Weighted measure algebras and uniform norms

by

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Abstract. Let $\omega$ be a weight on an LCA group $G$. Let $M(G, \omega)$ consist of the Radon measures $\mu$ on $G$ such that $\omega \mu$ is a regular complex Borel measure on $G$. It is proved that: (i) $M(G, \omega)$ is regular iff $M(G, \omega)$ has unique uniform norm property (UUNP) iff $L^1(G, \omega)$ has UUNP and $G$ is discrete; (ii) $M(G, \omega)$ has a minimum uniform norm iff $L^1(G, \omega)$ has UUNP; (iii) $M_00(G, \omega)$ is regular iff $M_00(G, \omega)$ has UUNP iff $L^1(G, \omega)$ has UUNP, where $M_00(G, \omega) := \{ \mu \in M(G, \omega) : \hat{\mu} = 0 \text{ on } \Delta(M(G, \omega)) \setminus \Delta(L^1(G, \omega))\}$.

1. Introduction. A uniform norm on a Banach algebra $(A, \| \cdot \|)$ is a (not necessarily complete) norm $\| \cdot \|$ on $A$ satisfying the square property $|a^2| = |a|^2$ ($a \in A$). A minimum uniform norm on $A$ is a minimum norm among all uniform norms on $A$. A Banach algebra $A$ has the Unique Uniform Norm Property (UUNP) if $A$ admits exactly one uniform norm [BhDe1]; in this case, the spectral radius $r(\cdot)$ is the only uniform norm on $A$. The perspective of UUNP in Banach algebras is discussed in [BhDe1] and [BhDe2]. By [BhDe2, Theorem 4.1] and [BhDe3], the Beurling algebra $L^1(G, \omega)$ has UUNP if and only if $L^1(G, \omega)$ is regular. On the other hand, either $L^1(G, \omega)$ has exactly one uniform norm or it has infinitely many uniform norms [BhDe4].

The present note is aimed at investigating UUNP in weighted measure algebra $M(G, \omega)$. In what follows, we briefly discuss preliminaries to fix up the notations.

Throughout let $G$ be an LCA group with the Haar measure $\lambda$. Let $\widehat{G}$ be the dual group of $G$. A generalized character on $G$ is a continuous homomorphism $\alpha : G \to (\mathbb{C} \setminus \{0\}, \times)$. Let $H(G)$ denote the set of all generalized characters on $G$ equipped with the compact-open topology. For $\alpha, \beta \in H(G)$, define $(\alpha + \beta)(s) = \alpha(s)\beta(s)$ ($s \in G$). If $G$ is compactly generated, then $H(G)$ is an LCA group [BhDe5]. Let $C_c(G)$ denote the set of all continuous

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functions on $G$ having compact supports equipped with the inductive limit topology $\tau$. Then $(C_c(G), \tau)$ is a commutative topological algebra with the convolution product. For $\alpha \in H(G)$, define

$$\varphi_\alpha(f) = \int_{G} f(s)\alpha(s)\,d\lambda(s) \quad (f \in C_c(G)).$$

Then $\varphi_\alpha \in \Delta(C_c(G))$, the Gelfand space of $C_c(G)$. Let $T : H(G) \to \Delta(C_c(G))$, $\alpha \mapsto \varphi_\alpha$. If $G$ is second countable and compactly generated, then $T$ is a homeomorphism [BhDe5].

By a *weight* on $G$, we mean a continuous function $\omega : G \to (0, \infty)$ such that $\omega(s+t) \leq \omega(s)\omega(t)$ ($s, t \in G$). The associated *Beurling algebra* is the convolution Banach algebra $L^1(G, \omega)$ of all measurable functions $f$ on $G$ satisfying $\|f\|_\omega := \int_G |f|\omega \,d\lambda < \infty$. An element $\alpha \in H(G)$ is an $\omega$-bounded generalized character if $|\alpha(s)| \leq \omega(s)$ ($s \in G$); let $H(G, \omega)$ denote the set of all such elements of $H(G)$. Let $T_1$ be the map $T$ restricted to $H(G, \omega)$. Then $T_1$ is a homeomorphism from $H(G, \omega)$ onto $\Delta(L^1(G, \omega))$ for any LCA group.

2. Weighted measure algebras and UUNP. A *Radon measure* on $G$ is a continuous linear functional on $C_c(G)$. Let $M_{\text{loc}}(G)$ denote the linear space of all Radon measures on $G$ (see [Da]). Let $L_{\text{loc}}(G)$ denote the space of all locally integrable, measurable functions on $G$. Then clearly $L_{\text{loc}}(G) \subseteq M_{\text{loc}}(G)$. Let $M(G)$ be the convolution Banach algebra of all complex regular Borel measures (necessarily finite) on $G$ with the total variation norm $\| \cdot \|$. For a weight $\omega$ on $G$, define

$$M(G, \omega) = \{ \mu \in M_{\text{loc}}(G) : \omega \mu \in M(G) \},$$

$$C_0(G, 1/\omega) = \{ f \in C(G) : f/\omega \in C_0(G) \},$$

$$C_b(G, 1/\omega) = \{ f \in C(G) : f/\omega \in C_b(G) \}.$$  

All are Banach spaces; the first one with norm $\| \mu \|_\omega := \| \omega \mu \|$ and the others with norm $\|f\|_{\omega^{-1}, \infty} := \|\omega^{-1}f\|_{\infty}$. It is clear that every $\mu \in M(G, \omega)$ is a continuous linear functional on $C_b(G, 1/\omega)$ by $\langle f, \mu \rangle = \int_G f/\omega \,d\nu$, where $\nu = \omega \mu \in M(G)$. For $\mu \in M(G, \omega)$ and $f \in C_b(G, 1/\omega)$, define

$$T_f^\mu : G \to \mathbb{C} \quad \text{as} \quad T_f^\mu(s) = \langle \tau_{-s}f, \mu \rangle \quad (s \in G).$$

Then $T_f^\mu \in C_b(G, 1/\omega)$ and $\|T_f^\mu\|_{\omega^{-1}, \infty} \leq \|\mu\|_{\omega} \|f\|_{\omega^{-1}, \infty}$. For $\mu, \nu \in M(G, \omega)$, the convolution product $\mu * \nu$ is defined as

$$\langle f, \mu * \nu \rangle = \langle T_f^\mu, \nu \rangle \quad (f \in C_b(G, 1/\omega)).$$

Then it is a routine verification that $\mu * \nu \in M(G, \omega)$, $\mu * \nu = \nu * \mu$ and $\|\mu * \nu\|_{\omega} \leq \|\mu\|_{\omega}\|\nu\|_{\omega}$. Thus $M(G, \omega)$ is a commutative Banach algebra. The pointmass measure $\delta_0$ is the identity of $M(G, \omega)$. The Beurling algebra
$L^1(G, \omega)$ is a closed ideal in $M(G, \omega)$; and by [Gh, Lemma 2.3], $M(G, \omega)$ is exactly the multiplier algebra of $L^1(G, \omega)$.

Let $\mu \in M_{\text{loc}}(G)$. Define $D_{L\mu} = \{\alpha \in H(G) : \int_G |\alpha(s)| d|\mu|(s) < \infty\}$.

When $D_{L\mu} \neq \emptyset$, $L\mu$ (also denoted by $\hat{\mu}$), defined as

$$(L\mu)(\alpha) = \hat{\mu}(\alpha) = \int_G \alpha(s) d\mu(s) \quad (\alpha \in D_{L\mu}),$$

is the Laplace transform of $\mu$. The following introduces weighted analogues of the classical transforms of harmonic analysis, thereby providing important tools for abelian weighted harmonic analysis.

**Proposition 2.1.** Let $\mu, \nu \in M_{\text{loc}}(G)$.

(i) $D_{L\mu} \neq \emptyset$ iff $\mu \in M(G, \omega)$ for some weight $\omega$ on $G$.

(ii) $D_{L\mu} \cap D_{L\nu} \neq \emptyset$ iff $\mu, \nu \in M(G, \omega)$ for some weight $\omega$ on $G$. In this case, $\mu * \nu \in M(G, \omega)$ and $D_{L\mu * \nu} \neq \emptyset$.

(iii) If $\alpha \in D_{L\mu}$, then $\alpha + \hat{G} \subseteq D_{L\mu}$.

(iv) If $\alpha \in D_{L\mu}$ and if $L\mu = 0$ on $\alpha + \hat{G}$, then $\mu = 0$.

(v) Let $\omega$ be a weight on $G$ and let $\mu \in M(G, \omega)$. Then $H(G, \omega) \subseteq D_{L\mu}$; the restriction of $L\mu$ to $H(G, \omega)$ is the generalized Fourier–Stieltjes transform of $\mu$.

(vi) Let $\omega$ be a weight on $G$ and let $f \in L^1(G, \omega)$. Then $H(G, \omega) \subseteq D_{Lf}$; the restriction of $Lf$ to $H(G, \omega)$ is the generalized Fourier transform of $f$.

(vii) Let $\omega$ be a weight on $G$ such that $\omega \geq 1$ on $G$ and let $\mu \in M(G, \omega)$. Then $\hat{G} \subseteq D_{L\mu}$; the restriction of $L\mu$ to $\hat{G}$ is the Fourier–Stieltjes transform of $\mu$.

(viii) Let $\omega$ be a weight on $G$ with $\omega \geq 1$ on $G$ and let $f \in L^1(G, \omega)$. Then $\hat{G} \subseteq D_{Lf}$; the restriction of $Lf$ to $\hat{G}$ is the Fourier transform of $f$.

**Proof.** (i) Let $\alpha \in D_{L\mu}$. Define $\omega(s) = |\alpha(s)| \ (s \in G)$. Then $\mu \in M(G, \omega)$. Conversely, let $\mu \in M(G, \omega)$ for some weight $\omega$ on $G$. By [BhDe3], $H(G, \omega) \neq \emptyset$. Let $\beta \in H(G, \omega)$. Then $\int_G |\beta(s)| d|\mu|(s) \leq \int_G \omega(s) d|\mu|(s) < \infty$. Hence $H(G, \omega) \subseteq D_{L\mu}$. In particular, $D_{L\mu} \neq \emptyset$.

(ii) Assume that $D_{L\mu} \cap D_{L\nu} \neq \emptyset$. Choose $\alpha \in D_{L\mu} \cap D_{L\nu}$. Define $\omega(s) = |\alpha(s)| \ (s \in G)$. Then $\mu, \nu \in M(G, \omega)$. Since $M(G, \omega)$ is a Banach algebra, $\mu * \nu \in M(G, \omega)$. By (i) above, $D_{L\mu * \nu} \neq \emptyset$. Conversely, let $\mu, \nu \in M(G, \omega)$ for some weight $\omega$ on $G$. Then, as in the proof of (i), $\emptyset \neq H(G, \omega) \subseteq D_{L\mu} \cap D_{L\nu}$.

(iii) For $\alpha \in D_{L\mu}$ and $\theta \in \hat{G}$, $\int_G |(\alpha + \theta)(s)| d|\mu|(s) = \int_G |\alpha(s)| d|\mu|(s) < \infty$. Hence $\alpha + \hat{G} \subseteq D_{L\mu}$. 


(iv) Let $\alpha \in D_{\mathcal{L}_{\mu}}$ be such that $\mathcal{L}_{\mu} = 0$ on $\alpha + \hat{G}$. Since $\alpha \in D_{\mathcal{L}_{\mu}}$, $\alpha \mu \in M(G)$. Now $\mathcal{L}(\alpha \mu)(\theta) = \mathcal{L}(\mu)(\alpha + \theta) = 0$ ($\theta \in \hat{G}$). Hence $\alpha \mu = 0$. Since $\alpha(s) \neq 0$ for any $s \in G$, $\mu = 0$.

(v) For $\beta \in H(G, \omega)$, $\int_{G} |\beta(s)| d|\mu|(s) \leq \int_{G} \omega(s) d|\mu|(s) < \infty$.

(vi) This can be proved as (v).

(vii) Let $\theta \in \hat{G}$. Then $\theta \in H(G)$ and $\int_{G} |\theta(s)| d|\mu|(s) \leq \int_{G} \omega(s) d|\mu|(s) < \infty$. Hence $\hat{G} \subseteq D_{\mathcal{L}_{\mu}}$.

(viii) This can be proved as in (vii).

**Proposition 2.2.** Let $\omega$ be a weight on $G$. Then there exists a weight $\tilde{\omega} \geq 1$ on $G$ such that $M(G, \tilde{\omega})$ is isometrically isomorphic to $M(G, \omega)$.

**Proof.** Since $H(G, \omega)$ is non-empty, choose $\alpha \in H(G, \omega)$. Define $\tilde{\omega}(s) = \omega(s)/|\alpha(s)|$ ($s \in G$). Then $\tilde{\omega}$ is a weight and $\tilde{\omega} \geq 1$ on $G$. Now define $T : M(G, \omega) \rightarrow M(G, \tilde{\omega})$ as $T(\mu) = \alpha \mu$. Then, for $\mu \in M(G, \omega)$,

$$||T(\mu)||_{\tilde{\omega}} = ||\alpha \mu||_{\tilde{\omega}} = \int_{G} \frac{\omega(s)}{|\alpha(s)|} d|\alpha \mu|(s) = \int_{G} \omega(s) d|\mu|(s) = ||\mu||_{\omega}.$$  

It is easy to see that $T$ is an algebra isomorphism.

**Proposition 2.3.** Let $\omega$ be a weight on $G$. Then the following are equivalent:

(i) $M(G, \omega)$ is regular;

(ii) $M(G, \omega)$ has UUNP;

(iii) $L^1(G, \omega)$ has UUNP and $G$ is discrete.

**Proof.** By Proposition 2.2, we can assume that $\omega \geq 1$ on $G$.

(i)⇒(ii). This is true for all semisimple, commutative Banach algebras.

(ii)⇒(iii). Since $M(G, \omega)$ is a dense subalgebra of $M(G)$, $M(G)$ has UUNP due to Theorem 2.4 in [BhDe2]. Hence, by [BhDe2, p. 233], $G$ must be discrete. In this case, $L^1(G, \omega) = M(G, \omega)$ has UUNP.

(iii)⇒(i). Since $G$ is discrete, $M(G, \omega) = L^1(G, \omega)$. By [BhDe2, Theorem 4.1] and [BhDe3], $L^1(G, \omega)$ is regular.

The following is the main result of the paper. It compares with the result that $L^1(G, \omega)$ has a minimum uniform norm if and only if $L^1(G, \omega)$ has UUNP [BhDe4, Theorem 1].

**Theorem 2.4.** $M(G, \omega)$ has a minimum uniform norm if and only if $L^1(G, \omega)$ has UUNP.

**Proof.** By Proposition 2.2, we can assume that $\omega \geq 1$ on $G$. Assume that $L^1(G, \omega)$ has UUNP. Then it is regular due to [BhDe2, Theorem 4.1]. Since $M(G, \omega)$ is the multiplier algebra of $L^1(G, \omega)$ and $\Delta(L^1(G, \omega)) \cong H(G, \omega)$ is a set of uniqueness for $M(G, \omega)$ by Proposition 2.1, it follows from [BhDe2,
Corollary 6.3] that $|\mu|_\infty := \sup\{|\hat{\mu}(\alpha)| : \alpha \in H(G, \omega)\}$ is the minimum uniform norm on $M(G, \omega)$.

Conversely, assume that $M(G, \omega)$ has a minimum uniform norm, say $|\cdot|_0$. Define $F = \{\alpha \in H(G, \omega) : \alpha$ is $|\cdot|_0$-continuous$\}$. Then $|f|_0 = \sup\{|\varphi_\alpha(f)| : \alpha \in F\} := |f|_F (f \in L^1(G, \omega))$. We break up the proof into three steps.

**Step I:** $|\cdot|_0$ is a minimum uniform norm on $L^1(G, \omega)$. Let $|\cdot|$ be any uniform norm on $L^1(G, \omega)$. Define

$$|\mu|_{op} = \sup\{|f \ast \mu| : f \in L^1(G, \omega) \text{ and } |f| \leq 1\} \quad (\mu \in M(G, \omega)).$$

Since $L^1(G, \omega)$ has a bounded approximate identity, $|\cdot|_{op}$ is a uniform norm on $M(G, \omega)$ and it is identical to $|\cdot|$ on $L^1(G, \omega)$. Since $|\cdot|_0$ is a minimum uniform norm on $M(G, \omega)$, $|\cdot|_0 \leq |\cdot|_{op}$ on $M(G, \omega)$ and hence $|\cdot|_0 \leq |\cdot|_{op} = |\cdot|$ on $L^1(G, \omega)$. Thus $|\cdot|_0$ is a minimum uniform norm on $L^1(G, \omega)$.

**Step II:** $F = \hat{G} + F$. Fix $\theta \in \hat{G}$. Define $|f|_{\theta + F} = |\theta f|_F$ on $L^1(G, \omega)$, where $\theta + F := \{\theta + \alpha : \alpha \in F\}$. Then $|\cdot|_{\theta + F}$ is a uniform norm on $L^1(G, \omega)$.

Since $|\cdot|_F (= |\cdot|_0)$ is the minimum uniform norm on $L^1(G, \omega)$, we have

$$|f|_F \leq |f|_{\theta + F} \quad (f \in L^1(G, \omega)).$$

This holds for each $\theta \in \hat{G}$. So $|f|_{\theta + F} \leq |\theta f|_{\theta + F} = |f|_F$. Hence $\theta + F \subseteq F$ by the definition of $F$ and $|\cdot|_F = |\cdot|_0$. Thus it follows that $F = \hat{G} + F$.

**Step III:** $F = H(G, \omega)$. Suppose this is not the case. Then there exist positive generalized characters $\alpha \in F$ and $\beta \in H(G, \omega) \setminus F$ because $F = \hat{G} + F$. Choose $t \in G$ such that $\beta(t) < \alpha(t)$. Let $U$ be an open neighbourhood of $t$ such that its closure is compact and $\beta(s) < \alpha(s)$ $(s \in U)$. Take $f = \chi_U$. Then $f \in L^1(G, \omega)$ because $\omega$ is continuous. Now

$$|f|_{\beta + \hat{G}} = \sup\{|\varphi_{\beta + \theta}(f)| : \theta \in \hat{G}\}$$

$$= \sup\left\{\left|\int \limits_G f(s)\beta(s)\theta(s) \, d\lambda(s)\right| : \theta \in \hat{G}\right\}$$

$$\leq \sup\left\{\int \limits_U \beta(s)|\theta(s)| \, d\lambda(s) : \theta \in \hat{G}\right\}$$

$$= \int \limits_U \beta(s) \, d\lambda(s) < \int \limits_U \alpha(s) \, d\lambda(s)$$

$$\leq \sup\{|\varphi_{\alpha + \theta}(f)| : \theta \in \hat{G}\} = |f|_{\alpha + \hat{G}} \leq |f|_F.$$  

Thus $|\cdot|_{\beta + \hat{G}}$ is a uniform norm and $|f|_{\beta + \hat{G}} < |f|_F$, which is a contradiction because the latter is the minimum uniform norm on $L^1(G, \omega)$. Thus $F = H(G, \omega)$.

By Step III, $|f|_0 = |f|_F = |f|_{H(G, \omega)} = r(f)$ $(f \in L^1(G, \omega))$, where $r(\cdot)$ denotes the spectral radius on $L^1(G, \omega)$. Thus the spectral radius is the only uniform norm on $L^1(G, \omega)$, and hence $L^1(G, \omega)$ has UUNP. $\blacksquare$
Example 2.5. For $\alpha \geq 0$, define $\omega_\alpha(s) = (1 + |s|)^\alpha$ $(s \in \mathbb{R})$. Then the Beurling algebra $L^1(\mathbb{R}, \omega_\alpha)$ has UUNP. Thus $M(\mathbb{R}, \omega_\alpha)$ has a minimum uniform norm, namely $\mu \mapsto |\hat{\mu}|_\infty$, but $M(\mathbb{R}, \omega_\alpha)$ fails to have UUNP. ■

Let

$$M_0(G, \omega) := \{ \mu \in M(G, \omega) : \hat{\mu} \in C_0(\Delta(L^1(G, \omega))) \}$$

and

$$M_{00}(G, \omega) := \{ \mu \in M(G, \omega) : \hat{\mu} = 0 \text{ on } \Delta(M(G, \omega)) \setminus \Delta(L^1(G, \omega)) \}$$

where $h(L^1(G, \omega))$ is the hull of $L^1(G, \omega)$ in $M(G, \omega)$. Then both are closed ideals in $M(G, \omega)$ and $L^1(G, \omega) \subseteq M_{00}(G, \omega) \subseteq M_0(G, \omega)$. These are weighted measure algebra analogues of $M_0(G)$ and $M_{00}(G)$ considered in [LN, p. 376]. Our feeling is that $M_{00}(G, \omega)$ is close in spirit to $L^1(G, \omega)$, whereas $M_0(G, \omega)$ is close to $M(G, \omega)$. The following supports this. It also compares with Proposition 2.3.

Proposition 2.6. Let $\omega$ be a weight on $G$. Then the following are equivalent:

(i) $M_{00}(G, \omega)$ is regular;
(ii) $M_{00}(G, \omega)$ has UUNP;
(iii) $L^1(G, \omega)$ has UUNP.

Proof. (i)$\Rightarrow$(ii). This is true for all semisimple, commutative, Banach algebras.

(ii)$\Rightarrow$(iii). Let $M_{00}(G, \omega)$ have UUNP. Let $| \cdot |$ be a uniform norm on $L^1(G, \omega)$. Define

$$|\mu|_{op} = \sup\{|f \ast \mu| : f \in L^1(G, \omega) \text{ and } |f| \leq 1\} \quad (\mu \in M_{00}(G, \omega)).$$

Then $| \cdot |_{op}$ is a uniform norm on $M_{00}(G, \omega)$ and it is identical to $| \cdot |$ on $L^1(G, \omega)$. Since $M_{00}(G, \omega)$ has UUNP, $| \cdot |_{op}$ is identical to the spectral radius of $M_{00}(G, \omega)$. Since $L^1(G, \omega)$ is an ideal in $M_{00}(G, \omega)$, $| \cdot | = | \cdot |_{op}$ is identical to the spectral radius on $L^1(G, \omega)$. Thus $L^1(G, \omega)$ has UUNP.

(iii)$\Rightarrow$(i). Assume that $L^1(G, \omega)$ has UUNP. By [BhDe2, Theorem 4.1], $L^1(G, \omega)$ is regular. Since $L^1(G, \omega) \subseteq M_{00}(G, \omega)$ and since $\Delta(L^1(G, \omega)) = \Delta(M_{00}(G, \omega))$, $M_{00}(G, \omega)$ is regular. ■

Conjecture. Motivated by the fact that $M_0(G)$ fails to have UUNP [BhDe2, p. 234], we conjecture that $M_0(G, \omega)$ has UUNP if and only if $L^1(G, \omega)$ has UUNP and $G$ is discrete.
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