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Bergman completeness of Zalcman type domains

by

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Abstract. We give an equivalent condition for Bergman completeness of Zalcman type domains. This also solves a problem stated by Pflug.

The main subject of this paper is a class of planar domains—the so-called Zalcman type domains. We give an equivalent condition for the Bergman completeness of a wide class of such domains. This answers a question raised in [10]. Moreover, this gives a rich collection of domains which are Bergman complete but not Bergman exhaustive, i.e. which are counterexamples to Kobayashi's conjecture (see [7]).

To begin with, let us recall some necessary notions and properties connected with potential theory in the complex plane (see e.g. [12]).

Let $\mathcal{P}(K)$ be the set of all probability Borel measures μ with supports in a compact set $K\subset\mathbb{C}$. We define the *logarithmic potential* of $\mu\in\mathcal{P}(K)$ by

$$p_\mu(z) := \int_K \log|z-w|\,d\mu(w), \qquad z\in\mathbb{C}.$$

A measure $\nu\in\mathcal{P}(K)$ is called the *equilibrium measure* of the set K if

$$I(\nu) = \sup\{I(\mu) : \mu\in\mathcal{P}(K)\},$$

where $I(\mu) := \int_K p_\mu(z)\,d\mu(z)$ is the *energy* of μ. The *logarithmic capacity* of a set $E\subset\mathbb{C}$ is the number

$$\operatorname{cap} E := \exp(\sup\{I(\mu) : \mu\in\mathcal{P}(K),\ K \text{ is a compact subset of } E\}).$$

For a compact set $K\subset\mathbb{C}$ and $\zeta\in\mathbb{C}\setminus K$, let

$$f_K(\zeta) := \begin{cases} \displaystyle\int_K \frac{d\mu_K(\lambda)}{\zeta-\lambda} & \text{if } \operatorname{cap} K>0,\\ 0 & \text{if } \operatorname{cap} K = 0,\end{cases}$$

where μ_K denotes the equilibrium measure of K.

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We set $\Delta(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C},\ r > 0$.

We will need the following properties:

(1) If $E_1 \subset E_2 \subset \mathbb{C}$, then $\operatorname{cap} E_1 \le \operatorname{cap} E_2$.

(2) If $B = \bigcup_{k=1}^{N} B_k$, where B_k are Borel sets in $\mathbb{C}$ and $\operatorname{diam} B \le d$ ($d > 0,\ N = 1, 2, \ldots, \infty$), then

$$\frac{1}{\log\left(\frac{d}{\operatorname{cap} B}\right)} \le \sum_{k=1}^{N} \frac{1}{\log\left(\frac{d}{\operatorname{cap} B_k}\right)}.$$

(3) $\operatorname{cap} \Delta(z, r) = \operatorname{cap} \partial\Delta(z, r) = r$.

(4) (Frostman's Theorem) Let μ be the equilibrium measure of a compact set K such that $\operatorname{cap} K > 0$. Then $p_\mu \ge \log \operatorname{cap} K$ on $\mathbb{C}$ and $p_\mu = \log \operatorname{cap} K$ on $K \setminus F$, where F is an F_σ-subset of ∂K such that $\operatorname{cap} F = 0$. Moreover, if $z \in \partial K$ is regular for the Dirichlet problem for the unbounded connected component of $\mathbb{C} \setminus K$, then $p_\mu(z) = \log \operatorname{cap} K$.

(5) (Wiener's criterion) Let $D \subset \mathbb{C}$ be a bounded domain and let $z_0 \in \partial D$. Fix $\theta \in (0, 1)$. Define $F_k := \{z \in \mathbb{C} \setminus D : \theta^{k+1} \le |z - z_0| < \theta^k\}$. Then z_0 is a regular point for the Dirichlet problem for D if and only if $\sum_{k=1}^{\infty} -k/\log \operatorname{cap} F_k = \infty$.

Let $L^2_{\mathrm{h}}(D)$ be the Hilbert space of square integrable functions holomorphic on $D \subset \mathbb{C}^n$ with the standard scalar product induced from $L^2(D)$ and norm $\|\cdot\|_D$. We define the *Bergman kernel* K_D and the function M_D for a bounded domain D by

$$K_D(z) := \sup\left\{\frac{|f(z)|^2}{\|f\|_D^2} : f \in L^2_{\mathrm{h}}(D),\ f \not\equiv 0\right\},$$
$$M_D(z; X) := \sup\left\{\frac{|f'(z)X|^2}{\|f\|_D^2} : f \in L^2_{\mathrm{h}}(D),\ f \not\equiv 0,\ f(z) = 0\right\}$$

for $z \in D$ and $X \in \mathbb{C}^n$. The *Bergman metric* β_D is given by the formula

$$\beta_D^2(z; X) := \sum_{j,k=1}^{n} \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \overline{z}_k} X_j \overline{X}_k, \qquad z \in D,\ X \in \mathbb{C}^n,$$

and the *Bergman distance* of $z, w \in D$ is

$$b_D(z, w) := \inf\{L_{\beta_D}(\alpha) \mid \alpha : [0, 1] \to D \text{ piecewise } \mathcal{C}^1,\ \alpha(0) = z,\ \alpha(1) = w\},$$

where $L_{\beta_D}(\alpha) := \int_0^1 \beta_D(\alpha(t), \alpha'(t))\, dt$.

If D is a planar domain, then $M_D(z; X) = |X|^2 M_D(z; 1)$ and $\beta_D(z; X) = |X|\beta_D(z; 1)$. For simplicity, we write $M_D(z) := M_D(z; 1)$ and $\beta_D(z) := \beta_D(z; 1)$. Recall that

$$\beta_D^2(z) = \frac{M_D(z)}{K_D(z)}, \qquad z \in D.$$

Let us also define, for $D \subset \mathbb{C}$,

$$\gamma_D(z) := \int_0^{1/4} \frac{d\delta}{\delta^3(-\log \operatorname{cap}(\overline{\Delta}(z,\delta) \setminus D))}, \qquad z \in D. \tag{6}$$

A bounded domain $D \subset \mathbb{C}^n$ is called *Bergman exhaustive at a point* $z_0 \in \partial D$ if $\lim_{D \ni z \to z_0} K_D(z) = \infty$. We say that D is *Bergman exhaustive* if it is Bergman exhaustive at each of its boundary points.

A bounded domain D is said to be *Bergman complete* if any Cauchy sequence with respect to the Bergman distance (a *Cauchy–Bergman sequence*) is convergent in the standard topology to a point of D.

We refer the reader to other publications for more properties of the Bergman kernel and metric (see e.g. [5]) and the function γ_D (see [14], [11]).

It is known that hyperconvexity implies both exhaustiveness (see [8]) and Bergman completeness (see [1] and [4]). But the converse is not true (see e.g. [2], [4]).

On the complex plane, if a domain is Bergman exhaustive, then it is also Bergman complete (see [3]). A classification of Bergman exhaustive planar domains is also known:

THEOREM 1 (see [14]). *Let D be a bounded domain in $\mathbb{C}$, $z_0 \in \partial D$. Then*

$$\lim_{D \ni z \to z_0} \gamma_D(z) = \infty \tag{7}$$

if and only if D is Bergman exhaustive at z_0.

Kobayashi [7] asked whether exhaustiveness is necessary for completeness. After a long period of uncertainty, the answer turned out to be negative (see [13]).

Let us now define a special type of plane domains—the so-called *Zalcman type* domains:

$$D := \Delta(0,1) \setminus \Big(\bigcup_{k=1}^{\infty} \overline{\Delta}(x_k, r_k) \cup \{0\} \Big), \tag{8}$$

where $x_k > x_{k+1} > 0$, $\lim_{k\to\infty} x_k = 0$, $\overline{\Delta}(x_k, r_k) \subset \Delta(0,1)$ and $\overline{\Delta}(x_k, r_k) \cap \overline{\Delta}(x_l, r_l) = \emptyset$ for $k \neq l$.

We also consider additional conditions for such domains:

$$\exists \theta_1 \in (0,1)\ \forall k \geq 1 : \quad \theta_1 \leq \frac{x_{k+1}}{x_k}; \tag{9}$$

$$\exists \theta_2 \in (\theta_1,1)\ \forall k \geq 1 : \quad \frac{x_{k+1}}{x_k} \leq \theta_2. \tag{10}$$

The following useful corollary follows from Theorem 1.

Corollary 2. *Let D be a domain given by* (8) *and satisfying* (9) *and* (10). *Then D is Bergman exhaustive if and only if*

$$\sum_{k=1}^{\infty} \frac{-1}{x_k^2 \log r_k} = \infty. \tag{11}$$

We prove the following

Theorem 3. *Let D be a domain given by* (8) *and satisfying* (9) *and* (10). *Then the following are equivalent*:

(i) *D is Bergman complete*,
(ii) $\sum_{k=1}^{\infty} 1/(x_k\sqrt{-\log r_k}) = \infty$.

The following problem was stated in [10]: which domains satisfying (8) with

$$x_k := 1/2^k \tag{12}$$

are Bergman complete? Theorem 3 gives an answer to that question.

Regarding the hyperconvexity and exhaustiveness of these domains, we know everything.

Theorem 4. *If D is a domain given by* (8) *and satisfying* (12), *then*:

(a) *D is hyperconvex if and only if*

$$\sum_{k=1}^{\infty} \frac{k}{-\log r_k} = \infty. \tag{13}$$

(b) *D is Bergman exhaustive if and only if*

$$\sum_{k=1}^{\infty} \frac{2^{2k}}{-\log r_k} = \infty.$$

Theorem 4 together with Theorem 3 gives us a rich collection of domains which are Bergman complete but not hyperconvex and, furthermore, Bergman complete but not Bergman exhaustive (they are simpler than those in [13]).

Incidentally, as a by-product of Theorem 3, we obtain (cf. Corollary 5 in [11])

Corollary 5. *Let D be a planar domain given by* (8).

(a) *If D satisfies* (10) *then*

$$\sum_{N=1}^{\infty} \frac{1}{x_N^2\sqrt{-\log r_N}} < \infty \ \Rightarrow\ \limsup_{0>x\to 0} \beta_D(x) < \infty.$$

(b) *If D satisfies* (9) *and* (10) *then*

$$\limsup_{0>x\to 0}\beta_D(x)<\infty \ \Rightarrow\ \limsup_{N\to\infty}\frac{1}{x_N^2\sqrt{-\log r_N}}<\infty.$$

For the proofs, we need the following lemma which is a straightforward corollary of Lemma 2 in [11].

LEMMA 6. *Given a bounded domain $D\subset\mathbb{C}$ and a number $\alpha\in(0,1)$, there is a constant $C>0$ such that for any compact set $K\subset\overline{\Delta}(0,\alpha)$ with $K\cap D=\emptyset$,*

$$\|f_K\|_D\le C\sqrt{-\log\operatorname{cap}(K)}.$$

Proof of Corollary 2. Notice that (7) holds for any $z_0\in\partial D\setminus\{0\}$. By Theorem 1, we only need to prove that (11) is equivalent to $\lim_{D\ni z\to 0}\gamma_D(z)=\infty$.

We have $\overline{\Delta}\big(x_{k+1}-\frac12 r_{k+1},\frac12 r_{k+1}\big)\subset\overline{\Delta}(0,\delta)\setminus D$ for $\delta\in(x_{k+1},x_k)$. Consequently, from the definition of γ_D and using (9) and (10), we obtain

$$\gamma_D(0)\ge\sum_{k=k_0}^{\infty}\int_{x_{k+1}}^{x_k}\frac{d\delta}{\delta^3(-\log\operatorname{cap}(\overline{\Delta}(0,\delta)\setminus D))}\ge\sum_{k=k_0}^{\infty}\int_{x_{k+1}}^{x_k}\frac{d\delta}{-\delta^3\log\frac12 r_{k+1}}$$
$$\ge\sum_{k=k_0}^{\infty}(x_k-x_{k+1})\frac{-1}{x_k^3\log\frac12 r_{k+1}}\ge C\sum_{k=k_0}^{\infty}\frac{-1}{x_{k+1}^2\log r_{k+1}}.$$

Above, k_0 is an integer such that $x_{k_0}<1/4$ and C is a numerical constant. Now, the divergence of the series in (11) implies that $\gamma_D(0)=\infty$. Due to the lower semicontinuity of γ_D (see [14]) we deduce that $\lim_{D\ni z\to 0}\gamma_D(z)=\infty$.

On the other hand, we have

$$\gamma_D(0)=\Big(\int_{x_1}^{1/4}+\sum_{k=1}^{\infty}\int_{x_{k+1}}^{x_k}\Big)\frac{d\delta}{\delta^3(-\log\operatorname{cap}(\overline{\Delta}(0,\delta)\setminus D))}$$
$$\le C_1+\sum_{k=1}^{\infty}\frac{x_k-x_{k+1}}{x_{k+1}^3}\sum_{j=k}^{\infty}\frac{1}{-\log r_j}\overset{(9)}{\le}C_1+C_2\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1}{x_k^2}\frac{1}{-\log r_j}$$
$$\overset{(10)}{\le}C_1+C_3\sum_{j=1}^{\infty}\frac{-1}{x_j^2\log r_j}.$$

Above, $C_1\ge 0$ and $C_2,C_3>0$ are constants. The last inequality holds due to (10):

$$\sum_{k=1}^{j}\frac{1}{x_k^2}\le\sum_{k=1}^{j}\frac{\theta_2^{2(j-k)}}{x_j^2}<\frac{1}{1-\theta_2^2}\frac{1}{x_j^2}.$$

Thus, if the series in (11) is convergent, then $\gamma_D(0)<\infty$. We can deduce directly from the definition of γ_D that $\gamma_D(y_1)\le\gamma_D(y_2)$ for $-1/4\le y_1\le$

$y_2 \le 0$. Hence, if $\gamma_D(0) < \infty$, then also $\limsup_{0>z\to 0} \gamma_D(z) < \infty$. This finishes the proof. ■

Proof of Theorem 3. (i)⇒(ii) (cf. the proof of Theorem 3 in [11]). Suppose that

$$\sum_{k=1}^{\infty} \frac{1}{x_k\sqrt{-\log r_k}} < \infty. \tag{14}$$

It is sufficient to show that there exists a $\delta > 0$ such that $\int_{-\delta}^{0} \beta_D(x)\,dx < \infty$.

Let us introduce some notations:

$$K_0 := \overline{\Delta}(0,1) \setminus \Delta(0, 1-\varepsilon_0), \qquad K_j := \overline{\Delta}(x_j, r_j), \qquad j \ge 1,$$
$$L_j := \bigcup_{k=j+1}^{\infty} \overline{\Delta}(x_k, r_k) \cup \overline{\Delta}(0, \varepsilon_j), \qquad j \ge 1,$$
$$\widetilde{D}_j := (D \cup \Delta(0, \varepsilon_j)) \cap \Delta(0, 1-\varepsilon_0),$$

where $\varepsilon_0 < 1/4$ is fixed. We choose $\varepsilon_j \in (0, x_{j+1})$ so small that

$$\frac{1}{-\log \operatorname{cap} L_j} < 2 \sum_{k=j+1}^{\infty} \frac{1}{-\log r_k} \tag{15}$$

(apply (2) and (3)) and such that $\partial\Delta(0,\varepsilon_j) \subset D$. For a compact set $B \subset \mathbb{C}$, let $p_B := p_{\mu_B}$ be the logarithmic potential. If $\operatorname{cap} B > 0$, we choose a function $\chi_B \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that

$$\chi_B(t) = \begin{cases} 1 & \text{if } t \le \log \operatorname{cap} B, \\ 0 & \text{if } t \ge \frac{1}{2}\log \operatorname{cap} B, \end{cases} \tag{16}$$

and

$$|\chi_B'(t)| \le \frac{4}{-\log \operatorname{cap} B}, \qquad t \in \mathbb{R}.$$

Let $\varphi_B := \chi_B \circ p_B$. Then $\varphi_B \equiv 1$ on B by Frostman's theorem. We will use the following lemma which will be proven later.

LEMMA 7. *Given a domain D as in Theorem* 3 *satisfying* (14), *there exists an integer N_0 such that for $j \ge N_0$*:

$$\operatorname{supp} \varphi_{K_j} \subset \overline{\Delta}(0, 1-2\varepsilon_0), \qquad \operatorname{supp} \varphi_{L_j} \subset \overline{\Delta}(0, 1-2\varepsilon_0); \tag{17}$$
$$\operatorname{supp} \varphi_{K_j} \cap \operatorname{supp} \varphi_{K_{j+1}} = \emptyset; \tag{18}$$
$$\operatorname{supp} \varphi_{K_j} \cap \operatorname{supp} \varphi_{L_j} = \emptyset; \tag{19}$$
$$\operatorname{dist}(-x_j, \operatorname{supp} \varphi_{L_j}) \ge \tfrac{1}{2}(1-\theta_2)x_j. \tag{20}$$

The behavior of the Bergman kernel and metric is a local property (see e.g. Theorem 6.3.5 in [5]). So, without loss of generality, we may assume that (17)–(20) hold for all $j \ge 1$.

Choose one more function $\varphi_0 \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that $\varphi_0 \equiv 1$ on K_0 and $\operatorname{supp}\varphi_0 \subset \overline{\Delta}(0, 1+\varepsilon_0) \setminus \Delta(0, 1-2\varepsilon_0)$. Then, by (17), we also have $\operatorname{supp}\varphi_0 \cap \bigcup_{j=1}^\infty (\operatorname{supp}\varphi_{K_j} \cup \operatorname{supp}\varphi_{L_j}) = \emptyset$.

Now,

$$\left|\frac{\partial\varphi_B}{\partial\overline{z}}(z)\right| = \left|\chi_B'(p_B(z))\,\frac{\partial p_B(z)}{\partial\overline{z}}\right| \le \frac{4|f_B(z)|}{-\log\operatorname{cap}B}, \qquad z \in D,$$

where $B = K_j$ or $B = L_j$ for $j \ge 1$, and

$$\left|\frac{\partial\varphi_0}{\partial\overline{z}}(z)\right| \le M, \qquad z \in D,$$

where $M > 0$ is a constant.

Now, take any $N \in \mathbb{N}$, choose $x \in [-x_{N-1}, -x_N]$, and put

$$\varphi := \varphi_0 + \varphi_{K_1} + \ldots + \varphi_{K_N} + \varphi_{L_N}.$$

Then $\varphi \equiv 1$ on $\partial\widetilde{D}_N$.

Take any $f \in L_{\mathrm{h}}^2(D)$, $f \not\equiv 0$. Using the Cauchy integral formula and the Green formula, we obtain

$$\begin{aligned}
|f'(x)| &= \frac{1}{2\pi}\left|\int_{\partial\widetilde{D}_N} \frac{f(\lambda)\,d\lambda}{(\lambda-x)^2}\right| = \frac{1}{2\pi}\left|\int_{\partial\widetilde{D}_N} \frac{f(\lambda)\varphi(\lambda)\,d\lambda}{(\lambda-x)^2}\right| \\
&= \frac{1}{\pi}\left|\int_{\widetilde{D}_N} \frac{f(\lambda)}{(\lambda-x)^2}\,\frac{\partial\varphi}{\partial\overline{\lambda}}(\lambda)\,dL^2(\lambda)\right| \\
&\le \frac{1}{\pi}\int_{\widetilde{D}_N} \frac{|f(\lambda)|}{|\lambda-x|^2}\left|\frac{\partial\varphi_0}{\partial\overline{\lambda}}(\lambda)\right| dL^2(\lambda) \\
&\quad + \sum_{j=1}^N \frac{1}{\pi}\int_{\widetilde{D}_N} \frac{|f(\lambda)|}{|\lambda-x|^2}\left|\frac{\partial\varphi_{K_j}}{\partial\overline{\lambda}}(\lambda)\right| dL^2(\lambda) \\
&\quad + \frac{1}{\pi}\int_{\widetilde{D}_N} \frac{|f(\lambda)|}{|\lambda-x|^2}\left|\frac{\partial\varphi_{L_N}}{\partial\overline{\lambda}}(\lambda)\right| dL^2(\lambda).
\end{aligned}$$

Now, we use the Cauchy–Schwarz inequality and the estimates $1/|\lambda - x| \le C_1/x_j$ for $\lambda \in \operatorname{supp}\varphi_{K_j}$, $j = 1, \ldots, N$, and $1/|\lambda - x| \le C_1/x_N$ for $\lambda \in \operatorname{supp}\varphi_{L_N}$ (due to (20)):

$$|f'(x)| \le C_2\|f\|_D\biggl(1 + \sum_{j=1}^N \frac{1}{x_j^2}\,\frac{\|f_{K_j}\|_D}{-\log\operatorname{cap}K_j} + \frac{1}{x_N^2}\,\frac{\|f_{L_N}\|_D}{-\log\operatorname{cap}L_N}\biggr).$$

Finally, using Lemma 6, we obtain

$$|f'(x)| \le C_3\|f\|_D\Bigl(1+\sum_{j=1}^{N}\frac{1}{x_j^2\sqrt{-\log\operatorname{cap}K_j}}+\frac{1}{x_N^2\sqrt{-\log\operatorname{cap}L_N}}\Bigr).$$

The constants $C_1, C_2, C_3 > 0$ above do not depend on N.

Thus

$$\sqrt{M_D(x)} \le C_3\Bigl(1+\sum_{j=1}^{N}\frac{1}{x_j^2\sqrt{-\log r_j}}+\frac{\sqrt{2}}{x_N^2}\sum_{j=N+1}^{\infty}\frac{1}{\sqrt{-\log r_j}}\Bigr) \tag{21}$$

owing to (15).

Now, let us move on to the final estimations:

$$\begin{aligned}\int_{-x_1}^{0}\sqrt{M_D(x)}\,dx &= \sum_{N=2}^{\infty}\int_{-x_{N-1}}^{-x_N}\sqrt{M_D(x)}\,dx\\ &\overset{(9)}{\le}\sum_{N=2}^{\infty}C_4\,x_N\sup_{x\in[-x_{N-1},-x_N]}\sqrt{M_D(x)}\end{aligned}$$

$$\le C_5\Bigl(\sum_{N=2}^{\infty}x_N+\sum_{N=2}^{\infty}x_N\sum_{j=1}^{N}\frac{1}{x_j^2\sqrt{-\log r_j}}+\sum_{N=2}^{\infty}\frac{1}{x_N}\sum_{j=N+1}^{\infty}\frac{1}{\sqrt{-\log r_j}}\Bigr),$$

where $C_4, C_5 > 0$ are some constants. The first series in brackets is finite because of (10). For the second series, we have

$$\begin{aligned}\sum_{N=1}^{\infty}x_N\sum_{j=1}^{N}\frac{1}{x_j^2\sqrt{-\log r_j}} &= \sum_{j=1}^{\infty}\Bigl(\sum_{N=j}^{\infty}x_N\Bigr)\frac{1}{x_j^2\sqrt{-\log r_j}}\\ &\le \frac{1}{1-\theta_2}\sum_{j=1}^{\infty}\frac{1}{x_j\sqrt{-\log r_j}}\overset{(14)}{<}\infty\end{aligned}$$

because (10) implies $\sum_{N=j}^{\infty}x_N \le \sum_{s=0}^{\infty}\theta_2^s x_j = x_j/(1-\theta_2)$. Notice also that

$$\sum_{N=1}^{j-1}\frac{1}{x_N}\le\sum_{s=1}^{j-1}\theta_2^s\,\frac{1}{x_j}<\frac{\theta_2}{1-\theta_2}\,\frac{1}{x_j}.$$

Thus,

$$\begin{aligned}\sum_{N=1}^{\infty}\frac{1}{x_N}\sum_{j=N+1}^{\infty}\frac{1}{\sqrt{-\log r_j}} &= \sum_{j=2}^{\infty}\Bigl(\sum_{N=1}^{j-1}\frac{1}{x_N}\Bigr)\frac{1}{\sqrt{-\log r_j}}\\ &\le \frac{\theta_2}{1-\theta_2}\sum_{j=1}^{\infty}\frac{1}{x_j\sqrt{-\log r_j}}\overset{(14)}{<}\infty.\end{aligned}$$

As a consequence, $\int_{-x_1}^{0}\beta_D(x)\,dx<\infty$ because K_D is separated from 0 on D.

(ii)⇒(i) (cf. the proof of Theorem 5 in [13]). Suppose that D is Bergman exhaustive at 0. Since D is Bergman exhaustive at any other point, it is Bergman complete by the result of Chen (see [3]).

Thus, we can assume that D is not Bergman exhaustive at 0. In view of Corollary 2, the series (11) is convergent. Hence, $\lim_{j\to\infty} 1/(x_j\sqrt{-\log r_j}) = 0$.

We are going to use an auxiliary lemma which will be proven later.

LEMMA 8. *Let D be a domain as in Theorem 3 with*

$$\sum_{j=1}^{\infty}\frac{1}{x_j\sqrt{-\log r_j}}=\infty \quad \textit{and} \quad \lim_{j\to\infty}\frac{1}{x_j\sqrt{-\log r_j}}=0.$$

If $\gamma:[0,1)\to D$ is a curve such that $\lim_{t\to 1}\gamma(t)=0$ and $\gamma|_{[0,t]}$ is piecewise $\mathcal{C}^1$ for all $t\in(0,1)$, then $\int_\gamma\sqrt{M_D(z)}\,dl(z)=\infty$.

Suppose that D is not Bergman complete. Then there exists a Cauchy–Bergman sequence $(z_k)_{k=1}^{\infty}\subset D$ such that $\lim_{k\to\infty} z_k=0$. We can choose the sequence such that $b_D(z_k,z_{k+1})<1/2^{k+1}$. We join each pair of points z_k, z_{k+1} by a $\mathcal{C}^1$-curve of L_{β_D}-length not greater than $1/2^k$. The curve which we obtain by gluing all the small pieces, say $\gamma:[0,1)\to D$, has a finite length with respect to the Bergman metric. We set $\gamma^\star:=\gamma([0,1))$.

Notice that the Bergman kernel K_D must be bounded on $\gamma^\star$. In fact, suppose the opposite. Then there is a sequence $(w_k)_{k=1}^{\infty}\subset\gamma^\star$ such that $\lim_{k\to\infty} w_k=0$ and $\lim_{k\to\infty} K_D(w_k)=\infty$. This sequence is also a Cauchy–Bergman sequence. Then, by the results of Pflug ([9]) and Chen (see [2], [3]), there is a subsequence $(w_{k_j})_{j=1}^{\infty}$ and a function $f\in L^2_{\mathrm{h}}(D)$ such that

$$\frac{|f(w_{k_j})|}{\sqrt{K_D(w_{k_j})}}\to 1,\qquad j\to\infty.$$

Because the functions from $L^2_{\mathrm{h}}(D)$ bounded in a neighborhood of 0 are dense in $L^2_{\mathrm{h}}(D)$ (see [3]), there exists a $g\in L^2_{\mathrm{h}}(D)$ such that $\|g-f\|_D\le 1/2$ and g is bounded near 0. Thus, also by the general properties of the Bergman kernel K_D, we have

$$\frac{|g(w_{k_j})|}{\sqrt{K_D(w_{k_j})}}\ge\frac{|f(w_{k_j})|}{\sqrt{K_D(w_{k_j})}}-\|f-g\|_D\ge\frac{|f(w_{k_j})|}{\sqrt{K_D(w_{k_j})}}-\frac{1}{2}.$$

Letting $j\to\infty$ yields a contradiction and we conclude that K_D is bounded on $\gamma^\star$.

Finally, we use Lemma 8, which leads to a contradiction:

$$\infty>\int_{\gamma^\star}\beta_D(z)\,dl(z)\ge\frac{1}{\sup_{\gamma^\star}\sqrt{K_D}}\int_{\gamma^\star}\sqrt{M_D(z)}\,dl(z)=\infty.$$

As a consequence, the domain D is Bergman complete. ■

Proof of Lemma 7. We see that

$$\operatorname{supp}\varphi_B \subset \left\{z\in\mathbb{C} : p_B(z) \le \tfrac{1}{2}\log\operatorname{cap} B\right\} \subset \{z\in\mathbb{C} : \operatorname{dist}(z,B)\le \sqrt{\operatorname{cap} B}\}$$

for a compact set B with $\operatorname{cap} B > 0$.

Let $\delta_0 > 0$ be small. For sufficiently large j (say $j \ge N_0 = N_0(\delta_0)$), we have

$$\begin{aligned}\operatorname{cap} K_j &= r_j \overset{(*)}{\le} \delta_0^2 x_j^2,\\ \operatorname{cap} L_j &\le \frac{-1}{\log\operatorname{cap} L_j} \overset{(15)}{\le} 2\sum_{k=j+1}^{\infty}\frac{-1}{\log r_k} \overset{(*)}{\le} 2\delta_0^2 x_{j+1}^2.\end{aligned}$$

Both the inequalities marked with $(*)$ hold since $\sum_{k=1}^{\infty} -1/(x_k^2\log r_k)$ is convergent (by (14)). The latter inequality is true because

$$\frac{1}{x_{j+1}^2}\sum_{k=j+1}^{\infty}\frac{-1}{\log r_k} \le \sum_{k=j+1}^{\infty}\frac{-1}{x_k^2\log r_k} \le \delta_0^2$$

if $j \ge 1$ is large enough.

Thus, we have

$$\begin{aligned}&\operatorname{supp}\varphi_{K_j} \subset K_j + \overline{\Delta}(0,\delta_0 x_j),\\ &\operatorname{supp}\varphi_{L_j} \subset L_j + \overline{\Delta}(0,\sqrt{2}\delta_0 x_{j+1}), \qquad j \ge N_0.\end{aligned}$$

Therefore, the conditions (17)–(20) are fulfilled provided that δ_0 is chosen small enough and N_0 is sufficiently large. Indeed, (17) is straightforward whereas (18), (19) and (20) follow from the inequalities, respectively:

$$\begin{aligned} r_j + \delta_0 x_j + r_{j+1} + \delta_0 x_{j+1} &< x_j - x_{j+1},\\ r_j + \delta_0 x_j + r_{j+1} + \sqrt{2}\delta_0 x_{j+1} &< x_j - x_{j+1},\\ \varepsilon_j + \sqrt{2}\delta_0 x_{j+1} + \tfrac{1}{2}(1-\theta_2)x_j &< x_j.\end{aligned}$$

The above inequalities can be obtained by the use of (10), lowering δ_0 if necessary. Recall that we have chosen $\varepsilon_j < x_{j+1}$ and $r_j \le \delta_0^2 x_j^2$. ∎

Proof of Lemma 8. Without loss of generality, we may assume that $|\gamma(0)| > x_1$ and

$$x_1\sqrt{-\log r_1} < x_j\sqrt{-\log r_j}, \qquad j > 1. \tag{23}$$

Fix $N \ge 2$ and take $z_0 \in D$ such that $x_{N+2} \le |z_0| \le x_{N+1}$. Define

$$f := f_{\overline{\Delta}(x_1,r_1)} - \frac{x_N - z_0}{x_1 - z_0} f_{\overline{\Delta}(x_N,r_N)}.$$

For a disk, we have the following formula:

$$f_{\overline{\Delta}(x,r)}(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{z - x - re^{it}} = \frac{1}{x-z}.$$

So, we can explicitly compute that

$$f(z_0) = 0, \qquad f'(z_0) = \frac{x_N - x_1}{(x_1 - z_0)^2 (x_N - z_0)}.$$

Using Lemma 6, we obtain the estimate

$$\begin{aligned} \|f\|_D &\le \|f_{\overline{\Delta}(x_1,r_1)}\| + \frac{|x_N - z_0|}{|x_1 - z_0|}\,\|f_{\overline{\Delta}(x_N,r_N)}\| \\ &\le C_1\Big(\sqrt{-\log r_1} + \frac{|x_N - z_0|}{|x_1 - z_0|}\sqrt{-\log r_N}\Big) \\ &\overset{(10),(23)}{\le} C_2 \frac{|x_N - z_0|}{|x_1 - z_0|}\sqrt{-\log r_N}, \end{aligned}$$

where $C_1, C_2 > 0$ are constants independent of N. Hence,

$$\begin{aligned} \sqrt{M_D(z_0)} &\ge \frac{|f'(z_0)|}{\|f\|_D} \ge \frac{x_1 - x_N}{C_2 |x_1 - z_0|\,|x_N - z_0|^2 \sqrt{-\log r_N}} \\ &\overset{(10)}{\ge} \frac{C_3}{x_N^2 \sqrt{-\log r_N}}, \end{aligned} \tag{24}$$

where $C_3 > 0$ is a constant. Finally,

$$\begin{aligned} \int_\gamma \sqrt{M_D(z)}\,dl(z) &\ge \sum_{N=2}^{\infty} \inf_{|z|\in[x_{N+2},x_{N+1}]} \sqrt{M_D(z)}\,(x_{N+1} - x_{N+2}) \\ &\ge C_3 \sum_{N=2}^{\infty} \frac{x_{N+1} - x_{N+2}}{x_N^2 \sqrt{-\log r_N}} \\ &\overset{(9),(10)}{\ge} C_4 \sum_{N=2}^{\infty} \frac{1}{x_N \sqrt{-\log r_N}} = \infty. \end{aligned}$$

This finishes the proof. ■

Proof of Theorem 4. (a) We know that hyperconvexity of a bounded domain is equivalent to the regularity of the Dirichlet problem (see e.g. [6, 12]). Applying Wiener's criterion (5) to the point $z_0 = 0$, we also get equivalence to (13). Note that for $\theta = 1/2$ in (5), due to the properties (1)–(3), we have

$$\frac{1}{-\log\big(\frac{1}{2} r_k\big)} \le \frac{1}{-\log \operatorname{cap} F_k} \le \frac{1}{-\log r_k} + \frac{1}{-\log r_{k+1}}$$

because

$$\Delta\Big(\frac{1}{2^k} - \frac{1}{2} r_k, \frac{1}{2} r_k\Big) \subset F_k \subset \Delta\Big(\frac{1}{2^{k+1}}, r_{k+1}\Big) \cup \Delta\Big(\frac{1}{2^k}, r_k\Big).$$

Then also

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{k}{-\log r_k}\le\sum_{k=1}^{\infty}\frac{k}{-\log\operatorname{cap}F_k}\le 2\sum_{k=1}^{\infty}\frac{k}{-\log r_k}.$$

(b) That is a consequence of Corollary 2. ∎

Proof of Corollary 5. (a) By (21), the convergence of the series $\sum_{N=1}^{\infty}1/(x_N^2\sqrt{-\log r_N})$ implies that $\sqrt{M_D(x)}<C$ for all $x\in[-1/2,0)$ and some numerical constant $C>0$. Thus, the Bergman metric β_D is also bounded on $[-1/2,0)$ because the Bergman kernel K_D is separated from 0 on D. Notice that we do not use (9) in the proof of (21).

(b) Suppose that $\limsup_{0>x\to 0}\beta_D(x)<\infty$. Then K_D must be bounded on $[-1/2,0)$ and (23) holds (reason as in the proof of Theorem 3, second part). To complete the proof, use (24). ∎

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