STUDIA MATHEMATICA 163 (1) (2004)

The Cauchy kernel for cones

by

SŁAWOMIR MICHALIK (Warszawa)

Abstract. A new representation of the Cauchy kernel $\mathcal{K}_\Gamma$ for an arbitrary acute convex cone Γ in $\mathbb{R}^n$ is found. The domain of holomorphy of $\mathcal{K}_\Gamma$ is described. An estimation of the growth of $\mathcal{K}_\Gamma$ near the singularities is given.

Introduction. The n-dimensional Cauchy kernel is a holomorphic function on $(\mathbb{C}\setminus\{0\})^n$ defined by

$$(\mathbb{C}\setminus\{0\})^n \ni (z_1,\ldots,z_n) \mapsto \frac{1}{z_1\ldots z_n}.$$

This function can be regarded as the Laplace transform of the characteristic function of $\overline{\mathbb{R}}^n_+$:

$$\frac{1}{z_1\ldots z_n} = (-1)^n \int\limits_{\overline{\mathbb{R}}^n_+} e^{\langle y,z\rangle}\,dy \quad \text{for } -\operatorname{Re} z \in \mathbb{R}^n_+ := (\mathbb{R}_+)^n.$$

Generalizing this formula, S. Bochner defined in [1] the Cauchy kernel $\mathcal{K}_\Gamma$ for an arbitrary cone $\Gamma \subset \mathbb{R}^n$ by

$$\mathcal{K}_\Gamma(z) := (-1)^n \int\limits_{\Gamma^*} e^{\langle y,z\rangle}\,dy \quad \text{for } \operatorname{Re} z \in -\operatorname{Int}(\Gamma^*), \tag{0.1}$$

where Γ^* is the dual cone to Γ. V. Vladimirov found and studied in [4]–[6] another representation of the Cauchy kernel,

$$\mathcal{K}_\Gamma(z) = (n-1)! \int\limits_{\Gamma^*\cap S^{n-1}} \frac{d\sigma}{\langle z,\sigma\rangle^n}, \tag{0.2}$$

and he proved that $\mathcal{K}_\Gamma$ has a holomorphic continuation to the set

$$\Omega_\Gamma := \bigcap_{\sigma\in\Gamma^*\cap S^{n-1}} \{z : \langle z,\sigma\rangle \neq 0\}.$$

2000 *Mathematics Subject Classification*: 32A26, 32D15.

Key words and phrases: Cauchy kernel, holomorphic continuation.

The paper was written during the author's work at the Institute of Mathematics, Polish Academy of Sciences.

Bochner's and Vladimirov's estimations of the domain of holomorphy of $\mathcal{K}_\Gamma$ are not optimal, because for the classical Cauchy kernel the set $(\mathbb{C} \setminus \{0\})^n$ is much larger than $\{\operatorname{Re} z \in -\mathbb{R}^n_+\}$ and $\Omega_{\overline{\mathbb{R}}^n_+}$. The aim of our paper is to give a better description of the domain of holomorphy of $\mathcal{K}_\Gamma$. To this end we make the following definitions. A set Γ is *fat* if $\Gamma \subset \overline{\operatorname{Int} \Gamma}$. A cone Γ is *acute* if $\operatorname{Int}(\Gamma^*) \neq \emptyset$. Moreover set

$$A(\Gamma) := \{a \in \partial\Gamma \cap S^{n-1} : \text{there exists a tangent space } V_a(\Gamma) \text{ to } \partial\Gamma \text{ at } a\},$$

$$H_{\mathbb{C}}(\Gamma) := \overline{\bigcup_{a \in A(\Gamma)} (V_a(\Gamma) + iV_a(\Gamma))}.$$

Now, we are ready to formulate the main result of the paper.

THEOREM 1. *Let Γ be an acute convex closed fat cone in $\mathbb{R}^n$. Then*:

(i) *The Cauchy kernel $\mathcal{K}_\Gamma$ has a holomorphic continuation to $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$.*

(ii) *For any $\alpha \in \mathbb{N}_0^n$ there exists a constant $C(n,\alpha) < \infty$, independent of Γ, such that for $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$,*

$$|D^\alpha \mathcal{K}_\Gamma(z)| \le C(n,\alpha)\mu_{n-1}(\Gamma \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n-|\alpha|}. \tag{0.3}$$

The proof is divided into three steps. We begin with the Cauchy kernel for simplicial cones. In this case, an explicit formula for $\mathcal{K}_\Gamma$ allows us to get (i) and (ii) immediately.

Next we prove that the Cauchy kernel for a polyhedral cone Γ is the sum of such kernels for simplicial cones in a simplicial division of Γ. This implies (i) for polyhedral cones (see Propositions 2.8 and 2.10). The inequality (0.3) for such cones is proved in Proposition 2.11.

In the last step, we prove that the Cauchy kernel for any acute convex closed fat cone is the limit of an almost uniformly convergent sequence of the Cauchy kernels for appropriate polyhedral cones. As a conclusion we obtain the assertions of Theorem 1 (see Propositions 3.2 and 3.3).

NOTATION. We use vector notation. In particular, if $a, b \in \mathbb{R}^n$ then $a < b$ means $a_i < b_i$ for $i = 1, \ldots, n$ and z^k means $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$ for $z \in \mathbb{C}^n$, $k \in \mathbb{N}_0^n$. If $z \in \mathbb{C}^n$, we write $\operatorname{Re} z := (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$ and $\operatorname{Im} z := (\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n)$.

Let $r \in \mathbb{R}$. We denote by $\mathbf{r}$ the n-tuple $(r, \ldots, r) \in \mathbb{R}^n$. If $\alpha \in \mathbb{N}_0^n$, we write $|\alpha| := |\alpha_1| + \ldots + |\alpha_n|$. The scalar product of $x, y \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is denoted by $\langle x, y \rangle := x_1 y_1 + \ldots + x_n y_n$.

We write D^α for the differential operator $\partial^\alpha / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. Let μ_{n-1} denote the Lebesgue measure on S^{n-1}.

A *cone* in $\mathbb{R}^n$ is a set $\Gamma \subset \mathbb{R}^n$ with the property that if $x \in \Gamma$, then $\lambda x \in \Gamma$ for all $\lambda > 0$. The *dual* cone to Γ is defined by

$$\Gamma^* := \{\xi : \langle \xi, y \rangle \ge 0 \text{ for every } y \in \Gamma\}.$$

For any cone $\varGamma$ in $\mathbb{R}^n$, set

$$H_{\mathbb{R}}(\varGamma) := \overline{\bigcup_{a\in A(\varGamma)} V_a(\varGamma)},$$

where $V_a(\varGamma)$ is the tangent space to $\partial\varGamma$ at a and the set $A(\varGamma)$ is defined in the previous section.

1. The Cauchy kernel for simplicial cones

DEFINITION 1.1. A cone $\varDelta \subset \mathbb{R}^n$ is called *simplicial* if

$$\varDelta := \{x \in \mathbb{R}^n : Ax \geq 0\} = \{x \in \mathbb{R}^n : (Ax)_j \geq 0 \text{ for } j = 1 \ldots n\} \tag{1.1}$$

for some non-singular matrix $A \in \mathrm{GL}(n, \mathbb{R})$.

Observe that the matrix A is unique modulo permutation of rows and multiplication of rows by positive constants.

It is easily seen that any simplicial cone can be described by

$$\varDelta(a^1, \ldots, a^n) := \{x \in \mathbb{R}^n : x = t_1 a^1 + \ldots + t_n a^n,\ t_i \geq 0,\ i = 1, \ldots, n\}$$

with linearly independent points $a^1, \ldots, a^n \in S^{n-1}$.

For a simplicial cone $\varDelta$ we have (see Exercise 8.7 in [3])

$$\mathcal{K}_{\varDelta}(z) = \frac{|\det A|}{(Az)^{\mathbf{1}}} \quad \text{with} \quad (Az)^{\mathbf{1}} := (Az)_1 \cdot \ldots \cdot (Az)_n. \tag{1.2}$$

The right-hand side of (1.2) does not depend on the choice of A in (1.1). By (1.2), $\mathcal{K}_{\varDelta}$ can be holomorphically continued to the set $\{z \in \mathbb{C}^n : (Az)_j \neq 0 \text{ for } j = 1, \ldots, n\}$. Note that

$$\mathbb{C}^n \setminus \bigcup_{j=1}^{n} \{z \in \mathbb{C}^n : (Az)_j = 0\} = \mathbb{C}^n \setminus H_{\mathbb{C}}(\varDelta).$$

By (1.2) we obtain

PROPOSITION 1.2. *Let $\varepsilon \in \{-1, 1\}^n$ and $\varDelta_\varepsilon := \{x \in \mathbb{R}^n : \varepsilon_i (Ax)_i \geq 0$ for $i = 1, \ldots, n\}$. Then*

$$\mathcal{K}_{\varDelta_\varepsilon}(z) = \operatorname{sgn}\varepsilon \cdot \mathcal{K}_{\varDelta}(z) \quad \textit{for } z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\varDelta) \tag{1.3}$$

with $\operatorname{sgn}\varepsilon := \operatorname{sgn}\varepsilon_1 \cdot \ldots \cdot \operatorname{sgn}\varepsilon_n$.

EXAMPLE 1.3. If we take $\varDelta := \overline{\mathbb{R}}^n_+$, we recover the classical Cauchy kernel

$$\mathcal{K}_{\overline{\mathbb{R}}^n_+}(z) = \frac{1}{z_1 \ldots z_n} \quad \text{for } z \in (\mathbb{C} \setminus \{0\})^n.$$

2. The Cauchy kernel for polyhedral cones

Definition 2.1. A set $W \subset \mathbb{R}^n$ is called a *polyhedral cone* if there exists $M \in \mathbb{N}$ such that

$$W = \Delta_1 \cup \ldots \cup \Delta_M,$$

where $\Delta_1, \ldots, \Delta_M$ are simplicial cones in $\mathbb{R}^n$ and $\mathrm{Int}(\Delta_i \cap \Delta_j) = \emptyset$ for $i \neq j,\ i, j = 1, \ldots, M$. The system $\mathcal{R} := \{\Delta_1, \ldots, \Delta_M\}$ is called a *simplicial division* of W.

Definition 2.2. Let W be a polyhedral cone in $\mathbb{R}^n$. By an $(n-1)$*-face* of W we mean a set $W^{n-1} := \partial W \cap V^{n-1}$ with non-empty interior in V^{n-1}, where V^{n-1} is some $(n-1)$-dimensional subspace of $\mathbb{R}^n$. By a k*-face* of W ($k = n-2, \ldots, 1$) we mean a set $W^k := \partial W^{k+1} \cap V^k$ with non-empty interior in V^k, where V^k is some k-dimensional subspace of $\mathbb{R}^n$. The set of k-faces of W is denoted by $\partial^{(k)} W$. A 1-face of W is called an *edge* of W.

For convenience we identify a point $a^i \in S^{n-1}$ with the edge $\Delta(a^i) = \{\lambda a^i : \lambda \geq 0\}$. Observe that every polyhedral convex cone W in $\mathbb{R}^n$ can be described by

$$W = W(a^1, \ldots, a^N) := \{x \in \mathbb{R}^n : x = t_1 a^1 + \ldots + t_N a^N \quad \text{for } t_i \geq 0,\ i = 1, \ldots, N\}$$

with edges $a^1, \ldots, a^N \in S^{n-1}$ and $N \geq n$.

Definition 2.3. Let W be a polyhedral cone in $\mathbb{R}^n$ with edges $a^1, \ldots, a^N$ ($n \leq N$). For any $a^i \in \partial^{(1)} W$ we set

$$\mathrm{Nb}_W(a^i) := \{a^j \in \{a^1, \ldots, a^N\} : \text{there exists } W^2 \in \partial^{(2)} W \text{ such that } \Delta(a^i, a^j) \subset W^2,\ \Delta(a^i, a^j) \cap \{a^1, \ldots, a^n\} = \{a^i, a^j\}\}.$$

Roughly, $\mathrm{Nb}_W(a^i)$ is the set of edges of W which are neighbours of a^i.

Lemma 2.4. *Let Γ_1, Γ_2 be closed cones in $\mathbb{R}^n$ such that $\Gamma_1 \cup \Gamma_2$ is a convex cone. Then $\Gamma_1^* \cup \Gamma_2^*$ is also a convex cone.*

The above lemma is well known, so we omit its proof.

Lemma 2.5. *Let W be a polyhedral cone in $\mathbb{R}^n$ with edges $a^1, \ldots, a^N$, where $N > n$. Then there exists an edge a^i such that*

$$\mathrm{Nb}_W(a^i) \neq \{a^1, \ldots, a^N\}. \tag{2.1}$$

Proof. Suppose that, on the contrary, $\mathrm{Nb}_W(a^i) = \{a^1, \ldots, a^N\}$ for every $i = 1, \ldots, N$ with $N > n$. Then $\Delta(a^{i_1}, \ldots, a^{i_k}) \in \partial^{(k)} W$ for $1 \leq i_1 < \ldots < i_k \leq N$ and $k < n$. In particular, every $(n-1)$-face of the simplicial cone $\Delta(a^1, \ldots, a^n)$ is simultaneously an $(n-1)$-face of W. Therefore

$$W(a^1, \ldots, a^N) = \Delta(a^1, \ldots, a^n)$$

and $N = n$. This contradicts our assumption that $N > n$. ∎

In the following the crucial role is played by the simplicial division constructed in the next lemma.

LEMMA 2.6. *Let $W := W(a^1,\dots,a^N)$ be an acute convex polyhedral cone in $\mathbb{R}^n$. Then there exists a simplicial division $\mathcal{R} = \{\Delta_j\}_{j=1}^K$ of W satisfying*:

(A) *If $\mathrm{Nb}_W(a^i) = \{a^1,\dots,a^N\}$ for some $i \in \{1,\dots,N\}$, then a^i is the edge of every simplicial cone Δ_j $(j=1,\dots,K)$.*

The simplicial division $\mathcal{R}$ satisfying (A) and constructed in the proof of this lemma will be called a *special division* of W. Observe that every edge of any simplicial cone in a special division $\mathcal{R}$ is also the edge of W. Conversely, every edge of W is also the edge of some simplicial cone in $\mathcal{R}$.

Proof. We construct $\mathcal{R}$ by induction on $N \geq n$.

1) For $N = n$, every cone $W(a^1,\dots,a^n)$ is a simplicial cone

$$W(a^1,\dots,a^n) = \Delta(a^1,\dots,a^n), \qquad \mathcal{R} = \{\Delta(a^1,\dots,a^n)\}$$

and obviously (A) holds.

2) Fix $k > n$ and suppose that for every N $(n \leq N < k)$ and every cone W_N with N edges $a^1,\dots,a^N$, there exists a simplicial division $\mathcal{R}$ of W_N satisfying (A). Consider any acute convex polyhedral cone W_k with k edges $a^1,\dots,a^k$. By Lemma 2.5 there exists an edge a^i satisfying (2.1). Put

$$W_{k-1} := W(a^1,\dots,\widehat{a^i},\dots,a^k) := W(a^1,\dots,a^{i-1},a^{i+1},\dots,a^k). \tag{2.2}$$

By the inductive assumption, there is a simplicial division $\{\Delta_m\}_{m=1}^{M'}$ of W_{k-1} satisfying (A). Set

$$W_{<k}^+ := \overline{W_k \setminus W_{k-1}}. \tag{2.3}$$

Notice that $W_{<k}^+$ is also a polyhedral cone in $\mathbb{R}^n$, with edges in $\mathrm{Nb}_{W_k}(a^i)$ and, by Lemma 2.5, the number of these edges is less than k. But generally $W_{<k}^+$ does not have to be convex. Let $1 \leq j_1 < \dots < j_l \leq k$ $(l < k)$ be indices different from i, satisfying

$$\mathrm{Nb}_{W_k}(a^i) = \{a^{j_1},\dots,a^{j_l},a^i\}, \tag{2.4}$$

and let

$$W_{<k}^- := W(a^{j_1},\dots,a^{j_l},-a^i). \tag{2.5}$$

Since

$$\mathrm{Nb}_{W_k}(a^i) = \mathrm{Nb}_{W_{<k}^+}(a^i) = \{a^{j_1},\dots,a^{j_l},a^i\},$$

we find that

$$\mathrm{Nb}_{W_{<k}^-}(-a^i) = \{a^{j_1},\dots,a^{j_l},-a^i\}.$$

The cone $W^-_{<k}$ has less than k edges. Hence, by the inductive assumption, we have a simplicial division $\{\Delta^-_m\}_{m=M'+1}^M$ of $W^-_{<k}$ satisfying (A), where

$$\Delta^-_m := \Delta(a^{j_1(m)}, \ldots, a^{j_{n-1}(m)}, -a^i), \tag{2.6}$$

with $\{a^{j_1(m)}, \ldots, a^{j_{n-1}(m)}\} \subset \{a^{j_1}, \ldots, a^{j_l}\}$ $(m = M'+1, \ldots, M)$. Define

$$\Delta_m := \Delta(a^{j_1(m)}, \ldots, a^{j_{n-1}(m)}, a^i) \quad \text{for } m = M'+1, \ldots, M. \tag{2.7}$$

Thus $\{\Delta_m\}_{m=M'+1}^M$ is a simplicial division of $W^+_{<k}$ and $\mathcal{R} := \{\Delta_m\}_{m=1}^M$ is a simplicial division of W_k.

A trivial verification shows that $\mathcal{R}$ satisfies condition (A). ∎

Lemma 2.7. *Let $W_k := W(a^1, \ldots, a^k)$ be an acute convex polyhedral cone in $\mathbb{R}^n$ $(k > n)$. Let W_{k-1}, $W^+_{<k}$, $W^-_{<k}$ be the cones defined by* (2.2)–(2.5). *Then*

$$W^-_{<k} \cap W^+_{<k} \subset W_{k-1} \subset W^-_{<k}. \tag{2.8}$$

Proof. Recall that, in the notation of the proof of Lemma 2.6,

$$W_{k-1} = W(a^1, \ldots, \widehat{a^i}, \ldots, a^k), \quad W^+_{<k} = \overline{W_k \setminus W_{k-1}},$$
$$\mathrm{Nb}_{W_k}(a^i) = \{a^{j_1}, \ldots, a^{j_l}, a^i\}, \quad W^-_{<k} = W(a^{j_1}, \ldots, a^{j_l}, -a^i).$$

Since $\mathrm{Nb}_{W^+_{<k}}(a^i) = \{a^{j_1}, \ldots, a^{j_l}, a^i\}$ and every edge of $W^+_{<k}$ belongs to $\mathrm{Nb}_{W^+_{<k}}(a^i)$, we have

$$W^+_{<k} \subset W(a^{j_1}, \ldots, a^{j_l}, a^i).$$

Therefore

$$W^+_{<k} \cap W^-_{<k} \subset W(a^{j_1}, \ldots, a^{j_l}, a^i) \cap W(a^{j_1}, \ldots, a^{j_l}, -a^i) = W(a^{j_1}, \ldots, a^{j_l}).$$

Clearly, $\{a^{j_1}, \ldots, a^{j_l}\} \subset \{a^1, \ldots, \widehat{a^i}, \ldots, a^k\}$. Hence

$$W(a^{j_1}, \ldots, a^{j_l}) \subset W(a^1, \ldots, \widehat{a^i}, \ldots, a^k) = W_{k-1},$$

and the left-hand inclusion in (2.8) holds.

To prove the right-hand inclusion, it is sufficient to show that $a^j \in W^-_{<k}$ for $j = 1, \ldots, k$, $j \neq i$. If $a^j \in \{a^{j_1}, \ldots, a^{j_l}\}$, this is trivial. Suppose that $a^j \notin \{a^{j_1}, \ldots, a^{j_l}\}$ (i.e. $a^j \notin \mathrm{Nb}_{W_k}(a^i)$). Then there exists $c > 0$ satisfying $a^j + ca^i \in W(a^{j_1}, \ldots, a^{j_l})$. Therefore $a^j \in W(a^{i_1}, \ldots, a^{i_l}, -a^i) = W^-_{<k}$. ∎

Proposition 2.8. *Let $W := W(a^1, \ldots, a^N)$ be an acute convex polyhedral cone in $\mathbb{R}^n$ and let $\mathcal{R} = \{\Delta_i\}_{i=1}^K$ be a special division of W. Then*

$$\mathcal{K}_W(z) = \sum_{i=1}^K \mathcal{K}_{\Delta_i}(z) \quad \textit{for } z \in \Omega_W \cap \bigcap_{j=1}^K (\mathbb{C}^n \setminus H_{\mathbb{C}}(\Delta_j)). \tag{2.9}$$

Furthermore, $\mathcal{K}_W$ has a holomorphic continuation to $\mathbb{C}^n \setminus H_{\mathbb{C}}(W)$.

Proof. We show (2.9) by induction on N.

1) For $N = n$, $W = \Delta(a^1, \dots, a^n)$, $\mathcal{R} = \{\Delta(a^1, \dots, a^n)\}$ and obviously (2.9) holds.

2) Fix $k > n$. Assuming (2.9) to hold for every N ($n \le N < k$) and every convex acute polyhedral cone W with N edges, we will prove it for any polyhedral cone W_k with k edges. Take W_{k-1}, $W^+_{<k}$, $W^-_{<k}$ as in the proof of Lemma 2.6. By Lemma 2.7 ,

$$W^-_{<k} \cap W^+_{<k} \subset W_{k-1} \subset W^-_{<k}.$$

Hence, by duality,

$$W^{-*}_{<k} \subset W^*_{k-1} \subset (W^-_{<k} \cap W^+_{<k})^*. \tag{2.10}$$

From Lemma 2.4 we conclude

$$(W^-_{<k} \cap W^+_{<k})^* = W^{-*}_{<k} \cup W^{+*}_{<k}. \tag{2.11}$$

Thus

$$W^*_{k-1} \subset W^{-*}_{<k} \cup W^{+*}_{<k}. \tag{2.12}$$

The cone $W^-_{<k} \cup W^+_{<k}$ is not acute, because it includes the line $a^i\mathbb{R}$. Therefore

$$\operatorname{Int}(W^{-*}_{<k} \cap W^{+*}_{<k}) \overset{(2.11)}{=} \operatorname{Int}(W^-_{<k} \cup W^+_{<k})^* = \emptyset. \tag{2.13}$$

Consequently,

$$\begin{aligned} W^*_k &\overset{(2.3)}{=} (W_{k-1} \cup W^+_{<k})^* = W^*_{k-1} \cap W^{+*}_{<k} \\ &= \overline{W^*_{k-1} \setminus (W^*_{k-1} \cap W^{-*}_{<k})}, \end{aligned} \tag{2.14}$$

where the last equality holds because

$$W^*_{k-1} \overset{(2.12)}{=} (W^*_{k-1} \cap W^{-*}_{<k}) \cup (W^*_{k-1} \cap W^{+*}_{<k})$$

and

$$\operatorname{Int}((W^*_{k-1} \cap W^{-*}_{<k}) \cap (W^*_{k-1} \cap W^{+*}_{<k})) \overset{(2.13)}{=} \emptyset.$$

Therefore

$$\begin{aligned} \mathcal{K}_{W_k}(z) &\overset{(0.2)}{=} (n-1)! \int\limits_{S^{n-1}\cap W^*_k} \frac{d\sigma}{\langle z, \sigma\rangle^n} \\ &\overset{(2.14)}{=} (n-1)! \int\limits_{S^{n-1}\cap W^*_{k-1}} \frac{d\sigma}{\langle z, \sigma\rangle^n} - (n-1)! \int\limits_{S^{n-1}\cap W^*_{k-1}\cap W^{-*}_{<k}} \frac{d\sigma}{\langle z, \sigma\rangle^n} \\ &\overset{(2.10)}{=} (n-1)! \int\limits_{S^{n-1}\cap W^*_{k-1}} \frac{d\sigma}{\langle z, \sigma\rangle^n} - (n-1)! \int\limits_{S^{n-1}\cap W^{-*}_{<k}} \frac{d\sigma}{\langle z, \sigma\rangle^n} \\ &\overset{(0.2)}{=} \mathcal{K}_{W_{k-1}}(z) - \mathcal{K}_{W^-_{<k}}(z). \end{aligned} \tag{2.15}$$

Let $\{\Delta_j\}_{j=1}^{M'}$ and $\{\Delta_j^-\}_{j=M'+1}^{M}$ be the special divisions of W_{k-1} and $W_{<k}^-$, constructed in the proof of Lemma 2.6. From the inductive assumption

$$\mathcal{K}_{W_{k-1}}(z)=\sum_{j=1}^{M'}\mathcal{K}_{\Delta_j}(z) \tag{2.16}$$

and

$$\mathcal{K}_{W_{<k}^-}(z)=\sum_{j=M'+1}^{M}\mathcal{K}_{\Delta_j^-}(z). \tag{2.17}$$

By Proposition 1.2 we obtain

$$\mathcal{K}_{\Delta_j^-}(z)=-\mathcal{K}_{\Delta_j}(z)\quad \text{for } j=M'+1,\dots,M \tag{2.18}$$

with Δ_j^- and Δ_j defined by (2.6) and (2.7). Finally,

$$\begin{aligned}\mathcal{K}_{W_k}(z)&\overset{(2.15)}{=}\mathcal{K}_{W_{k-1}}(z)-\mathcal{K}_{W_{<k}^-}(z)\overset{(2.16),(2.17)}{=}\sum_{j=1}^{M'}\mathcal{K}_{\Delta_j}(z)-\sum_{j=M'+1}^{M}\mathcal{K}_{\Delta_j^-}(z)\\&\overset{(2.18)}{=}\sum_{j=1}^{M'}\mathcal{K}_{\Delta_j}(z)+\sum_{j=M'+1}^{M}\mathcal{K}_{\Delta_j}(z)=\sum_{j=1}^{M}\mathcal{K}_{\Delta_j}(z).\end{aligned}$$

This means that (2.9) holds on $\Omega_W\cap\bigcap_{j=1}^{K}(\mathbb{C}^n\setminus H_{\mathbb{C}}(\Delta_j))$.

Now, we are in a position to prove that $\mathcal{K}_W$ can be holomorphically continued to the set $\mathbb{C}^n\setminus H_{\mathbb{C}}(W)$. Denote by $\{V_1,\dots,V_{m'}\}$ a set of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ satisfying $\mathrm{Int}_{V_l}(V_l\cap\partial W)\neq\emptyset$ for $l=1,\dots,m'$. A trivial verification shows that

$$\bigcup_{l=1}^{m'}(V_l+iV_l)=H_{\mathbb{C}}(W).$$

Similarly, let $\{V_1,\dots,V_{m'},V_{m'+1},\dots,V_m\}$ be a set of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ satisfying $\bigcup_{j=1}^{K}\mathrm{Int}_{V_l}(V_l\cap\partial\Delta_j)\neq\emptyset$ for $l=1,\dots,m$. Hence $\bigcup_{l=1}^{m}(V_l+iV_l)=\bigcup_{j=1}^{K}H_{\mathbb{C}}(\Delta_j)$. Clearly, $\sum_{j=1}^{K}\mathcal{K}_{\Delta_j}$ is a holomorphic function on

$$\bigcap_{j=1}^{K}(\mathbb{C}^n\setminus H_{\mathbb{C}}(\Delta_j))=\Big\{z\in\mathbb{C}^n: z\notin\bigcup_{l=1}^{m}(V_l+iV_l)\Big\}.$$

Therefore, using (2.9), we see that $\mathcal{K}_W$ can be holomorphically continued to this set. Put

$$V_l^-:=V_l\setminus\bigcup_{j=1}^{l-1}V_j\quad \text{for } l=m'+1,\dots,m.$$

We have

$$\bigcup_{l=m'+1}^{m} V_l^- = \Big(\bigcup_{l=m'+1}^{m} V_l\Big) \setminus \bigcup_{j=1}^{m'} V_j. \tag{2.19}$$

It is sufficient to show that $\mathcal{K}_W$ can be holomorphically continued to some open neighbourhood of $V_l^- + iV_l^-$ for $l = m'+1, \ldots, m$. Indeed, then $\mathcal{K}_W$ can be holomorphically continued to

$$\begin{aligned}\mathbb{C}^n \setminus H_{\mathbb{C}}(W) &= \Big\{z \in \mathbb{C}^n : z \notin \bigcup_{j=1}^{m}(V_j + iV_j)\Big\} \\ &\qquad \cup \Big\{z \in \mathbb{C}^n : z \in \bigcup_{l=m'+1}^{m} (V_l^- + iV_l^-)\Big\} \\ &\overset{(2.19)}{=} \Big\{z \in \mathbb{C}^n : z \notin \bigcup_{j=1}^{m'}(V_j + iV_j)\Big\}.\end{aligned}$$

In the first step we take $l := m$. Then there exists $(b_1, \ldots, b_n) \in \mathbb{R}^n \setminus \{0\}$ such that

$$V_m = \{x \in \mathbb{R}^n : b_1x_1 + \ldots + b_nx_n = 0\}.$$

Suppose that $b_n \neq 0$. Applying (1.2) we see that

$$\sum_{\Delta\in\mathcal{R}} \mathcal{K}_\Delta(z) = \frac{1}{b_1z_1 + \ldots + b_nz_n} \cdot \frac{w_m(z)}{v_m(z)}, \tag{2.20}$$

where w_m, v_m are polynomials with real coefficients such that $v_m(z) \neq 0$ for $z \in V_m^- + iV_m^-$. Write $U_m := (V_m^- \cap (W \cup -W)) + i(V_m^- \cap (W \cup -W))$. Clearly

$$b_1z_1 + \ldots + b_nz_n = 0 \quad \text{ for } z \in U_m.$$

Since $\mathcal{K}_W$ is holomorphic on Ω_W, it is also holomorphic on some open neighbourhood of the non-empty set U_m. Thus, by (2.20), w_m vanishes on U_m. Define a map $\varphi_m : \mathbb{C}^{n-1} \to V_m + iV_m$ by

$$\varphi_m(\widetilde{z}_1, \ldots, \widetilde{z}_{n-1}) := \Big(\widetilde{z}_1, \ldots, \widetilde{z}_{n-1}, -\frac{b_1}{b_n}\,\widetilde{z}_1 - \ldots - \frac{b_{n-1}}{b_n}\,\widetilde{z}_{n-1}\Big),$$

and $\widetilde{w}_m : \mathbb{C}^{n-1} \to \mathbb{C}$ by

$$\widetilde{w}_m(\widetilde{z}_1, \ldots, \widetilde{z}_{n-1}) := w_m(\varphi_m(\widetilde{z}_1, \ldots, \widetilde{z}_{n-1})).$$

We see that $\widetilde{w}_m$ is a polynomial which vanishes on an open set in $\mathbb{C}^{n-1}$. Thus $\widetilde{w}_m(t) \equiv 0$ on $\mathbb{C}^{n-1}$. This means that the polynomial w_m vanishes on $V_m + iV_m$. Therefore the right-hand side of (2.20) is a holomorphic function on some open neighbourhood of $V_m^- + iV_m^-$.

Next we repeat the procedure taking successively $l := m-1, m-2, \ldots$ $\ldots, l+1$. After $m - m'$ steps we conclude that $\mathcal{K}_W$ can be holomorphically continued to $\mathbb{C}^n \setminus H_{\mathbb{C}}(W)$. ∎

Proposition 2.9. *Let W be an acute convex polyhedral cone in $\mathbb{R}^n$ and let*

$$V := \{x \in \mathbb{R}^n : F(x) := b_1x_1 + \ldots + b_nx_n = 0\}$$

be an $(n-1)$-dimensional subspace in $\mathbb{R}^n$ dividing W into

$$W^1 := W \cap \{x \in \mathbb{R}^n : F(x) \geq 0\}, \qquad W^2 := W \cap \{x \in \mathbb{R}^n : F(x) \leq 0\}.$$

Then

$$\mathcal{K}_W(z) = \mathcal{K}_{W^1}(z) + \mathcal{K}_{W^2}(z) \text{ for } z \in (\mathbb{C}^n \setminus H_{\mathbb{C}}(W^1)) \cap (\mathbb{C}^n \setminus H_{\mathbb{C}}(W^2)). \tag{2.21}$$

Proof. We first construct a special division of W^1. To do this take edges $a^1, \ldots, a^M$ with $W = W(a^1, \ldots, a^N)$, $W^1 = W(a^1, \ldots, a^L, a^{N+1}, \ldots, a^M)$ and $W^2 = W(a^{K+1}, \ldots, a^M)$ for $1 \leq K \leq L < N \leq M$. In other words,

$$a^1, \ldots, a^K \in W^1 \setminus V, \qquad a^{K+1}, \ldots, a^L, a^{N+1}, \ldots, a^M \in W^1 \cap V. \tag{2.22}$$

Observe that for $K = 1$, by (2.22), a^1 is a unique edge of W^1 which does not belong to V. Hence we have

$$\mathrm{Nb}_{W^1}(a^1) = \{a^1, \ldots, a^L, a^{N+1}, \ldots, a^M\}. \tag{2.23}$$

Fix $K > 1$ and consider a cone $\widetilde{W} := W(a^1, \ldots, a^K, a^{i_1} \ldots, a^{i_{n-1}})$ with edges $a^{i_1}, \ldots, a^{i_{n-1}} \in \{a^{K+1}, \ldots, a^L, a^{N+1}, \ldots, a^M\}$ such that $\Delta(a^{i_1}, \ldots, a^{i_{n-1}})$ is a simplicial cone in V. Then $\{a^{i_1}, \ldots, a^{i_{n-1}}\} \subset \mathrm{Nb}_{\widetilde{W}}(a^{i_j})$ for $j = 1, \ldots, n-1$. Therefore Lemma 2.5 shows that there exists $j \in \{1, \ldots, K\}$ satisfying $\mathrm{Nb}_{\widetilde{W}}(a^j) \neq \{a^1, \ldots, a^K, a^{i_1}, \ldots, a^{i_{n-1}}\}$. Consequently,

$$\mathrm{Nb}_{W^1}(a^j) \neq \{a^1, \ldots, a^L, a^{N+1}, \ldots, a^M\}. \tag{2.24}$$

Without loss of generality we can assume that (2.24) is satisfied for $j = 1$. As in Lemma 2.6 we construct the division of W^1 with respect to a^1 into $W^{1+}_{<M'}$ ($M' := M + L - N$) and $W^1_{M'-1} = W(a^2, \ldots, a^L, a^{N+1}, \ldots, a^M)$. In the next step we replace W^1 by $W^1_{M'-1}$ and a^1 by a^2. After $K-1$ such steps we have the division of W^1 into $W^{1+}_{<M'}, \ldots, W^{1+}_{<M'-K+2}$ and $W^1_{M'-K+1} = W(a^K, \ldots, a^L, a^{N+1}, \ldots, a^M)$. By Lemma 2.7, as in the proof of Proposition 2.8, we have

$$\mathcal{K}_{W^1}(z) \overset{(2.15)}{=} \mathcal{K}_{W^1_{M'-K+1}}(z) - \mathcal{K}_{W^{1-}_{<M'-K+2}}(z) - \ldots - \mathcal{K}_{W^{1-}_{<M'}}(z). \tag{2.25}$$

We next construct a division of W similar to the above. Denote by $W' := W(a^1, \ldots, a^M)$ the cone $W = W^1 \cup W^2$ with edges $a^1, \ldots, a^N$ and *pseudo-edges* $a^{N+1}, \ldots, a^M$. We extend Definition 2.3 by writing $\mathrm{Nb}_{W'}(a^i)$ for the set of edges and pseudo-edges of W' which are neighbours of the edge a^i ($i = 1, \ldots, N$). By (2.22) we have

$$\mathrm{Nb}_{W^1}(a^i) = \mathrm{Nb}_{W'}(a^i) \quad \text{for } i = 1, \ldots, K. \tag{2.26}$$

As in Lemma 2.6 we divide W' with respect to a^1 into $W'^{+}_{<M} \overset{(2.26)}{=} W^{1+}_{<M'}$ and $W'_{M-1} \overset{(2.26)}{=} W^1_{M'-1} \cup W^2 = W(a^2, \ldots, a^M)$. Observe that inclusions (2.8) in Lemma 2.7 are satisfied for W'_{M-1}, $W'^{+}_{<M}$ and $W'^{-}_{<M}$. Hence, as in the proof of Proposition 2.8,

$$\mathcal{K}_{W'}(z) \overset{(2.15)}{=} \mathcal{K}_{W'_{M-1}}(z) - \mathcal{K}_{W'^{-}_{<M}}(z).$$

After $K-1$ such steps we have the division of W' into $W^{1+}_{<M'}, \ldots, W^{1+}_{<M'-K+2}$ and $W'_{M-K+1} = W(a^K, \ldots, a^M)$, which satisfies

$$\mathcal{K}_{W'}(z) \overset{(2.15)}{=} \mathcal{K}_{W'_{M-K+1}}(z) - \mathcal{K}_{W^{1-}_{<M'-K+2}}(z) - \ldots - \mathcal{K}_{W^{1-}_{<M'}}(z). \tag{2.27}$$

In the last step, by (2.23) we divide W'_{M-K+1} into $W'^{+}_{<M-K+1} = W^1_{M'-K+1}$ and $W'_{M-K} = W^2$ such that

$$\mathcal{K}_{W'_{M-K+1}}(z) \overset{(2.15)}{=} \mathcal{K}_{W'_{M-K}}(z) - \mathcal{K}_{W'^{-}_{<M-K+1}}(z).$$

By convexity of $W'^{+}_{<M-K+1}$ we obtain

$$\begin{aligned} \mathcal{K}_{W'_{M-K+1}}(z) &= \mathcal{K}_{W'_{M-K}}(z) + \mathcal{K}_{W'^{+}_{<M-K+1}}(z) \\ &= \mathcal{K}_{W^2}(z) + \mathcal{K}_{W^1_{M'-K+1}}(z). \end{aligned} \tag{2.28}$$

Finally, by (2.25), (2.27), (2.28) we have

$$\mathcal{K}_W(z) = \mathcal{K}_{W'}(z) = \mathcal{K}_{W^1}(z) + \mathcal{K}_{W^2}(z). \ \blacksquare$$

Applying Proposition 2.9, one can improve Proposition 2.8.

PROPOSITION 2.10. *Let W be an acute convex polyhedral cone in $\mathbb{R}^n$ and $\mathcal{R} = \{\Delta_j\}_{j=1}^K$ be any simplicial division of W. Then*

$$\mathcal{K}_W(z) = \sum_{j=1}^K \mathcal{K}_{\Delta_j}(z) \quad \text{for } z \in \Omega_W \cap \bigcap_{j=1}^K (\mathbb{C}^n \setminus H_{\mathbb{C}}(\Delta_j))$$

and $\mathcal{K}_W$ can be holomorphically continued to $\mathbb{C}^n \setminus H_{\mathbb{C}}(W)$.

Proof. Fix any simplicial division $\{\Delta_j\}_{j=1}^M$ of W. Observe that, according to Proposition 2.9, every set $\{V_1, \ldots, V_k\}$ of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ meeting $\operatorname{Int} W$ divides W into closed and convex polyhedral cones $\mathcal{R}_W := \{W_1, \ldots, W_L\}$ satisfying $W = W_1 \cup \ldots \cup W_L$ and $\operatorname{Int}(W_i \cap W_j) = \emptyset$ for $i \neq j$. Thus, by Proposition 2.9,

$$\mathcal{K}_W(z) = \sum_{j=1}^L \mathcal{K}_{W_j}(z). \tag{2.29}$$

In particular, we can assume that $\{W_j\}_{j=1}^L$ is a subdivision of $\{\Delta_j\}_{j=1}^M$. This means that we can divide $\mathcal{R}_W$ into $\mathcal{R}_W^1, \ldots, \mathcal{R}_W^M$ such that

$$\Delta_j = \bigcup_{W' \in \mathcal{R}_W^j} W' \quad \text{for } j = 1, \ldots, M.$$

Again applying Proposition 2.9, we have

$$\mathcal{K}_{\Delta_j}(z) = \sum_{W' \in \mathcal{R}_W^j} \mathcal{K}_{W'}(z) \quad \text{for } j = 1, \ldots, M. \tag{2.30}$$

We conclude that

$$\mathcal{K}_W(z) \overset{(2.29)}{=} \sum_{W' \in \mathcal{R}_W} \mathcal{K}_{W'}(z) = \sum_{j=1}^M \sum_{W' \in \mathcal{R}_W^j} \mathcal{K}_{W'}(z) \overset{(2.30)}{=} \sum_{j=1}^M \mathcal{K}_{\Delta_j}(z). \blacksquare$$

The next proposition describes the behaviour of $\mathcal{K}_W$ near the set of singularities.

PROPOSITION 2.11. *Let W be an acute convex polyhedral cone in $\mathbb{R}^n$. Then for every $\alpha \in \mathbb{N}_0^n$ there exists $C(n,\alpha) < \infty$ such that*

$$|D^\alpha \mathcal{K}_W(z)| \le C(n,\alpha) \mu_{n-1}(W \cap S^{n-1}) (\operatorname{dist}(z, H_{\mathbb{C}}(W)))^{-n-|\alpha|}. \tag{2.31}$$

for $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(W)$.

Proof. We have (see [6, p. 159])

$$\operatorname{dist}(z, H_{\mathbb{C}}(W)) = \inf_{\sigma \in W^* \cap S^{n-1}} \langle z, \sigma \rangle \quad \text{for } z \in W + iW. \tag{2.32}$$

Fix $\alpha \in \mathbb{N}_0^n$. Then for $z \in \operatorname{Int}(W + iW)$ we obtain

$$\begin{aligned} |D^\alpha \mathcal{K}_W(z)| &\overset{(0.2)}{=} (n + |\alpha| - 1)! \Big| \int_{W^* \cap S^{n-1}} \frac{\sigma^\alpha d\sigma}{\langle z, \sigma \rangle^{n+|\alpha|}} \Big| \\ &\le (n + |\alpha| - 1)! \mu_{n-1}(W^* \cap S^{n-1}) \sup_{\sigma \in W^* \cap S^{n-1}} \langle z, \sigma \rangle^{-n-|\alpha|} \\ &\overset{(2.32)}{=} (n + |\alpha| - 1)! \mu_{n-1}(W^* \cap S^{n-1}) (\operatorname{dist}(z, H_{\mathbb{C}}(W)))^{-n-|\alpha|}. \end{aligned} \tag{2.33}$$

Consider a simplicial cone Δ and a matrix A corresponding to Δ. Fix $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\Delta)$. Then there exists $\widetilde{z} \in \operatorname{Int}(\Delta + i\Delta)$ such that

$$A(\operatorname{Re} \widetilde{z})_j = |A(\operatorname{Re} z)_j|, \quad A(\operatorname{Im} \widetilde{z})_j = |A(\operatorname{Im} z)_j| \tag{2.34}$$

for $j = 1, \ldots, n$. So

$$\operatorname{dist}(z, H_{\mathbb{C}}(\Delta)) = \operatorname{dist}(\widetilde{z}, H_{\mathbb{C}}(\Delta)). \tag{2.35}$$

Estimating the Cauchy kernel we obtain

$$(2.36)\qquad |D^\alpha \mathcal{K}_\Delta(z)| \overset{(1.2)}{=} \left| D^\alpha \frac{|\det A|}{(Az)^{\mathbf{1}}} \right| \overset{(2.34)}{=} \left| D^\alpha \frac{|\det A|}{(A\widetilde{z})^{\mathbf{1}}} \right| \overset{(1.2)}{=} |D^\alpha \mathcal{K}_\Delta(\widetilde{z})|$$
$$\overset{(2.33)}{\le} (n+|\alpha|-1)!\mu_{n-1}(\Delta^* \cap S^{n-1})(\operatorname{dist}(\widetilde{z}, H_{\mathbb{C}}(\Delta)))^{-n-|\alpha|}$$
$$\overset{(2.35)}{=} (n+|\alpha|-1)!\mu_{n-1}(\Delta^* \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(\Delta)))^{-n-|\alpha|}.$$

Therefore

$$(2.37)\qquad |D^\alpha \mathcal{K}_\Delta(z)| \overset{(1.3)}{=} \min_{\varepsilon\in\{-1,1\}^n} |D^\alpha \mathcal{K}_{\Delta_\varepsilon}(z)|$$
$$\overset{(2.36)}{\le} \min_{\varepsilon\in\{-1,1\}^n} \mu_{n-1}(\Delta^*_\varepsilon \cap S^{n-1})(n+|\alpha|-1)!(\operatorname{dist}(z, H_{\mathbb{C}}(\Delta)))^{-n-|\alpha|}.$$

Observe that every edge of the simplicial cone Δ is perpendicular to some $(n-1)$-dimensional subspace of $\mathbb{R}^n$ containing an $(n-1)$-face of Δ^*. On the other hand, every $(n-1)$-dimensional subspace containing an $(n-1)$-face of Δ is perpendicular to some edge of Δ^*. This means that all angles between appropriate subspaces from $H_{\mathbb{R}}(\Delta)$ and $H_{\mathbb{R}}(\Delta^*)$ are preserved. Hence

$$(2.38)\qquad \min_{\varepsilon\in\{-1,1\}^n} \mu_{n-1}(\Delta^*_\varepsilon \cap S^{n-1}) = \min_{\varepsilon\in\{-1,1\}^n} \mu_{n-1}(\Delta_\varepsilon \cap S^{n-1}).$$

Finally, we have

$$(2.39)\qquad |D^\alpha \mathcal{K}_\Delta(z)|$$
$$\overset{(2.37)}{\le} (n+|\alpha|-1)! \min_{\varepsilon\in\{-1,1\}^n} \mu_{n-1}(\Delta^*_\varepsilon \cap S^{n-1})(\operatorname{dist}(z, H(\Delta)))^{-n-|\alpha|}$$
$$\overset{(2.38)}{=} (n+|\alpha|-1)! \min_{\varepsilon\in\{-1,1\}^n} \mu_{n-1}(\Delta_\varepsilon \cap S^{n-1})(\operatorname{dist}(z, H(\Delta)))^{-n-|\alpha|}$$
$$\le (n+|\alpha|-1)!\mu_{n-1}(\Delta \cap S^{n-1})(\operatorname{dist}(z, H(\Delta)))^{-n-|\alpha|}.$$

So, we see that (2.31) holds for simplicial cones.

Consider an acute convex polyhedral cone W and fix $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(W)$. Suppose that $\{\Delta_i\}_{i=1}^m$ is a simplicial division of W such that $z \notin \bigcup_{j=1}^m H_{\mathbb{C}}(\Delta_j)$. By Proposition 2.10,

$$(2.40)\qquad |D^\alpha \mathcal{K}_W(z)| \le \sum_{j=1}^m |D^\alpha \mathcal{K}_{\Delta_j}(z)|$$
$$\overset{(2.39)}{\le} \sum_{j=1}^m (n+|\alpha|-1)!\mu_{n-1}(\Delta_j \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(\Delta_j)))^{-n-|\alpha|}$$
$$\le (n+|\alpha|-1)!\mu_{n-1}(W \cap S^{n-1})\Big(\operatorname{dist}\Big(z, \bigcup_{j=1}^m H_{\mathbb{C}}(\Delta_i)\Big)\Big)^{-n-|\alpha|}.$$

The proof will be completed by showing that there exists $C < \infty$ (independent of W and z) such that for every $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\Delta)$ we can find a simplicial division $\{\Delta_j\}_{j=1}^m$ of W satisfying

$$\operatorname{dist}(z, H_{\mathbb{C}}(W)) \leq C \operatorname{dist}\Big(z, \bigcup_{j=1}^m H_{\mathbb{C}}(\Delta_j)\Big). \tag{2.41}$$

Fix $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(W)$. Let $V_z \subset H_{\mathbb{R}}(W)$ be an $(n-1)$-dimensional subspace in $\mathbb{R}^n$ with the property that

$$\operatorname{dist}(z, H_{\mathbb{C}}(W)) = \operatorname{dist}(z, V_z + iV_z). \tag{2.42}$$

Without loss of generality we can assume, that

$$\operatorname{dist}(\operatorname{Im} z, V_z) \leq \operatorname{dist}(\operatorname{Re} z, V_z). \tag{2.43}$$

If $\operatorname{Re} z \notin \operatorname{Int}(W \cup -W)$ then we can find a simplicial division $\{\Delta_j\}_{j=1}^m$ of W (depending on $\operatorname{Re} z$) satisfying

$$\operatorname{dist}(\operatorname{Re} z, V_z) = \operatorname{dist}\Big(\operatorname{Re} z, \bigcup_{j=1}^m H_{\mathbb{R}}(\Delta_j)\Big).$$

Then

$$\begin{aligned}\operatorname{dist}(z, H_{\mathbb{C}}(W)) &\overset{(2.42)}{=} \operatorname{dist}(z, V_z + iV_z) \overset{(2.43)}{\leq} 2\operatorname{dist}(\operatorname{Re} z, V_z) \\ &\leq 2\operatorname{dist}\Big(\operatorname{Re} z, \bigcup_{j=1}^m H_{\mathbb{R}}(\Delta_j)\Big) \leq 2\operatorname{dist}\Big(z, \bigcup_{j=1}^m H_{\mathbb{C}}(\Delta_j)\Big)\end{aligned}$$

and (2.41) holds.

Assume now that $\operatorname{Re} z \in \operatorname{Int} W$ (for $\operatorname{Re} z \in \operatorname{Int}(-W)$ the proof is similar). Let

$$r := \operatorname{dist}(\operatorname{Re} z, V_z) = \operatorname{dist}(\operatorname{Re} z, \partial W)$$

and let

$$\Gamma(r) := \{x \in \mathbb{R}^n : x = \lambda\sigma \text{ for } \lambda \geq 0,\ \sigma \in S^{n-1},\ \operatorname{dist}(\sigma, \operatorname{Re} z) \leq r\}$$

be a light cone, which is contained in W. Denote by $\widetilde{\Delta}_1 \subset \Gamma(r)$ a simplicial cone such that the distance $\widetilde{r} := \operatorname{dist}(\operatorname{Re} z, \partial\widetilde{\Delta}_1)$ is maximal and $\operatorname{Re} z \in \widetilde{\Delta}_1$. The fraction $r/\widetilde{r}$ is finite and only depends on the dimension of the space (is independent of W and z). Furthermore, let $\{\widetilde{\Delta}_i\}_{i=2}^k$ be a simplicial division of $W \setminus \operatorname{Int} \widetilde{\Delta}_1$ such that

$$\operatorname{dist}(\operatorname{Re} z, H_{\mathbb{R}}(\widetilde{\Delta}_1)) \leq \operatorname{dist}(\operatorname{Re} z, H_{\mathbb{R}}(\widetilde{\Delta}_j)) \quad \text{for } j = 2, \ldots, m.$$

Hence $\{\widetilde{\Delta}_j\}_{j=1}^m$ is a simplicial division of W satisfying

$$\operatorname{dist}(\operatorname{Re} z, V_z) \leq D(n) \operatorname{dist}\Big(\operatorname{Re} z, \bigcup_{j=1}^m H_{\mathbb{R}}(\widetilde{\Delta}_j)\Big). \tag{2.44}$$

Therefore

$$\begin{aligned}\operatorname{dist}(z, H_{\mathbb{C}}(W)) &\overset{(2.42)}{=} \operatorname{dist}(z, V_z + iV_z) \overset{(2.43)}{\leq} 2\operatorname{dist}(\operatorname{Re} z, V_z)\\ &\overset{(2.44)}{\leq} 2D(n)\operatorname{dist}\Big(\operatorname{Re} z, \bigcup_{j=1}^{m} H_{\mathbb{R}}(\widetilde{\Delta}_j)\Big) \leq 2D(n)\operatorname{dist}\Big(z, \bigcup_{j=1}^{m} H_{\mathbb{C}}(\widetilde{\Delta}_j)\Big)\end{aligned}$$

and (2.41) also holds. ∎

3. The Cauchy kernel for acute convex cones. In this section we will consider acute closed convex fat cones in $\mathbb{R}^n$. Recall

RADEMACHER'S THEOREM (see Theorem 3.2 in [2]). *Every Lipschitz function $f : U \to \mathbb{R}$, with $U \subset \mathbb{R}^n$ open, is differentiable almost everywhere (in the sense of Lebesgue measure).*

After an appropriate rotation, the boundary of every convex closed fat cone in $\mathbb{R}^n$ is locally the graph of a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. Therefore, by the above theorem we have

COROLLARY 3.1. *Let Γ be an acute convex closed fat cone in $\mathbb{R}^n$. Then for almost all $a \in \partial\Gamma$ there exists a tangent space to $\partial\Gamma$ at a, denoted by $V_a(\Gamma)$.*

PROPOSITION 3.2. *Let Γ be an acute convex closed fat cone in $\mathbb{R}^n$. Then:*

(i) *There exists a sequence $\{W_k\}_{k=1}^{\infty}$ of acute convex polyhedral cones such that for $k \to \infty$, $\mathcal{K}_{W_k}$ converges almost uniformly in Ω_Γ to $\mathcal{K}_\Gamma$.*

(ii) *$\mathcal{K}_\Gamma$ can be holomorphically continued to $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$.*

Proof. If Γ is a polyhedral cone, then we can take $W_k := \Gamma$ for $k = 1, 2, \ldots$

Assume that Γ is not a polyhedral cone. We will construct a sequence $\{W_k\}_{k=1}^{\infty}$ of polyhedral cones in the following way:

1. Let W_1 be a simplicial cone containing Γ with the following property: every $(n-1)$-face of W_1 is contained in some tangent space $V_a(\Gamma)$. Since n distinct $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ define a simplicial cone, W_1 exists. Set $A_\Gamma^1 := A_\Gamma \cap \partial W_1$.

2. Assume that we have defined $W_1, \ldots, W_{k-1}$ and $A_\Gamma^1, \ldots, A_\Gamma^{k-1}$. Set $A_\Gamma^{k-1} := A_\Gamma \cap \partial W_{k-1}$ and $F := \partial\Gamma \cap S^{n-1}$. Because F is compact and Γ is not a polyhedral cone, there exists $x_k \in F$ such that

$$\operatorname{dist}(x_k, A_\Gamma^{k-1}) = \sup_{x \in F} \operatorname{dist}(x, A_\Gamma^{k-1}) > 0.$$

By Corollary 3.1, A_Γ is dense in F. Hence, we can choose a point $a_k \in A_\Gamma$ such that

$$\operatorname{dist}(a_k, A_\Gamma^{k-1}) > \frac{9}{10}\operatorname{dist}(x_k, A_\Gamma^{k-1}). \tag{3.1}$$

The tangent space $V_{a_k}(\Gamma)$ divides the cone W_{k-1}, as in Proposition 2.9, into cones: W_{k-1}^1 and W_{k-1}^2. We can assume that $\Gamma \subset W_{k-1}^1$. We obtain a polyhedral cone $W_k := W_{k-1}^1$. Define $A_\Gamma^k := A_\Gamma \cap W_k$. From the construction of W_k, we have

$$A_\Gamma^{k-1} \subset A_\Gamma^k, \qquad H_{\mathbb{C}}(W_k) = H_{\mathbb{C}}(W_{k-1}) \cup (V_{a_k}(\Gamma) + iV_{a_k}(\Gamma))$$

and $\mathcal{K}_{W_k}$ is a holomorphic function on $\mathbb{C}^n \setminus H_{\mathbb{C}}(W_k)$.

Observe that the sequence $\{W_k\}_{k=1}^\infty$ of acute convex polyhedral cones constructed above satisfies

$$W_1 \supset W_2 \supset \ldots \supset \Gamma, \qquad A_\Gamma^1 \subset A_\Gamma^2 \subset \ldots \subset A_\Gamma, \tag{3.2}$$

$$\mathbb{C}^n \setminus H_{\mathbb{C}}(W_1) \supset \mathbb{C}^n \setminus H_{\mathbb{C}}(W_2) \supset \ldots \supset \mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma). \tag{3.3}$$

We show that $A_\Gamma^\infty := \bigcup_{k=1}^\infty A_\Gamma^k$ is dense in A_Γ. Assume that, on the contrary, there exist $\varepsilon > 0$ and $x \in A_\Gamma$ satisfying $\operatorname{dist}(x, A_\Gamma^\infty) > 10\varepsilon/9$. By the choice of $\{a_k\}_{k=2}^\infty$ we get

$$\operatorname{dist}(a_m, a_l) \ge \operatorname{dist}(a_m, A_\Gamma^{m-1}) \overset{(3.1)}{\ge} \varepsilon \quad \text{for } l = 2, 3, \ldots \text{ and } m > l.$$

This leads to a contradiction, because all points $\{a_k\}_{k=1}^\infty$ are contained in a compact sphere S^{n-1}.

Thus A_Γ^∞ is dense in A_Γ and by Corollary 3.1, A_Γ^∞ is also dense in $\partial\Gamma \cap S^{n-1}$. This means that the cones $\{W_k\}_{k=1}^\infty$ satisfy $\bigcap_{k=1}^\infty W_k = \Gamma$ and

$$\mu_{n-1}((W_k \setminus \Gamma) \cap S^{n-1}) \to 0 \quad \text{as } k \to \infty, \tag{3.4}$$

where μ_{n-1} is the Lebesgue measure on S^{n-1}. We also have

$$\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma) = \operatorname{Int}\Big(\bigcap_{k=1}^\infty (\mathbb{C}^n \setminus H_{\mathbb{C}}(W_k))\Big).$$

Hence, passing to the dual cones, we have

$$W_1^* \subset W_2^* \subset \ldots \subset \Gamma^*, \qquad \Gamma^* = \bigcup_{k=1}^\infty W_k^*.$$

Thus

$$\mu_{n-1}((\Gamma^* \setminus W_k^*) \cap S^{n-1}) \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

Now we show that the sequence of holomorphic functions $\mathcal{K}_{W_k}$ is almost uniformly convergent in $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$. Fix $\varepsilon > 0$ and $K \Subset \mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$. It is sufficient to show that there exists $m \in \mathbb{N}$ such that for every $k > l > m$,

$$\sup_{z \in K} |\mathcal{K}_{W_l}(z) - \mathcal{K}_{W_k}(z)| \le \varepsilon. \tag{3.6}$$

By (3.3) the functions $\mathcal{K}_{W_l}$ and $\mathcal{K}_{W_k}$ are holomorphic on K. Take any $z \in K$.

Combining Proposition 2.9 with the construction of the cones $\{W_k\}_{k\in\mathbb{N}}$ we have

$$\mathcal{K}_{W_l}(z) - \mathcal{K}_{W_k}(z) = \sum_{j=l}^{k-1} \mathcal{K}_{W_j^2}(z). \tag{3.7}$$

By Proposition 2.11 for fixed $j \in \{1, \dots, k-1\}$ we obtain

$$|\mathcal{K}_{W_j^2}(z)| \le C(n)\mu_{n-1}(W_j^2 \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(W_j^2)))^{-n}. \tag{3.8}$$

Since $H_{\mathbb{C}}(W_j^2) \subset H_{\mathbb{C}}(\Gamma)$, we deduce that

$$(\operatorname{dist}(z, H_{\mathbb{C}}(W_j^2)))^{-n} \le (\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n}. \tag{3.9}$$

As K is compact in $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$, there exists $M < \infty$ such that

$$\sup_{z\in K} (\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n} \le M. \tag{3.10}$$

Furthermore, we have

$$\sum_{j=l}^{k-1} \mu_{n-1}(W_j^2 \cap S^{n-1}) = \mu_{n-1}(W_l \cap S^{n-1}) - \mu_{n-1}(W_k \cap S^{n-1}). \tag{3.11}$$

Combining (3.4) and (3.2) we find m such that for every $k > l > m$,

$$\mu_{n-1}(W_l \cap S^{n-1}) - \mu_{n-1}(W_k \cap S^{n-1}) \le \frac{\varepsilon}{C(n)M}. \tag{3.12}$$

Finally,

$$\begin{aligned}
\sup_{z\in K} |\mathcal{K}_{W_l}(z) - \mathcal{K}_{W_k}(z)| &\overset{(3.7)}{=} \sup_{z\in K} \Big|\sum_{j=l}^{k-1} \mathcal{K}_{W_j^2}(z)\Big| \le \sup_{z\in K} \sum_{j=l}^{k-1} |\mathcal{K}_{W_j^2}(z)| \\
&\overset{(3.8)}{\le} \sup_{z\in K} \sum_{j=l}^{k-1} C(n)\mu_{n-1}(W_j^2 \cap S^{n-1}) \cdot (\operatorname{dist}(z, H_{\mathbb{C}}(W_j^2)))^{-n} \\
&\overset{(3.9)}{\le} C(n) \sup_{z\in K} \Big(\sum_{j=l}^{k-1} \mu_{n-1}(W_j^2 \cap S^{n-1})\Big) \cdot (\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n} \\
&\overset{(3.10),(3.11)}{\le} C(n)(\mu_{n-1}(W_l \cap S^{n-1}) - \mu_{n-1}(W_k \cap S^{n-1})) \cdot M \\
&\overset{(3.12)}{<} C(n)\frac{\varepsilon}{C(n)M} \cdot M = \varepsilon,
\end{aligned}$$

that is, (3.6) holds.

So, we can define

$$F(z) := \lim_{k\to\infty} \mathcal{K}_{W_k}(z) \quad \text{for } z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma). \tag{3.13}$$

By almost uniform convergence, F is holomorphic on $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$. On the other hand,

$$\begin{aligned} F(z) &\overset{(3.13)}{=} (n-1)! \lim_{k\to\infty} \int\limits_{W_k^* \cap S^{n-1}} \frac{d\sigma}{\langle z, \sigma\rangle^n} \\ &\overset{(3.5)}{=} (n-1)! \int\limits_{\Gamma^* \cap S^{n-1}} \frac{d\sigma}{\langle z, \sigma\rangle^n} = \mathcal{K}_\Gamma(z) \end{aligned}$$

on Ω_Γ. Thus the Cauchy kernel $\mathcal{K}_\Gamma$ has a holomorphic continuation to $\mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$. ∎

Proposition 3.3. *Let Γ be an acute convex closed fat cone in $\mathbb{R}^n$. Then for any $\alpha \in \mathbb{N}_0^n$ there exists $C(n,\alpha) < \infty$ such that for $z \in \mathbb{C}^n \setminus H_{\mathbb{C}}(\Gamma)$,*

$$|D^\alpha \mathcal{K}_\Gamma(z)| \le C(n,\alpha)\mu_{n-1}(\Gamma \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n-|\alpha|}. \tag{3.14}$$

Proof. By Proposition 3.2 for any acute convex closed fat cone Γ there exists a sequence of acute convex polyhedral cones such that

$$\begin{aligned} |D^\alpha \mathcal{K}_\Gamma(z)| & \\ &\overset{(3.13)}{=} \lim_{k\to\infty} |D^\alpha \mathcal{K}_{W_k}(z)| \\ &\overset{(2.31)}{\le} \lim_{k\to\infty} C(n,\alpha)\mu_{n-1}(W_k \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(W_k)))^{-n-|\alpha|} \\ &\overset{(3.4),(3.3)}{\le} C(n,\alpha)\mu_{n-1}(\Gamma \cap S^{n-1})(\operatorname{dist}(z, H_{\mathbb{C}}(\Gamma)))^{-n-|\alpha|}. \ \blacksquare \end{aligned} \tag{3.15}$$

Acknowledgments. I would like to thank my teacher G. Łysik for fruitful discussions, careful reading of the manuscript and many helpful suggestions.

References

[1] S. Bochner, *Group invariance of Cauchy's formula in several variables*, Ann. of Math. 45 (1944), 687–707.

[2] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Stud. Adv. Math., CRC Press, 1992.

[3] Z. Szmydt and B. Ziemian, *The Mellin Transformation and Fuchsian Type Partial Differential Equations*, Kluwer, Dordrecht, 1992.

[4] V. S. Vladimirov, *A generalization of the Cauchy–Bochner integral representation*, Math. USSR-Izv. 3 (1969), 87–104.

[5] —, *On Cauchy–Bochner representation*, ibid. 6 (1972), 529–535.

[6] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.

Faculty of Mathematics and Natural Sciences
College of Science
Cardinal Stefan Wyszyński University
Dewajtis 5
01-815 Warszawa, Poland
E-mail: slawek@impan.gov.pl

Received March 28, 2002
Revised version October 14, 2002 (4919)