STUDIA MATHEMATICA 163 (1) (2004)

Weyl's and Browder's theorems for operators satisfying the SVEP

by

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Abstract. We study Weyl's and Browder's theorem for an operator T on a Banach space such that T or its adjoint has the single-valued extension property. We establish the spectral mapping theorem for the Weyl spectrum, and we show that Browder's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$. Also, we give necessary and sufficient conditions for such T to obey Weyl's theorem. Weyl's theorem in an important class of Banach space operators is also studied.

1. Introduction. Throughout this paper, X denotes an infinite-dimensional complex Banach space, $\mathcal{L}(X)$ the algebra of all bounded linear operators on X and $\mathcal{K}(X)$ its ideal of compact operators. For an operator $T \in \mathcal{L}(X)$, write T^* for its adjoint; $N(T)$ for its null space; $R(T)$ for its range; $\sigma(T)$ for its spectrum; $\sigma_{\rm su}(T)$ for its surjective spectrum; $\sigma_{\rm ap}(T)$ for its approximate spectrum; and $\sigma_{\rm p}(T)$ for its point spectrum.

From [29] we recall that for $T \in \mathcal{L}(X)$, the *ascent* $a(T)$ and the *descent* $d(T)$ are given by $a(T) = \inf\{n \geq 0 : N(T)^n = N(T)^{n+1}\}$ and $d(T) = \inf\{n \geq 0 : R(T)^n = R(T)^{n+1}\}$, respectively; the infimum over the empty set is taken to be ∞. If the ascent and descent of $T \in \mathcal{L}(X)$ are both finite, then $a(T) = d(T) = p$, $X = N(T)^p \oplus R(T)^p$ and $R(T)^p$ is closed.

An operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $R(T)$ is closed and either $\dim N(T)$ or $\operatorname{codim} R(T)$ is finite. For such an operator the *index* is defined by $\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T)$, and if the index is finite, T is said to be *Fredholm*. Also, an operator $T \in \mathcal{L}(X)$ is said to be *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent. For $T \in \mathcal{L}(X)$, the essential spectrum $\sigma_{\rm e}(T)$, the Weyl spectrum $\sigma_{\rm w}(T)$ and the Browder spectrum $\sigma_{\rm b}(T)$ are defined by

$$\sigma_{\rm e}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$
$$\sigma_{\rm w}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

2000 *Mathematics Subject Classification*: 47A53, 47A10, 47B20.

Key words and phrases: Weyl's theorem, single-valued extension property, quasinilpotent part, analytic core.

$$\sigma_{\rm b}(T) = \mathbb{C} \setminus \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

The Weyl and Browder spectra have the important property of being the largest subsets of the spectrum remaining invariant under arbitrary compact and commuting compact perturbations, respectively. Indeed, for every operator $T \in \mathcal{L}(X)$ we have ([27] and [18])

$$\sigma_{\rm w}(T) = \bigcap\{\sigma(T+K) : K \in \mathcal{K}(X)\}$$
$$\sigma_{\rm b}(T) = \{\sigma(T+K) : K \in \mathcal{K}(X) \text{ and } KT = TK\};$$

consequently,

$$\sigma_{\rm e}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T).$$

For a subset K of $\mathbb{C}$, we shall write $\operatorname{iso} K$ for its *isolated points* and $\operatorname{acc} K = K \setminus \operatorname{iso} K$ for its *accumulation points.*

A complex number λ is said to be a *Riesz point* of $T \in \mathcal{L}(X)$ if $\lambda \in \operatorname{iso}\sigma(T)$ and the spectral projection corresponding to the set $\{\lambda_0\}$ has finite-dimensional range. The set of all Riesz points of T will be denoted by $\Pi_0(T)$. It is known that if $T \in \mathcal{L}(X)$ and $\lambda \in \sigma(T)$, then $\lambda \in \Pi_0(T)$ if and only if $T - \lambda$ is Fredholm of finite ascent and descent (see [4]); consequently, $\sigma_{\rm b}(T) = \sigma(T) \setminus \Pi_0(T)$.

The set of complex numbers $\lambda \in \operatorname{iso}\sigma(T)$ for which $N(T-\lambda)$ is non-zero and finite-dimensional is denoted by $\Pi_{00}(T)$.

Now, let us introduce some basic notions from local spectral theory. Let T be a bounded linear operator on X. We say that T has the *single-valued extension property*, SVEP for brevity, if for every non-empty open set $U \subseteq \mathbb{C}$, the only analytic solution of the equation $(T-\lambda)f(\lambda) = 0$ for $\lambda \in U$ is the zero function. For an element $x \in X$, let $\varrho_T(x)$ be the *local resolvent set* of T at x, defined as the union of all open subsets $\mathcal{U}$ of $\mathbb{C}$ such that there exists an analytic function $f : \mathcal{U} \to \mathbb{C}$ which satisfies $(T-\lambda)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$. The *local spectrum* of T at x is defined by $\sigma_T(x) = \mathbb{C} \setminus \varrho_T(x)$. Also, for a subset F of $\mathbb{C}$, the corresponding *analytic spectral subspace* of T is the linear subspace $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. The operator T is said to satisfy *Dunford's condition* (C) if for every closed subset F of $\mathbb{C}$, $X_T(F)$ is closed. We note that every operator satisfying (C) has the SVEP (Proposition 1.2.19 of [17]).

Definition. Let T be a bounded operator on X. We will say that

(i) *Weyl's theorem holds for* T if $\sigma_{\rm w}(T) = \sigma(T) \setminus \Pi_{00}(T)$,
(ii) *Browder's theorem holds for* T if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T) = \sigma(T) \setminus \Pi_0(T)$.

We note that if Weyl's theorem holds for $T \in \mathcal{L}(X)$, then so does Browder's theorem. More precisely, A. Barnes has proved in [2] that

$$(1.1)\qquad \sigma_{\rm w}(T) \subseteq \sigma(T) \setminus \Pi_{00}(T) \quad \text{if and only if} \quad \Pi_{00}(T) = \Pi_0(T)$$

and

$$\sigma_{\mathrm{w}}(T) \supseteq \sigma(T) \setminus \Pi_{00}(T) \quad \text{if and only if} \quad \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T). \tag{1.2}$$

The investigation of operators obeying Weyl's theorem was initiated by Hermann Weyl, who proved that for a self-adjoint operator T on Hilbert space we have $\sigma(T) = \sigma_{\mathrm{w}}(T) \setminus \Pi_{00}(T)$. Later, many mathematicians have been interested in this problem and Weyl's result was extended to Toeplitz operators by L. A. Coburn [6], to p-hyponormal operators [5], and to some Banach space operators [24]. Weyl's theorem may fail for the square of an operator T when it holds for T (see Example 1 of [25]). In [10], it was shown that if T is p-hyponormal then $f(T)$ satisfies Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ denotes the space of all analytic functions on an open neighbourhood of $\sigma(T)$. Moreover, if for an operator T there exists a non-zero complex polynomial p such that $p(T)$ is p-hyponormal then Weyl's theorem holds for T (see [9]).

In the present paper, we study Weyl's and Browder's theorem for an operator T such that T or T^* has the SVEP. In Section 2, we prove that for such an operator the spectral mapping theorem for $\sigma_{\mathrm{w}}(T)$ holds, and $f(T)$ satisfies Browder's theorem for every $f \in \mathcal{H}(\sigma(T))$; also we give several necessary and sufficient conditions for T to obey Weyl's theorem. In Section 3, we consider an important class of operators on a Banach space X, $\mathcal{P}(X)$, that contains most of the operators studied in the literature in connection with Weyl's and Browder's theorems, and we prove that if there exists a function $h \in \mathcal{H}(\sigma(T))$ not identically constant on any connected component of its domain, and such that $h(T) \in \mathcal{P}(X)$, then Weyl's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$.

2. Weyl's and Browder's theorems. Before stating our results, we need to introduce the following notions.

We shall say that an operator $T \in \mathcal{L}(X)$ is *semi-regular* if $R(T)$ is closed and $N(T) \subseteq R(T^n)$ for every $n \in \mathbb{N}$. The *semi-regular resolvent set* is defined by $\text{s-reg}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is semi-regular}\}$; we note that $\text{s-reg}(T) = \text{s-reg}(T^*)$ is an open subset of $\mathbb{C}$ ([22]). As an immediate consequence of Theorem 2.7 of [22], we derive the following lemma.

LEMMA 2.1. *Let $T \in \mathcal{L}(X)$.*

(i) *If T has the SVEP then* $\text{s-reg}(T) = \varrho_{\mathrm{ap}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$.
(ii) *If T^* has the SVEP then* $\text{s-reg}(T) = \varrho_{\mathrm{su}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{su}}(T)$.
(iii) *If both T and T^* have the SVEP then* $\text{s-reg}(T) = \varrho(T) := \mathbb{C} \setminus \sigma(T)$.

For an operator $T \in \mathcal{L}(X)$, we let

$$\varrho_{\mathrm{SF}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is semi-Fredholm}\}$$

denote the *semi-Fredholm resolvent set*.

PROPOSITION 2.2. *Let* $T \in \mathcal{L}(X)$.

(i) *If* T *has the SVEP then* $\operatorname{ind}(T-\lambda) \le 0$ *for every* $\lambda \in \varrho_{\mathrm{SF}}(T)$.
(ii) *If* T^* *has the SVEP then* $\operatorname{ind}(T-\lambda) \ge 0$ *for every* $\lambda \in \varrho_{\mathrm{SF}}(T)$.

Proof. (i) Let $\lambda \in \varrho_{\mathrm{SF}}(T)$. From the Kato decomposition [14, Theorem 4], it follows that there exists $\delta > 0$ for which $\{\mu \in \mathbb{C} : 0 < |\mu-\lambda| < \delta\} \subseteq \text{s-reg}(T)$. Since T has the SVEP, $\text{s-reg}(T) = \varrho_{\mathrm{ap}}(T)$. Therefore, $N(T-\mu) = \{0\}$, and so $\operatorname{ind}(T-\mu) \le 0$, for $0 < |\mu-\lambda| < \delta$. Thus, by the continuity of the index we get $\operatorname{ind}(T-\lambda) \le 0$.

(ii) By duality. ∎

Before stating our next theorem, we recall that the spectral mapping theorem holds for the Browder spectrum, but may fail to hold for the Weyl spectrum. However, if $T \in \mathcal{L}(X)$ and $f \in \mathcal{H}(\sigma(T))$ we have the inclusion $\sigma_{\mathrm{w}}(f(T)) \subseteq f(\sigma_{\mathrm{w}}(T))$, by Theorem 2 of [12].

THEOREM 2.3. *If* T *or its adjoint has the SVEP, then* $f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T))$ *for every* $f \in \mathcal{H}(\sigma(T))$.

Proof. This follows from Proposition 2.2 and [28, Theorem 2]. ∎

Let $T \in \mathcal{L}(X)$ be an operator. The *analytic core* of T is the subspace

$$K(T) := \{x \in X : \exists (x_n)_{n\ge 0} \subseteq X \text{ and } \exists c > 0 \text{ such that } x = x_0,\\ Tx_{n+1} = x_n \text{ and } \|x_n\| \le c^n\|x\| \text{ for all } n \ge 0\}.$$

The *quasi-nilpotent part* of T is the subspace

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Both subspaces, which will be of particular importance in this paper, have been introduced in [20] and were thoroughly studied by M. Mbekhta in [20], [22] and [23] (see also [1] and [11]). The following facts are easy to verify: $T(K(T)) = K(T)$; $N(T^n) \subseteq H_0(T)$ for every $n \in \mathbb{N}$; if $x \in X$, then $x \in H_0(T)$ if and only if $Tx \in H_0(T)$; if T is invertible then $H_0(T) = \{0\}$.

THEOREM 2.4 ([20, Théorème 1.6]). *For* $T \in \mathcal{L}(X)$, *the following conditions are equivalent*:

(i) λ *is an isolated point of* $\sigma(T)$.
(ii) $X = H_0(T-\lambda) \oplus K(T-\lambda)$, *where* $H_0(T-\lambda) \ne \{0\}$ *and the direct sum is topological.*

Moreover, λ *is a pole of the resolvent of* T *of order* d *if and only if* $H_0(T-\lambda) = N(T-\lambda)^d$ *and* $K(T-\lambda) = R(T-\lambda)^d$.

In [21], M. Mbekhta introduced and studied an important subclass of $\mathcal{L}(X)$ defined as those operators $T \in \mathcal{L}(X)$ for which $K(T) = \{0\}$. It was shown that for such operators, the spectrum is connected and the SVEP holds.

THEOREM 2.5. *Let $T \in \mathcal{L}(X)$. If there exists a complex number $\lambda_0 \in \operatorname{acc} \sigma(T)$ such that $K(T - \lambda_0) = \{0\}$ or $K(T^* - \lambda_0) = \{0\}$, then both $f(T)$ and $f(T^*)$ satisfy Weyl's theorem, for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Since Weyl's theorem is translation-invariant, we may assume that $\lambda_0 = 0$. Suppose that $K(T) = \{0\}$ or $K(T^*) = \{0\}$. Then from [21, Propositions 2.1 and 2.6], it follows that T or T^* has the SVEP, and $\sigma(T) = \sigma_{\rm w}(T) = \sigma_{\rm w}(T^*)$ is connected and contains 0. In particular, $\sigma(T)$ does not have any isolated point, because otherwise, $\sigma(T) = \{0\}$, which contradicts the fact that $0 \in \operatorname{acc} \sigma(T)$. Let $f \in \mathcal{H}(\sigma(T))$. Since the identity operator obeys Weyl's theorem, we may assume that the function f is non-constant. Hence $f(\sigma(T)) = \sigma(f(T)) = \sigma(f(T^*))$ is a connected subset of $\mathbb{C}$ without isolated points, and therefore $\Pi_{00}(f(T)) = \Pi_{00}(f(T^*)) = \operatorname{iso} \sigma(f(T)) = \emptyset$. Moreover, by Theorem 2.3, we also have

$$\sigma(f(T)) = f(\sigma(T)) = f(\sigma_{\rm w}(T)) = \sigma_{\rm w}(f(T)) = \sigma_{\rm w}(f(T^*)).$$

Consequently, $f(T)$ and $f(T^*)$ obey Weyl's theorem. ∎

Much of what follows is based on the following lemma:

LEMMA 2.6. *Let $T \in \mathcal{L}(X)$ be a semi-Fredholm operator. Then $H_0(T)$ is closed if and only if $\dim H_0(T)$ is finite.*

Proof. Assume that $H_0(T)$ is closed. Since T is semi-Fredholm, the Kato decomposition provides two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, X_1 is finite-dimensional, $T_{|X_1}$ is nilpotent and $T_{|X_2}$ is semi-regular. Therefore $X_1 \subseteq H_0(T)$ and $H_0(T) = X_1 \oplus H_0(T) \cap X_2$. Since $H_0(T) \cap X_2 = H_0(T_{|X_2})$ is closed and $T_{|X_2}$ is semi-regular, it follows that $H_0(T) \cap X_2 = \{0\}$ (see [20]). Thus $H_0(T) = X_1$ is finite-dimensional. The other implication is trivial. ∎

REMARK. Let $T \in \mathcal{L}(X)$. As immediate consequences of Theorem 2.4 and Lemma 2.6 we derive the following well known assertions:

(i) $\Pi_0(T) = \operatorname{iso} \sigma(T) \cap \varrho_{\rm e}(T) \subseteq \Pi_{00}(T)$, where $\varrho_{\rm e}(T) = \mathbb{C} \setminus \sigma_{\rm e}(T)$.

(ii) $\sigma_{\rm b}(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T)$.

(iii) If Weyl's theorem holds for T then so does Browder's theorem. Indeed, if T satisfies Weyl's theorem, then $\Pi_{00}(T) \subseteq \mathbb{C} \setminus \sigma_{\rm w}(T) \subseteq \varrho_{\rm e}(T)$ and so $\Pi_{00}(T) \subseteq \operatorname{iso} \sigma(T) \cap \varrho_{\rm e}(T) = \Pi_0(T)$. Thus, $\Pi_0(T) = \Pi_{00}(T)$ and $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$.

We recall that if $T \in \mathcal{L}(X)$ and $H_0(T)$ is closed, then $T_{|H_0(T)}$ is quasi-nilpotent (see [23]).

LEMMA 2.7. *Let $T \in \mathcal{L}(X)$ be a non-invertible Fredholm operator of index 0. Then $H_0(T)$ is closed if and only if $0 \in \Pi_0(T)$.*

Proof. Suppose that $H_0(T)$ is closed. Then $T_{|H_0(T)}$ is quasi-nilpotent. Moreover, from Lemma 2.6 it follows that $H_0(T)$ is finite-dimensional, and hence $T_{|H_0(T)}$ is nilpotent. Therefore, there exists an integer $d \geq 1$ for which $H_0(T) = N(T)^d$, and consequently, T has finite ascent. Since T^d and T^{d+1} are Fredholm of index 0, we get $\operatorname{codim} R(T)^d = \dim N(T)^d = \dim N(T)^{d+1} = \operatorname{codim} R(T)^{d+1}$, which implies that $R(T)^d = R(T)^{d+1}$. Now, T has finite ascent and descent, so $0 \in \operatorname{iso} \sigma(T) \cap \varrho_{\mathrm{e}}(T) = \Pi_0(T)$. Conversely, if $0 \in \Pi_0(T)$, then $H_0(T)$ is closed by Theorem 2.4. ∎

Let $T \in \mathcal{L}(X)$. We denote by $\sigma_{\mathrm{p}}^{\mathrm{f}}(T)$ the set of all eigenvalues of T of finite multiplicity; evidently $\Pi_{00}(T) \subseteq \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$.

PROPOSITION 2.8. *Let T be a bounded operator on X.*

(i) *If $H_0(T-\lambda)$ is closed for every $\lambda \in \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, then T satisfies Browder's theorem.*

(ii) *If $H_0(T-\lambda)$ is finite-dimensional for every $\lambda \in \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, then T satisfies Weyl's theorem.*

Proof. (i) Since $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, it follows from the preceding lemma that $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) \subseteq \Pi_0(T)$, and hence $\sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \Pi_0(T) \subseteq \sigma_{\mathrm{w}}(T)$. The other inclusion is obvious. Therefore, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$ and Browder's theorem holds for T.

(ii) From part (i), T satisfies Browder's theorem, and hence it suffices to show that $\Pi_{00}(T) = \Pi_0(T)$. Let $\lambda \in \Pi_{00}(T)$. Then $\lambda \in \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, and by hypothesis, $H_0(T-\lambda)$ is finite-dimensional. Moreover, by Theorem 2.4, we have $X = H_0(T-\lambda) \oplus K(T-\lambda)$, and since the restriction of $T-\lambda$ to $K(T-\lambda)$ is invertible, we deduce that $T-\lambda$ is Fredholm. Thus $\lambda \in \Pi_0(T)$ by part (i) of the preceding Remark. The other inclusion is clear, therefore T satisfies Weyl's theorem. ∎

In general, Weyl's theorem need not hold for an operator satisfying the SVEP:

EXAMPLE 1. Consider the operator T_1 defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T_1(x_1, x_2, \ldots) = (x_2/2, x_3/3, \ldots)$. Then T_1 is quasi-nilpotent, and hence has the SVEP, and $\Pi_{00}(T_1) = \{0\}$. Consequently, T_1 does not satisfy Weyl's theorem, because $\sigma(T_1) \setminus \Pi_{00}(T_1) = \emptyset$ and $\sigma_{\mathrm{w}}(T_1) = \{0\}$.

However, for Browder's theorem we have the following result.

THEOREM 2.9. *If $T \in \mathcal{L}(X)$ or its adjoint has the SVEP, then Browder's theorem holds for $f(T)$, for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Let us show first that Browder's theorem holds for T. Let $\lambda_0 \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Then $T-\lambda_0$ is Fredholm and by the Kato decomposition we can choose $\delta > 0$ for which $U := \{\lambda \in \mathbb{C} : 0 < |\lambda - \lambda_0| < \delta\} \subseteq \text{s-reg}(T) \cap \varrho_{\mathrm{e}}(T)$.

From the continuity of the index and the fact that $\operatorname{ind}(T-\lambda_0)=0$, it follows that

$$\operatorname{ind}(T-\lambda)=0 \quad \text{if } |\lambda-\lambda_0|<\delta. \tag{2.1}$$

If T (resp. T^*) has the SVEP then $\text{s-reg}(T)=\varrho_{\text{ap}}(T)$ (resp. $\text{s-reg}(T)=\varrho_{\text{su}}(T)$) and so $U\subseteq\varrho(T)$, by (2.1). Hence $\lambda_0\in\operatorname{iso}\sigma(T)\cap\varrho_{\text{e}}(T)=\Pi_0(T)$. The other inclusion is trivial. Thus Browder's theorem holds for T.

Now, if $f\in\mathcal{H}(\sigma(T))$ then Theorem 2.3 implies that

$$\sigma_{\text{w}}(f(T))=f(\sigma_{\text{w}}(T))=f(\sigma_{\text{b}}(T))=\sigma_{\text{b}}(f(T)).$$

Therefore Browder's theorem holds for $f(T)$. ∎

In [2], Barnes showed that Browder's theorem holds for an operator if and only if it holds for its adjoint. Therefore, we can add to the conclusion of Theorem 2.9 that also $f(T^*)$ satisfies Browder's theorem.

It is interesting to note that in contrast to Browder's theorem, Weyl's theorem does not pass from an operator to its adjoint even if it has the SVEP. Indeed, if we consider the operator T_2 on $\ell^2(\mathbb{N})$ defined by $T_2(x_1,x_2,\ldots)=(0,x_1/2,x_2/3,\ldots)$, then T_2 is the adjoint of the operator T_1 introduced in Example 1. Being quasi-nilpotent, T_2 has the SVEP, and since $\Pi_{00}(T_2)=\emptyset$, it follows that T_2 satisfies Weyl's theorem. However, as we have shown, $T_2^*=T_1$ does not satisfy Weyl's theorem.

COROLLARY 2.10. *If $T\in\mathcal{L}(X)$ or its adjoint has the SVEP, then*

(i) *Weyl's theorem holds for T if and only if $\Pi_0(T)=\Pi_{00}(T)$,*
(ii) *Weyl's theorem holds for T^* if and only if $\Pi_0(T^*)=\Pi_{00}(T^*)$.*

Proof. This is a straightforward consequence of Theorem 2.9, [2, Theorem 6] and the equivalences (1.1) and (1.2). ∎

The following lemma, which is required to obtain a useful characterizations of Riesz points, was established in [3], and we give here the proof for completeness.

LEMMA 2.11. *Let $T\in\mathcal{L}(X)$ be an operator of finite descent. The following assertions are equivalent*:

(i) *$H_0(T)$ is closed.*
(ii) *There exists a positive integer p for which $H_0(T)=N(T^p)$.*
(iii) *T has finite ascent.*

Proof. (i)⇒(ii). Assume that $H_0(T)$ is closed. Then $T_0=T_{|H_0(T)}$ is quasi-nilpotent. On the other hand, $R(T)^d=R(T)^{d+1}$ where $d=d(T)$, hence if $x\in R(T_0)^d$ then $x=T^dy=T^{d+1}z$, for $y\in H_0(T)$ and $z\in X$, therefore $y-Tz\in N(T)^d\subseteq H_0(T)$ and so $Tz\in H_0(T)$, i.e. $z\in H_0(T)$; this implies that $R(T_0)^d=R(T_0)^{d+1}$. It follows that T_0 is a quasi-nilpotent operator of finite descent, and hence, by Corollary 10.6 of [29], T_0 is nilpotent.

(ii)⇒(i) and (ii)⇒(iii) are obvious.

(iii)⇒(ii). If $a(T)$ is finite, we have $X = N(T)^p \oplus R(T)^p$, where $p = a(T) = d(T)$, and $R(T)^p$ is closed. Since $T_{|R(T)^p}$ is invertible, we have $H_0(T_{|R(T)^p}) = \{0\}$, and hence $H_0(T) = N(T)^p$. ∎

For an operator $T \in \mathcal{L}(X)$, the *reduced modulus* is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in X \text{ and } d(x, N(T)) = 1\};$$

obviously $\gamma(T) > 0$ if and only if $R(T)$ is closed, and $\gamma(T) = \|T^{-1}\|^{-1}$ if T is invertible (see [14]).

PROPOSITION 2.12. *For $T \in \mathcal{L}(X)$, the following conditions are equivalent*:

(i) $\lambda \in \Pi_0(T)$.
(ii) $\lambda \in \Pi_{00}(T)$ *and* $R(T-\lambda)$ *is closed.*
(iii) $\lambda \in \operatorname{iso}\sigma(T)$ *and* $H_0(T-\lambda)$ *is finite-dimensional.*
(iv) $\lambda \in \operatorname{iso}\sigma(T)$ *and* $K(T-\lambda)$ *is finite-codimensional.*
(v) $\lambda \in \Pi_{00}(T)$ *and there is* $d \geq 1$ *for which* $H_0(T-\lambda) = N(T-\lambda)^d$.
(vi) $\lambda \in \Pi_{00}(T)$ *and there is* $d \geq 1$ *for which* $K(T-\lambda) = R(T-\lambda)^d$.
(vii) $\lambda \in \Pi_{00}(T)$ *and* $T-\lambda$ *has finite descent.*
(viii) $\lambda \in \Pi_{00}(T)$ *and* λ *is a pole of the resolvent of* T.
(ix) $\lambda \in \Pi_{00}(T)$ *and* γ *is discontinuous at* λ.

Proof. Without loss of generality, we may consider $\lambda = 0$.

(i)⇒(ii). This is straightforward.

(ii)⇒(iii). If we assume (ii), it follows that T is semi-Fredholm and $H_0(T)$ is closed. Thus $\dim H_0(T)$ is finite, by Lemma 2.6.

(iii)⇔(iv). This follows easily from Theorem 2.4.

(iii)⇒(v). Since $\dim H_0(T)$ is finite, $T_{|H_0(T)}$ is nilpotent and hence $H_0(T) = N(T)^d$ for some integer $d \geq 1$. Moreover, $H_0(T)$ is not trivial, because $0 \in \operatorname{iso}\sigma(T)$, therefore $N(T)$ is non-zero and finite-dimensional. Thus $0 \in \Pi_{00}(T)$.

(v)⇒(vi). This is an immediate consequence of Theorem 2.4 and the fact that $T(K(T)) = K(T)$.

(vi)⇒(vii). If $R(T)^d = K(T)$ then $R(T)^{d+1} = T(R(T)^d) = T(K(T)) = K(T) = R(T)^d$.

(vii)⇒(viii). We have $H_0(T)$ closed and $d(T)$ finite, and Lemma 2.11 shows that $a(T)$ is finite. Thus, 0 is a pole of the resolvent by Theorem 10.2 of [29].

(viii)⇒(i). By Theorem 2.4, $X = N(T-\lambda)^d \oplus R(T-\lambda)^d$ and $R(T-\lambda)^d$ is closed. Since $N(T-\lambda)$ is finite-dimensional, we can show easily that also $N(T-\lambda)^d$ is finite-dimensional, therefore $(T-\lambda)^d$ is Fredholm. Thus, $T-\lambda$ is Fredholm and $\lambda \in \varrho_e(T) \cap \operatorname{iso}\sigma(T) = \Pi_0(T)$.

(ii)⇒(ix). If we assume (ii), and hence (v) and (vi), then $N(T)$ is nonzero and $N(T) \cap R(T)^d = \{0\}$. Therefore T is not semi-regular, and Theorem 4.1 of [22] ensures that γ is discontinuous at 0.

(ix)⇒(ii). Let $\delta > 0$ be such that $T - \lambda$ is invertible if $0 < |\lambda| < \delta$. We have $\gamma(T-\lambda) = \|(T-\lambda)^{-1}\|^{-1}$ for $0 < |\lambda| < \delta$. Since T is not invertible, $\lim_{\lambda\to 0} \gamma(T-\lambda) = 0$. But γ is discontinuous at 0, thus $\gamma(T) > 0$, and hence $R(T)$ is closed. ∎

The following theorem follows immediately from Corollary 2.10 and Proposition 2.12.

THEOREM 2.13. *For $T \in \mathcal{L}(X)$ such that T or its adjoint has the SVEP, the following conditions are equivalent*:

(i) *Weyl's theorem holds for T.*
(ii) *$R(T-\lambda)$ is closed for every $\lambda \in \Pi_{00}(T)$.*
(iii) *$H_0(T-\lambda)$ is finite-dimensional for every $\lambda \in \Pi_{00}(T)$.*
(iv) *$K(T-\lambda)$ is finite-codimensional for every $\lambda \in \Pi_{00}(T)$.*
(v) *For every $\lambda \in \Pi_{00}(T)$, there exists $d \geq 1$ for which $H_0(T-\lambda) = N(T-\lambda)^d$.*
(vi) *For every $\lambda \in \Pi_{00}(T)$, there exists $d \geq 1$ for which $K(T-\lambda) = R(T-\lambda)^d$.*
(vii) *$T-\lambda$ has finite descent for every $\lambda \in \Pi_{00}(T)$.*
(viii) *Every $\lambda \in \Pi_{00}(T)$ is a pole of the resolvent of T.*
(ix) *γ is discontinuous at every $\lambda \in \Pi_{00}(T)$.* ∎

For an operator satisfying the SVEP, the equivalence between (i), (ii), (iii) and (ix) has been established recently by R. Curto and Y. Han in [8]. However, the arguments we have used are different.

We recall that an operator $T \in \mathcal{L}(X)$ is said to be *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T.

COROLLARY 2.14. *Let T be a bounded operator on X for which there exist an integer $d \geq 1$ and a constant $c > 0$ such that*

$$\|(T-\lambda)^{-1}\| \leq \frac{c}{\operatorname{dist}(\lambda, \sigma(T))^d} \quad \text{for all } \lambda \notin \sigma(T). \tag{2.2}$$

If T or T^* has the SVEP then Weyl's theorem holds for $f(T)$, for every $f \in \mathcal{H}(\sigma(T))$.

Proof. First we claim that for all $\mu \in \operatorname{iso}\sigma(T)$, $H_0(T-\mu) = N(T-\mu)^d$. To show this, let $\mu \in \operatorname{iso}\sigma(T)$. Then $H_0(T-\mu)$ is closed, by Theorem 2.4. Let T_0 be the restriction of T to $H_0(T-\mu)$. From (2.2), we can easily see that for λ in a small deleted neighbourhood of μ, we have

$$|\lambda-\mu|^d \|(T_0-\lambda)^{-1}\| \leq c. \tag{2.3}$$

Consider a sufficiently small circle C of radius ε and centre μ. Since $T_0 - \mu$ is quasi-nilpotent, the Riesz functional calculus gives

$$(T_0 - \mu)^d = \frac{1}{2\pi i} \int_C (\lambda - \mu)^d (\lambda - T_0)^{-1}\, d\lambda,$$

and hence, by (2.3), $\|(T_0 - \mu)^d\| \le c\varepsilon$, for every small $\varepsilon > 0$. This implies that $H_0(T-\mu) = N(T_0-\mu)^d \subseteq N(T-\mu)^d \subseteq H_0(T-\mu)$, and so $H_0(T-\mu) = N(T-\mu)^d$. Therefore, we deduce that T is isoloid and satisfies Weyl's theorem, by assertion (v) of the preceding theorem. Consequently, by Theorem 2.3 and [28, Proposition 2], it follows that, for $f \in \mathcal{H}(\sigma(T))$,

$$\sigma_{\mathrm{w}}(f(T)) = f(\sigma(T_w)) = f(\sigma(T) \setminus \Pi_{00}(T)) = \sigma(f(T)) \setminus \Pi_{00}(f(T)),$$

and therefore $f(T)$ obeys Weyl's theorem. ■

3. Applications. Now let us introduce the class $\mathcal{P}(X)$ formed by the operators $T \in \mathcal{L}(X)$ such that for every complex number λ there exists an integer $d_\lambda \ge 1$ for which $H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}$. This class is considerably large, it contains every totally paranormal and subscalar operator, and consequently, every M-hyponormal, p-hyponormal and log-hyponormal operator; see the examples given at the end of this section. In [1] (see also [16]), it was shown that if for an operator $T \in \mathcal{L}(X)$, $H_0(T-\lambda)$ is closed for every complex number λ, then T has the SVEP. Therefore, the SVEP is shared by all the operators in $\mathcal{P}(X)$.

The main result of this section is the following:

THEOREM 3.1. *Let $T \in \mathcal{L}(X)$. If there exists a function $h \in \mathcal{H}(\sigma(T))$ not identically constant in any connected component of its domain, and such that $h(T) \in \mathcal{P}(X)$, then Weyl's theorem holds for both $f(T)$ and $f(T^*)$, for every $f \in \mathcal{H}(\sigma(T))$.*

We note that Theorem 3.1 may fail if the function h is not assumed to be not identically constant in any connected component of its domain. To show this, let T_1 be the quasi-nilpotent operator on $\ell^2(\mathbb{N})$ introduced in Example 1; then T_1 does not satisfy Weyl's theorem. However, if we let h be the identity function on $\mathbb{C}$, then $h(T_1) = I$ belongs to $\mathcal{P}(\ell^2(\mathbb{N}))$.

Before giving the proof of Theorem 3.1, some results are to be considered first.

We recall that an operator $T \in \mathcal{L}(X)$ is called a *quasi-affine transform* of $S \in \mathcal{L}(X)$ if there exists $A \in \mathcal{L}(X)$, injective with dense range, such that $SA = AT$; we then write $T \prec S$. If $T \prec S$ and $S \prec T$ then we say that T and S are *quasi-similar*.

LEMMA 3.2. *Let T and S be bounded operators on X.*

(i) *If $S \in \mathcal{P}(X)$ and $T \prec S$ then $T \in \mathcal{P}(X)$.*

(ii) *If $T \in \mathcal{P}(X)$ and Y is a closed T-invariant subspace of X, then $T_{|Y} \in \mathcal{P}(Y)$.*

Proof. (i) Let $A \in \mathcal{L}(X)$ be injective with dense range such that $AT = SA$, and let $x \in H_0(T-\lambda)$. We have

$$\|(S-\lambda)^n Ax\|^{1/n} = \|A(T-\lambda)^n x\|^{1/n} \le \|A\|^{1/n}\|(T-\lambda)^n x\|^{1/n},$$

hence $\lim_{n\to\infty}\|(S-\lambda)^n Ax\|^{1/n} = 0$ and $Ax \in H_0(S-\lambda)$. Since $S \in \mathcal{P}(X)$, $H_0(S-\lambda) = N(S-\lambda)^d$ for some integer $d \ge 1$, therefore $A(T-\lambda)^d x = (S-\lambda)^d Ax = 0$, and so $x \in N(T-\lambda)^d$, because A is injective. Thus, $H_0(T-\lambda) = N(T-\lambda)^d$ and $T \in \mathcal{P}(X)$.

(ii) Let $\lambda \in \mathbb{C}$. There exists a positive integer d such that $H_0(T-\lambda) = N(T-\lambda)^d$. Since

$$H_0(T_{|Y}-\lambda) \subseteq H_0(T-\lambda) \cap Y = N(T-\lambda)^d \cap Y = N(T_{|Y}-\lambda)^d,$$

we get $H_0(T_{|Y}-\lambda) = N(T_{|Y}-\lambda)^d$, as desired. ∎

The first assertion of the next proposition is a special case of Theorem 3.1, and it will be required for proving that theorem.

PROPOSITION 3.3. *For $T \in \mathcal{P}(X)$, the following assertions hold*:

(i) *Weyl's theorem holds for both T and T^*.*

(ii) *If Y is a T-invariant closed subspace of X, then Weyl's theorem holds for $T_{|Y}$.*

Proof. (i) Since T has the SVEP, Theorem 2.13 implies that Weyl's theorem holds for T. Let us show that T^* satisfies Weyl's theorem. By Corollary 2.10, it suffices to prove that $\Pi_{00}(T^*) = \Pi_0(T^*)$. Let $\lambda \in \Pi_{00}(T^*)$ and let $d \ge 1$ be such that $H_0(T-\lambda) = N(T-\lambda)^d$. It follows that $\lambda \in \operatorname{iso}\sigma(T)$ and $X = N(T-\lambda)^d \oplus K(T-\lambda)$, hence $R(T-\lambda)^d = (T-\lambda)^d(K(T-\lambda)) = K(T-\lambda)$ is closed, and so is $R(T^*-\lambda)^d$. On the other hand, since $N(T^*-\lambda)$ is finite-dimensional, we can show by an easy inductive argument that also $N(T^*-\lambda)^d$ is finite-dimensional. Therefore $(T^*-\lambda)^d$ is semi-Fredholm, and hence so is $T^*-\lambda$. Finally, $H_0(T^*-\lambda)$ is closed, because $\lambda \in \operatorname{iso}\sigma(T^*)$; then by Lemma 2.6 we deduce that $H_0(T^*-\lambda)$ is finite-dimensional. Thus $\lambda \in \Pi_0(T^*)$, by Proposition 2.12. The other inclusion is clear.

(ii) Straightforward from Lemma 3.2 and assertion (i). ∎

The following example shows that assertion (ii) of the preceding proposition may fail for an arbitrary operator even if it has the SVEP.

EXAMPLE 2. Let T_1 be as in Example 1, and let S be the unilateral left shift on $\ell^2(\mathbb{N})$ given by $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Define T on $X := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = T_1 \oplus S$. Now, T_1 is quasi-nilpotent, and since S has the SVEP and $\sigma(S)$ is the closed unit disc (see [17]), it follows that also T has the SVEP and $\sigma(T)$ is the closed unit disc. Therefore T does not have any isolated point, and hence, by Corollary 2.10, Weyl's theorem holds for T. However, the restriction T_1 of T does not satisfy Weyl's theorem.

In the next result, we establish one of the closure properties of the class $\mathcal{P}(X)$.

THEOREM 3.4. *For $T \in \mathcal{L}(X)$, the following assertions are equivalent*:

(i) $T \in \mathcal{P}(X)$.

(ii) $f(T) \in \mathcal{P}(X)$ *for every* $f \in \mathcal{H}(\sigma(T))$.

(iii) *There exists a function $f \in \mathcal{H}(\sigma(T))$ not identically constant in any component of its domain such that $f(T) \in \mathcal{P}(X)$.*

To prove this theorem, we need the following lemma:

LEMMA 3.5. *Let T be a bounded operator on X, and let p be a complex polynomial.*

(i) *If λ_0 is a complex number such that $p(\lambda_0) \neq 0$, then*

$$H_0(T-\lambda_0) \cap N(p(T)) = \{0\}.$$

(ii) *If, in addition, T has the SVEP, then*

$$H_0(p(T)) = H_0(T-\lambda_1) \oplus H_0(T-\lambda_2) \oplus \ldots \oplus H_0(T-\lambda_n),$$

where $\lambda_1, \ldots, \lambda_n$ are the distinct roots of the polynomial p.

Proof. (i) Suppose that there exists a non-zero element x in $H_0(T-\lambda_0) \cap N(p(T))$, and let $p(\lambda_0) - p(T) = q(T)(\lambda_0 - T)$ where q is a polynomial. It follows that $q(T)(\lambda_0 - T)x = p(\lambda_0)x$, and hence, for all n, $[q(T)(\lambda_0 - T)]^n x = p(\lambda_0)^n x$. Therefore

$$|p(\lambda_0)|\, \|x\|^{1/n} \leq \|q(T)^n\|^{1/n} \|(T-\lambda_0)^n x\|^{1/n} \quad \text{for all } n \geq 0.$$

Since x is a non-zero vector of $H_0(T-\lambda_0)$, we obtain $p(\lambda_0) = 0$, the desired contradiction.

(ii) If $x \in H_0(p(T))$, then, by Proposition 1.3 of [21], there exists an analytic function f such that $x = (p(T)-\mu)f(\mu)$ for $\mu \in \mathbb{C} \setminus \{0\}$. Hence, for $\lambda \in \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_n\}$,

$$x = (p(T) - p(\lambda))f(p(\lambda)) = (T-\lambda)Q(T,\lambda)f(p(\lambda)),$$

where Q is a polynomial of T and λ. Consequently, $\sigma_T(x) \subseteq \{\lambda_1, \ldots, \lambda_n\}$, and so

$$x \in X_T(\{\lambda_1, \ldots, \lambda_n\}) = \bigoplus_{i=1}^{n} X_T(\{\lambda_i\}),$$

by part (g) of [17, Proposition 1.2.16]. Since T has the SVEP, Proposition 1.3 of [21] implies that $X_T(\{\lambda_i\}) = H_0(T-\lambda_i)$. Therefore $H_0(p(T)) \subseteq \bigoplus_{i=1}^{n} H_0(T-\lambda_i)$. Conversely, since each λ_i is a root of p, we can easily show that $H_0(T-\lambda_i) \subseteq H_0(p(T))$ for $1 \leq i \leq n$. ■

Proof of Theorem 3.4. (i)⇒(ii). Suppose that $T \in \mathcal{P}(X)$, and consider an analytic function f on Ω, where $\sigma(T) \subseteq \Omega$. Let α be an arbitrary complex number. If $\alpha \notin f(\sigma(T)) = \sigma(f(T))$, then $f(T) - \alpha$ is invertible and hence $H_0(f(T) - \alpha) = N(f(T) - \alpha) = \{0\}$. Therefore, we may assume that $\alpha \in f(\sigma(T))$. Let $g := f - \alpha$ on Ω. Suppose first that g has only finitely many zeros in $\sigma(T)$. Then $g(\lambda) = p(\lambda)h(\lambda)$, where h is analytic on Ω and without zeros in $\sigma(T)$, while p is a polynomial of the form $p(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)^{\alpha_i}$ with distinct roots $\lambda_1, \ldots, \lambda_n \in \sigma(T)$. It follows that $g(T) = p(T)h(T)$ and $h(T)$ is invertible, so that

$$H_0(g(T)) = H_0(p(T)) = \bigoplus_{i=1}^{n} H_0(T - \lambda_i),$$

by part (ii) of the preceding lemma. On the other hand, since $T \in \mathcal{P}(X)$, we can choose a positive integer d such that $H_0(T - \lambda_i) = N(T - \lambda_i)^d$ for $1 \le i \le n$. In particular, for $1 \le i \le n$, this implies that $T - \lambda_i$ has finite ascent, therefore $H_0(T - \lambda_i) = N(T - \lambda_i)^{d\alpha_i}$. Finally, we get

$$\begin{aligned} H_0(g(T)) &= \bigoplus_{i=1}^{n} N(T - \lambda_i)^{d\alpha_i} = N\Big(\prod_{i=1}^{n}(T - \lambda_i)^{d\alpha_i}\Big) \\ &= N(p(T)^d) = N(g(T)^d). \end{aligned}$$

Thus $g(T) \in \mathcal{P}(X)$.

Now if g has infinitely many zeros in $\sigma(T)$, then there exist two disjoint open subsets of $\mathbb{C}$, Ω_1 and Ω_2, such that $\Omega = \Omega_1 \cup \Omega_2$, $g = 0$ on Ω_1, and g has only finitely many zeros on Ω_2. It follows that $\sigma(T) = F_1 \cup F_2$, where F_1 and F_2 are two closed disjoint subsets of $\mathbb{C}$ and $F_i \subseteq \Omega_i$ for $i = 1, 2$. Therefore, the spectral decomposition provides two closed T-invariant subspaces X_1, X_2 for which $X = X_1 \oplus X_2$, $\sigma(T_{|X_1}) = F_1$ and $\sigma(T_{|X_2}) = F_2$; in particular $g(T)_{|X_1} = g(T_{|X_1}) = 0$. Since $T_{|X_2} \in \mathcal{P}(X_2)$, by Lemma 3.2, and g has only finitely many zeros in $\sigma(T_{|X_2})$, the same argument as above leads to $g(T_{|X_2}) \in \mathcal{P}(X_2)$, and consequently, $H_0(g(T)_{|X_2}) = N(g(T)^k_{|X_2})$ for some integer $k \ge 1$. Finally,

$$H_0(f(T) - \alpha) = H_0(g(T)) = X_1 \oplus N(g(T)^k_{|X_2}) = N(g(T)^k) = N(f(T) - \alpha)^k,$$

which completes the proof of (i)⇒(ii).

(ii)⇒(iii) is obvious.

(iii)⇒(i). Consider $\lambda_0 \in \sigma(T)$ and let $\alpha = f(\lambda_0)$. Since f is non-constant on each connected component of its domain, it follows that $f(\lambda) - \alpha = (\lambda - \lambda_0)^r p(\lambda) g(\lambda)$, where p is a complex polynomial such that $p(\lambda_0) \ne 0$ and g is an analytic function which does not vanish in $\sigma(T)$. Therefore, $f(T) - \alpha = (T - \lambda_0)^r p(T) g(T)$ and $g(T)$ is invertible. On the other hand, by hypothesis, there exists $d \ge 1$ such that $H_0(f(T) - \alpha) = N(f(T) - \alpha)^d$.

Hence

$$H_0(T-\lambda_0) \subseteq H_0(f(T)-\alpha) = N(T-\lambda_0)^{dr} \oplus N(p(T)^d),$$

and since $N(T-\lambda_0)^{dr} \subseteq H_0(T-\lambda_0)$ and $H_0(T-\lambda_0) \cap N(p(T)^d) = \{0\}$, by part (i) of the preceding lemma, we conclude that $H_0(T-\lambda_0) = N(T-\lambda_0)^{dr}$, as desired. ∎

Proof of Theorem 3.1. From Theorem 3.4, it follows that $f(T) \in \mathcal{P}(X)$ for every $f \in \mathcal{H}(\sigma(T))$, and therefore Weyl's theorem holds for both $f(T)$ and $f(T^*)$, by Proposition 3.3(i). ∎

COROLLARY 3.6. *If $T \in \mathcal{P}(X)$ then both $f(T)$ and $f(T^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

As an immediate consequence of Lemma 3.2 and Corollary 3.6, we have:

COROLLARY 3.7. *If $T \in \mathcal{P}(X)$, then for every $S \in \mathcal{L}(X)$ such that $S \prec T$, Weyl's theorem holds for both $f(S)$ and $f(S^*)$ whenever $f \in \mathcal{H}(\sigma(T))$.*

As we have shown in Example 2, Weyl's theorem does not pass from an operator to its restriction to a closed invariant subspace. However, for the class $\mathcal{P}(X)$, by Proposition 3.3 and Corollary 3.6, we have:

COROLLARY 3.8. *Let $T \in \mathcal{P}(X)$. If Y is a T-invariant closed subspace of X, then Weyl's theorem holds for $f(T_{|Y})$ and $f(T_{|Y})^*$, for every $f \in \mathcal{H}(\sigma(T))$.*

We end this section by these examples:

EXAMPLE 3. We show that $\mathcal{P}(X)$ contains every subscalar operator. First, recall from [7] and [17] that an operator $T \in \mathcal{L}(X)$ is said to be *generalized scalar* if there exists a continuous algebra homomorphism $\Phi : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{L}(X)$ such that $\Phi(1) = I$ and $\Phi(Z) = T$, where $\mathcal{C}^\infty(\mathbb{C})$ denotes the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$, and Z stands for the identity function on $\mathbb{C}$. An operator similar to the restriction of a generalized scalar operator to a closed invariant subspace is called *subscalar*. It is well known that a subscalar operator has Dunford's property (C).

Now, to show that a subscalar operator $T \in \mathcal{L}(X)$ belongs to $\mathcal{P}(X)$, by Lemma 3.2 we may assume that T is generalized scalar. Consider a continuous algebra homomorphism $\Phi : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{L}(X)$ such that $\Phi(1) = I$ and $\Phi(Z) = T$, and let $\lambda \in \mathbb{C}$. Since T satisfies Dunford's condition (C), it follows that T has the SVEP and $H_0(T-\lambda) = X_T(\{\lambda\})$ is closed (see [21]). On the other hand, for $f \in \mathcal{C}^\infty(\mathbb{C})$, $\Phi(f)(H_0(T-\lambda)) \subseteq H_0(T-\lambda)$, because $T = \Phi(Z)$ commutes with $\Phi(f)$. Now, if we consider the continuous algebra homomorphism $\widetilde{\Phi} : \mathcal{C}^\infty(\mathbb{C}) \to \mathcal{L}(H_0(T-\lambda))$ defined by $\widetilde{\Phi}(f) = \Phi(f)_{|H_0(T-\lambda)}$ for $f \in \mathcal{C}^\infty(\mathbb{C})$, we deduce that $T_{|H_0(T-\lambda)}$ is generalized subscalar. Hence, by

[17, Proposition 1.5.10], $T_{|H_0(T-\lambda)} - \lambda$ is nilpotent. Thus there exists $d \geq 1$ such that $H_0(T-\lambda) = N(T-\lambda)^d$.

The examples that follow have received a systematic treatment in the literature; we recapture most of the results that have been established.

Example 4. If $T \in \mathcal{L}(X)$ is a spectral operator of finite type (see [7] and [24]), then T is generalized scalar, by Theorem 3.6 of [7], hence $T \in \mathcal{P}(X)$, and consequently, Weyl's theorem holds for $f(T)$ and $f(T^*)$, for every $f \in \mathcal{H}(\sigma(T))$.

Example 5. Let $T \in \mathcal{L}(X)$ be a *totally paranormal operator* (see [15]), i.e. $\|(T-\lambda)x\|^2 \leq \|(T-\lambda)^2 x\| \, \|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in X$. By an easy inductive argument we see that $\|(T-\lambda)x\|^n \leq \|(T-\lambda)^n x\| \, \|x\|^{n-1}$ for all $x \in X$, $\lambda \in \mathbb{C}$ and $n \geq 1$. Hence $H_0(T-\lambda) = N(T-\lambda)$ for all $\lambda \in \mathbb{C}$, and thus $T \in \mathcal{P}(X)$. Therefore, by Theorem 3.4, $\mathcal{P}(X)$ contains the class of *algebraically totally paranormal operators* defined as those operators $S \in \mathcal{L}(X)$ for which there exists a non-constant polynomial p such that $p(T)$ is totally paranormal.

Let H denote a complex Hilbert space.

Example 6. If $T \in \mathcal{L}(H)$ is *M-hyponormal* (see [10]), i.e. there exists $M > 0$ such that $TT^* \leq MT^*T$, then it follows from Proposition 2.4.9 of [17] that T is subscalar and so $T \in \mathcal{P}(H)$.

Example 7. If $T \in \mathcal{L}(H)$ is *p-hyponormal* (see [10]), i.e. there exists $0 < p \leq 1$ such that $(T^*T)^p \geq (TT^*)^p$ (if $p = 1$, then T is called *hyponormal*), then T is subscalar, by [19, Corollary 2]. Therefore, by Theorem 3.4, $T \in \mathcal{P}(H)$. More generally, $\mathcal{P}(H)$ contains the class of *algebraically p-hyponormal operators*, i.e. those $S \in \mathcal{L}(H)$ for which there exists a non-constant complex polynomial q such that $q(S)$ is p-hyponormal.

Example 8. If T is *log-hyponormal* (see [10]), T is invertible and $\log(T^*T) \geq \log(TT^*)$, then T is subscalar, by [19, Corollary 2], and so $T \in \mathcal{P}(H)$.

In [15], [13] (see also [9]), it was established that if T is algebraically totally paranormal or algebraically hyponormal, then Weyl's theorem holds for $f(T)$, for every $f \in \mathcal{H}(\sigma(T))$. In fact, Theorem 3.1 generalizes these results.

Acknowledgments. I wish to thank Professors Mostafa Mbekhta and Mohamed Zohry for interesting discussions as well as their useful suggestions for the improvement of this paper.

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Received May 6, 2003
Revised version June 21, 2003 (5197)