 when N_f satisfies the conditions (E1)–(E2).

(ii) One cannot immediately apply the known scheme e.g. of [1–3, 9, 18] to prove Theorem 7.1, since the Orlicz space X is nonseparable [21, Subsections 1.8, 1.4], [24, Theorem 8.14] by our choice of X in (17) and so it is impossible to establish the existence of a "global" continuous selection $u : X \to Z'$ of $N_f : X \to 2^{Z'}$ (cf. Theorem 3.3). Later on, we shall modify this scheme to prove Theorem 7.1.

(iii) A sufficient condition guaranteeing that the multivalued superposition (Nemytskiĭ) operator N_f acts as desired in the condition (E1) is the following inequality:

$$\sup_{w\in f(s,u)} |w|(\ln(e + \sup_{w\in f(s,u)} |w|))^{1/2} \le a(s) + be^{c|u|^{\varrho-\varepsilon}} \qquad (\text{for a.a. } s \in \Omega \text{ and all } u \in \mathbb{R}^m)$$

with some $a \in L_1(\Omega,\mathbb{R})$, $b \in [0,\infty)$, $c \in (0,\infty)$, and $\varepsilon \in (0, \varrho - 1)$.

(iv) The analogous existence results are valid for more complicated nonlinear inclusions in Orlicz spaces such as multivalued versions of nonlinear elliptic boundary value problems, which were studied e.g. in [3, 27] and references cited therein.

Proof of Theorem 7.1. We divide the proof of Theorem 7.1 into six steps.

STEP 1. As is well known (see e.g. [17, Section 8.2, Theorem 5.8]), the operator L generated by the Laplacian $-\Delta_m$ is continuous and invertible from H_0^1 into H^{-1}, and

$$\langle Lx, x\rangle \geq \alpha\|x\|_1^2 \quad (x \in H_0^1) \tag{20}$$

for some $\alpha > 0$. Recall that the solvability of (16) in H_0^1 means the existence of $x \in H_0^1$ and $y \in N_f(x)$ such that $y \in H^{-1}$ and $Lx = y$. Via (18) together with the condition (E1), we see that x is a solution of the problem (16) if and only if x is a solution of the inclusion $x \in L^{-1}N_f(x)$ in X. Since H_0^1 is compactly embedded in X, the linear inverse operator $L^{-1} : Z' \to X$ is compact.

STEP 2. Suppose that there exist $x \in X$, $z \in Z'$, $0 \leq \lambda \leq 1$, and $Q(x) \in Z'$ such that $x = L^{-1}(z)$, $z = \lambda Q(x)$ and

$$\langle Q(x), x\rangle \leq \gamma\langle x, x\rangle + d, \tag{21}$$

where $d := \|\delta(\cdot)\|_{L_1}$. We claim that

$$\|x\|_1 \leq d^{1/2}c^{-1/2}, \quad \|x\|_X \leq r = r(c, d, h) < \infty, \tag{22}$$

where $c := \alpha(\mu_\Delta - \gamma)\mu_\Delta^{-1} \in (0, \infty)$, and r depends on c, d and on the norm h of the continuous embedding $H_0^1 \subset X$.

To see this, by [3, Lemma 4, (16)] together with $0 \leq \gamma < \mu_\Delta$ in the condition (E3) we see that $c \in (0, \infty)$ and

$$\gamma\langle x, x\rangle \leq \langle Lx, x\rangle - c\|x\|_1^2. \tag{23}$$

By (21) and (23) we get

$$\langle Q(x), x\rangle + c\|x\|_1^2 \leq \langle Lx, x\rangle + d.$$

Then by the same proof of [3, Lemma 3] we obtain (22).

STEP 3. Since $X = L_\Psi$ is a split space (see e.g. [6, Lemmas 2.5 and 4.2]) and $Z' = L_{\Phi^*}$ is perfect, i.e. $Z' = (Z')''$ (see e.g. [24, Theorem 13.18]) by (17), applying [4, Theorem 5] for N_f satisfying (E1)–(E2), we find that N_f is bounded in Z' on bounded subsets of X. Hence,

$$r_* := \sup\{\|y\|_{Z'} : y \in N_f(x),\ \|x\|_X \leq r\} < \infty. \tag{24}$$

STEP 4. Since $L^{-1} : Z' \to X$ is a compact linear operator, the set

$$C := \operatorname{clco}\{L^{-1}y : \|y\|_Y \leq r_* + 1\} \tag{25}$$

is, by the well-known Mazur theorem [19], a compact metric subspace of X, where $\operatorname{clco}(D)$ is the convex closure of D. Since the Orlicz space Z' is regular (see e.g. [21, Subsections 1.4, 1.6, 1.8, 1.9], [24, Theorem 8.14]), we can apply Theorem 3.3 to find a continuous single-valued selection $Q_1 : C \to Z'$

for $N_f|_C : C \to \mathrm{BdCl}(Z')$. Hence, by (19), we conclude that $Q_1(x)(s) \in f(s, x(s))$ a.e. $(x \in C)$ and

$$\langle Q_1(x), x\rangle \le \gamma\langle x, x\rangle + d \quad (\forall x \in C), \tag{26}$$

where $d = \|\delta(\cdot)\|_{L_1}$.

By the Dugundji–Tietze–Urysohn theorem [22, Theorem 18.1], we can find a continuous nonlinear projection R from X onto the closed convex set C such that $R(x) = x$ $(x \in C)$. Then the continuous single-valued map $\widetilde{Q}_1 := Q_1 R : X \to Z'$ is a continuous extension of $Q_1 : C \to Z'$.

Step 5. We claim that

$$y \ne \lambda \widetilde{Q}_1 L^{-1}(y) \quad (0 \le \lambda \le 1,\ \|y\|_{Z'} = r_* + 1). \tag{27}$$

To see this, suppose that on the contrary there exist $z \in Z'$ and λ such that $z = \lambda \widetilde{Q}_1 L^{-1}(z)$, $\|z\|_Y = r_* + 1$, and $0 \le \lambda \le 1$. Then, $x := L^{-1}(z) \in C$, and so $\widetilde{Q}_1(x) = Q_1(x)$. Therefore, by (26), $Q(x) := \widetilde{Q}_1(x) = Q_1(x)$ satisfies (21) and $z = \lambda Q(x)$. Applying the result of Step 2, we get $\|x\|_X \le r$ (see (22)). From (24) together with $Q_1(x) \in N_f(x)$ we obtain

$$\|z\|_{Z'} = \|\lambda Q_1(x)\|_{Z'} \le \|Q_1(x)\|_{Z'} \le r_* < \infty,$$

which contradicts the equality $\|z\|_{Z'} = r_* + 1$.

Step 6. Since the linear inverse operator $L^{-1} : Z' \to X$ is compact, the single-valued operator $A := \widetilde{Q}_1 L^{-1} : Z' \to Z'$ is clearly completely continuous. By (27) together with the classical Leray–Schauder principle (see e.g. [22, Theorem 42.1]) it follows that A has a fixed point z_* with $\|z_*\|_{Z'} < r_* + 1$, i.e. $z_* = A(z_*) = \widetilde{Q}_1 L^{-1}(z_*)$.

Using arguments analogous to those from Step 5, we get

$$\|z_*\|_{Z'} < r_* + 1, \qquad x_* := L^{-1}(z_*) \in C, \qquad z_* = \widetilde{Q}_1(x_*) = Q_1(x_*),$$

and therefore since $Q_1(x_*) \in N_f(x_*)$ we get $x_* \in L^{-1} N_f(x_*)$. Via Step 1 we deduce that x_* is a solution of the system (16). ∎

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