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Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals

by

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Abstract. We consider post-critically finite self-similar fractals with regular harmonic structures. We first obtain effective resistance estimates in terms of the Euclidean metric, which in particular imply the embedding theorem for the domains of the Dirichlet forms associated with the harmonic structures. We then characterize the domains of the Dirichlet forms.

1. Introduction. Let $(K, \{F_i\}_{i=1}^M)$ be a post-critically finite (p.c.f.) self-similar fractal in $\mathbb{R}^n$ $(n \geq 1)$ with a regular harmonic structure $(H, \mathbf{r})$, and let $(\mathcal{E}, \mathcal{D})$ be the Dirichlet form associated with $(H, \mathbf{r})$. Let R be the effective resistance determined by the form $(\mathcal{E}, \mathcal{D})$. In this paper, we are concerned with the following problems:

(1) What is the relationship between R and the Euclidean metric?

(2) How to characterize the domain $\mathcal{D}$ of $\mathcal{E}$?

These two problems are important in studying the dynamical aspects of fractals, such as PDE's, Brownian motions, heat kernels, and function spaces on fractals.

Recall that the answer to the first problem above is obvious if K is a bounded open interval in $\mathbb{R}$ and $\mathcal{E}$ is the classical energy form with respect to the Lebesgue measure,

$$\mathcal{E}(f,g) = \frac{1}{2} \int_K \nabla f \cdot \nabla g \, dx. \tag{1.1}$$

As a matter of fact, there exists some $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x-y| \leq R(x,y) \leq c|x-y|. \tag{1.2}$$

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The second inequality in (1.2) follows by using the definition of R (see (2.9) below) and the Sobolev embedding theorem:

$$|f(x) - f(y)| \le c^{1/2}|x-y|^{1/2}\mathcal{E}(f,f)^{1/2} \tag{1.3}$$

(see for example [1, formula (9), p. 98]). The first inequality in (1.2) also follows; this is because for any $x_0 < y_0$ in K, letting

$$f_0(x) = \begin{cases} 0 & \text{if } x \le x_0, \\ (x-x_0)/(y_0-x_0) & \text{if } x_0 \le x \le y_0, \\ 1 & \text{if } x \ge y_0, \end{cases}$$

we obtain

$$\mathcal{E}(f_0,f_0) = \frac{1}{2}\int_K |f_0'(x)|^2\,dx = \frac{1}{2}\,|y_0-x_0|^{-1},$$

and then use the definition (2.8) below (cf. [18, Sect. 1.6]). Note that for the higher-dimensional case, there does not exist such an elegant estimate for R as in (1.2).

The second problem above is also easy for the classical case: if K is an open domain in $\mathbb{R}^n$ ($n \ge 1$) and $\mathcal{E}$ is as in (1.1), then the domain $\mathcal{D}$ of $\mathcal{E}$ is just $W^{1,2}(K)$, the usual Sobolev space on K.

However, for the fractal case, the above two problems are non-trivial. Recall that for problem (1), if K is a nested fractal, there exists a geodesic metric d on K, and Barlow [2, Lemma 8.17] obtained the following relationship between R and d:

$$R(x,y) \sim d(x,y)^{\theta},$$

where $\theta = \log\varrho/\log\gamma$, and ϱ,γ are the *resistance* and *shortest path* scaling factors, respectively. If K is a Sierpiński gasket in $\mathbb{R}^2$, Strichartz [18, Sect. 1.6] obtained a relationship between R and the Euclidean metric:

$$R(x,y) \sim |x-y|^{d_w-d_f}, \tag{1.4}$$

where $d_f = \log 3/\log 2$ and $d_w = \log 5/\log 2$ are, respectively, the Hausdorff and walk dimensions of the Sierpiński gasket. Under certain mild conditions, we shall obtain in Section 3 a relationship between R and the Euclidean metric for p.c.f. fractals with regular harmonic structures. (If the harmonic structure is not regular, then $R(x,y)$ may be infinite for some points x and y, but $|x-y| < \infty$ for any $x,y \in K$ since K is bounded. So R cannot be controlled from above by the Euclidean metric. Therefore, the estimate (1.4) fails.) In particular, we show that (1.4) holds with different exponents for a certain class of nested fractals. (One may use the heat kernel estimates in [12] to derive (1.4) for some nested fractals—but this is another story.)

As for problem (2), the first result was obtained by Jonsson [9] for the Sierpiński gasket K in $\mathbb{R}^n$. It was shown that the domain $\mathcal{D}$ of the energy form $\mathcal{E}$ is equivalent to a Sobolev-type space on K, that is,

$$\mathcal{D} \simeq W^{\beta/2,2}(\mu) := \{f \in L^2(K,\mu) : W_{\beta/2,2}(f) < \infty\}, \tag{1.5}$$

where μ is the $\alpha := \log(n+1)/\log 2$-dimensional Hausdorff measure on K, $\beta = \log(n+3)/\log 2$ is the walk dimension, and

$$W_{\beta/2,2}(f) = \sup_{0<r<1} r^{-(\alpha+\beta)} \int_K \int_{B(x,r)} |f(y) - f(x)|^2 \, d\mu(y) \, d\mu(x). \tag{1.6}$$

Here $B(x,r) = \{y \in K : |y-x| < r\}$ is the ball of radius r and center x in K under the Euclidean metric. (Note that Jonsson used $\mathrm{Lip}(\beta/2,2,\infty)(K,\mu)$ to denote the space $W^{\beta/2,2}(\mu)$.) Pietruska-Pałuba generalized Jonsson's result to a certain class of nested fractals in $\mathbb{R}^n$ (see [16]).

On the other hand, one can characterize the domain $\mathcal{D}$ of the energy $\mathcal{E}$ with the help of heat kernel estimates. Assume that the heat kernel (or transition density) $p(t,x,y)$ exists on K, and satisfies

$$t^{-\alpha/\beta}\Phi_1(t^{-1/\beta}|x-y|) \le p(t,x,y) \le t^{-\alpha/\beta}\Phi_2(t^{-1/\beta}|x-y|) \tag{1.7}$$

for all $x, y \in K$ and $0 < t < 1$, where $\alpha, \beta > 0$ and $\Phi_i \ge 0$ is continuous and decreasing on $[0,\infty)$ for $i = 1,2$. Under certain mild assumptions on Φ_1 and Φ_2, one can show that the domain $\mathcal{D}$ of the Dirichlet form $(\mathcal{E},\mathcal{D})$ associated with the heat kernel $p(t,x,y)$ is equivalent to $W^{\beta/2,2}(\mu)$, where μ is the α-measure on K (that is, $\mu(B(x,r)) \sim r^\alpha$); see [17] for the Euclidean case and [4] for metric spaces. However, it is rather complicated to obtain heat kernel estimates like (1.7) (see [7, 13] for p.c.f. fractals with regular harmonic structures).

In Section 4, we characterize the domain $\mathcal{D}$ of the energy $\mathcal{E}$ on p.c.f. fractals with regular harmonic structures. We avoid using the heat kernel estimates, and present a direct proof. We do follow the technique in [9], but there are some new twists in our proof. The effective metric R and the self-similar measure μ with the standard weights will be used in defining the function spaces $W^{\beta/2,2}(\mu)$. We mention in passing that a closely related problem was studied in [11], where the domain of the Laplacian on p.c.f. self-similar sets was characterized.

Notation. The constants in this paper sometimes change from line to line while they are all denoted by the same letter c. The integers M, M_i and constants c_i are fixed for $i \ge 0$. For two non-negative functions f, g, by $f \sim g$ we mean that there is some $c > 0$ such that $c^{-1} f \le g \le c f$.

2. Preliminaries

2.1. *p.c.f. fractals and Dirichlet forms.* We first recall the concept of p.c.f. fractals introduced by Kigami [10].

Let $M \geq 2$ be an integer, and set $S = \{1, \ldots, M\}$. Let $W_* = \bigcup_{m \geq 0} S^m$ be the collection of all finite words. Let (X, d) be a complete metric space, and let $\{F_i\}_{i=1}^M$ be a family of strict contractions on (X, d). Then there exists a unique non-empty compact subset K of X such that

$$K = \bigcup_{i=1}^{M} F_i(K) \tag{2.1}$$

(see [8] or [3]). For any word $w = i_1 \cdots i_m \in S^m$, any sequence $\{p_i\}_{i=1}^M$ of positive numbers and any function $f : K \to \mathbb{R}$, denote by $|w| = m$ the length of w, and set

$$F_w = F_{i_1} \circ \cdots \circ F_{i_m}, \qquad K_w = F_w(K),$$
$$p_w = p_{i_1} \cdots p_{i_m}, \qquad f_w = f \circ F_w.$$

For the empty word w, set $p_w = 1$ and $F_w = \mathrm{id}$. Write $B(x_0, r) = \{y \in K : |y - x_0| < r\}$ for $x_0 \in K$ and $r > 0$.

Define a continuous surjection $\pi : S^{\mathbb{N}} \to K$ by

$$\{\pi(w)\} = \bigcap_{m \geq 1} F_{i_1 \cdots i_m}(K)$$

for any infinite word $w = i_1 i_2 \cdots \in S^{\mathbb{N}}$. Let

$$\mathcal{C} = \bigcup_{i \neq j} (K_i \cap K_j), \qquad \Gamma = \pi^{-1}(\mathcal{C}), \qquad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\Gamma),$$

where $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ is the shift map defined by

$$\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 \cdots .$$

If $\mathcal{P}$ is finite, the triple $(K, S, \{F_i\}_{i \in S})$ is termed a *post-critically finite* self-similar set (see [10, Definition 1.3.13, p. 23]). Let

$$V_0 = \pi(\mathcal{P}), \qquad V_m = \bigcup_{w \in S^m} F_w(V_0) \quad (m \geq 1), \qquad V_* = \bigcup_{m \geq 0} V_m. \tag{2.2}$$

If $(K, \{F_i\}_{i=1}^M)$ is a p.c.f. fractal, then $V_m \subset V_{m+1}$ $(m \geq 0)$. From now on we assume that $(K, \{F_i\}_{i=1}^M)$, or simply K, is a p.c.f. fractal.

We now recall how to construct a Dirichlet form on a p.c.f. fractal K. Let V_0 be as in (2.2), and $\ell(V_0) = \{f : V_0 \to \mathbb{R}\}$ be the collection of all real functions on V_0. Let $H = (H_{pq})_{p,q \in V_0}$ be a *Laplace matrix*, or simply a Laplace, on V_0, that is, for any $f, g \in \ell(V_0)$,

- $H_{pq} = H_{qp} \geq 0$ for any $p \neq q \in V_0$;
- $(f, Hg) := \sum_{p \in V_0} f(p) (\sum_{q \in V_0} H_{pq} g(q)) \leq 0$;
- $Hf = 0$ if and only if f is constant.

Given a Laplace H on V_0, and a family $\mathbf{r} = \{r_i\}_{i=1}^M$ of positive numbers, we define an *energy form* $\mathcal{E}_m$ on V_m for $m \geq 0$ by

$$\begin{aligned} &\mathcal{E}_0(f,g) = -(f, Hg), \\ &\mathcal{E}_m(f,g) = \sum_{w \in S^m} r_w^{-1} \mathcal{E}_0(f_w, g_w) \quad (m \geq 1), \end{aligned} \tag{2.3}$$

for $f, g : V_m \to \mathbb{R}$. In what follows, we write $\mathcal{E}_m(f) := \mathcal{E}_m(f,f)$ for simplicity. If there exists a pair $(H, \mathbf{r})$ such that the variational problem

$$\min\{\mathcal{E}_1(g) : g|_{V_0} = f\} = \mathcal{E}_0(f) \tag{2.4}$$

is solvable for any $f \in \ell(V_0)$, then we say that $(H, \mathbf{r})$ is a *harmonic structure* for K. If in addition $r_i < 1$ for all $1 \leq i \leq M$, then the harmonic structure is said to be *regular* (see [10, Definition. 3.1.2, p. 69]).

From now on we assume K has a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$.

Note that (2.4) implies that the sequence $\{\mathcal{E}_m(f)\}_{m \geq 0}$ is non-decreasing in m for any $f : V_* \to \mathbb{R}$. Let

$$\mathcal{E}(f) := \lim_{m \to \infty} \mathcal{E}_m(f), \tag{2.5}$$

$$\mathcal{D} = \{f \in C(K) : \mathcal{E}(f) < \infty\}, \tag{2.6}$$

where $C(K)$ is the space of all continuous functions on K. It is known that $(\mathcal{E}, \mathcal{D})$ defined as in (2.5) and (2.6) is a local, regular, irreducible Dirichlet form on $L^2(K, \mu)$ for any Borel measure μ which charges every set of the form K_w for $w \in S^m$ (see [2, Theorem 7.14, p. 99] or [10, Theorem 3.4.6, p. 92]). Clearly $\mathcal{E}$ is *self-similar*: for any $f \in \mathcal{D}$, we have $f \circ F_i \in \mathcal{D}$ for each i, and

$$\mathcal{E}(f) = \sum_{i \in S} r_i^{-1} \mathcal{E}(f \circ F_i). \tag{2.7}$$

We call $\{r_i\}_{i=1}^M$ the *weights* of the energy $\mathcal{E}$.

In order to characterize the domain $\mathcal{D}$ of the form $(\mathcal{E}, \mathcal{D})$, we need the *effective resistance* R on K. Let $R : K \times K \to [0, \infty]$ be defined by $R(x,x) = 0$ for $x \in K$, and

$$R(x,y)^{-1} = \inf\{\mathcal{E}(f) : f(x) = 0,\ f(y) = 1\} \tag{2.8}$$

for any $x \neq y \in K$. Note that (2.8) is equivalent to

$$R(x,y) = \sup\{|f(x) - f(y)|^2 / \mathcal{E}(f) : \mathcal{E}(f) > 0\} \tag{2.9}$$

for $x \neq y \in K$. It turns out that R is a metric on K, and the topology induced by R is equal to the original topology on K (see [10, Theorem 3.3.4, p. 85] or [2, Proposition 7.18, p. 101]).

2.2. *Partition.* The idea of partition on p.c.f. fractals (going back to Hambly [6]) will be useful in our analysis. Let $\mathbf{a} = \{a_i\}_{i=1}^M$ be a family of numbers with $0 < a_i < 1$ for each i. For $0 < \lambda < 1$, define

$$\Lambda_{\mathbf{a}}(\lambda) = \{w = i_1 \cdots i_m : a_w \le \lambda < a_{i_1} \cdots a_{i_{m-1}}\} \tag{2.10}$$

with the convention that $a_\emptyset = 1$. We call $\Lambda_{\mathbf{a}}(\lambda)$ the *partition* with respect to $\mathbf{a}$ and λ. Note that the set $\Lambda_{\mathbf{a}}(\lambda)$ is finite; this is easily seen since $0 < \lambda < 1$ and $0 < a_i < 1$ for each i. For simplicity, for $\tau, w \in \Lambda_{\mathbf{r}}(\lambda)$ write

$$\tau \sim w \quad \text{if } K_\tau \cap K_w \neq \emptyset.$$

For $x, y \in K$, we write $x \sim y$ if $x, y \in K_w$ for some $w \in \Lambda_{\mathbf{a}}(\lambda)$. Clearly, by (2.1) and (2.7), we see that, for any partition $\Lambda_{\mathbf{a}}(\lambda)$,

$$K = \bigcup_{w \in \Lambda_{\mathbf{a}}(\lambda)} K_w, \quad \mathcal{E}(f) = \sum_{w \in \Lambda_{\mathbf{a}}(\lambda)} r_w^{-1} \mathcal{E}(f_w) \quad (f \in \mathcal{D}). \tag{2.11}$$

For $f : V_* \to \mathbb{R}$ and $0 < \lambda < 1$, define

$$\mathcal{E}_\lambda(f) := \sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} r_w^{-1} \mathcal{E}_0(f_w). \tag{2.12}$$

PROPOSITION 2.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$. Then $\{\mathcal{E}_\lambda(f)\}$ is increasing as $\lambda \searrow 0$ for any f, and*

$$\lim_{\lambda \to 0} \mathcal{E}_\lambda(f) = \mathcal{E}(f), \quad f \in \mathcal{D}. \tag{2.13}$$

Proof. Let $0 < \lambda_1 < \lambda_2 < 1$. Then the partition $\Lambda_{\mathbf{r}}(\lambda_2)$ is a "father" of $\Lambda_{\mathbf{r}}(\lambda_1)$, that is, any word $w \in \Lambda_{\mathbf{r}}(\lambda_1)$ can be written as $w = \tau w'$ with $\tau \in \Lambda_{\mathbf{r}}(\lambda_2)$ and $w' \in W_*$, with w' being possibly an empty word. Indeed, let

$$w = i_1 \cdots i_m \in \Lambda_{\mathbf{r}}(\lambda_1) \setminus \Lambda_{\mathbf{r}}(\lambda_2) \quad (m \ge 1).$$

Then $\lambda_2 \ge r_{i_1} \cdots r_{i_{m-1}}$; otherwise we would have

$$r_{i_1} \cdots r_{i_m} \le \lambda_1 < \lambda_2 < r_{i_1} \cdots r_{i_{m-1}},$$

and so $w = i_1 \cdots i_m \in \Lambda_{\mathbf{r}}(\lambda_2)$ by the definition, a contradiction. Let $1 \le k \le m-1$ be an integer such that

$$r_{i_1} \cdots r_{i_k} \le \lambda_2 < r_{i_1} \cdots r_{i_{k-1}}.$$

This implies $\tau := i_1 \cdots i_k \in \Lambda_{\mathbf{r}}(\lambda_2)$. Setting $w' := i_{k+1} \cdots i_m$, we see that $w = \tau w'$ with $\tau \in \Lambda_{\mathbf{r}}(\lambda_2)$. This shows that $\Lambda_{\mathbf{r}}(\lambda_2)$ is a father of $\Lambda_{\mathbf{r}}(\lambda_1)$. Therefore, for $f \in \mathcal{D}$,

$$\mathcal{E}_{\lambda_1}(f) = \sum_{w \in \Lambda_{\mathbf{r}}(\lambda_1)} r_w^{-1} \mathcal{E}_0(f_w) \ge \sum_{\tau \in \Lambda_{\mathbf{r}}(\lambda_2)} r_\tau^{-1} \mathcal{E}_0(f_\tau) = \mathcal{E}_{\lambda_2}(f),$$

proving that $\{\mathcal{E}_\lambda(f)\}$ is decreasing in λ for any f. Here, the inequality follows from both the harmonic structure and the post-critical finiteness of K.

Finally, for $0 < \lambda < 1$, letting

$$m_1 = m_1(\mathbf{r}, \lambda) = \min\{|w| : w \in \Lambda_{\mathbf{r}}(\lambda)\},$$
$$m_2 = m_2(\mathbf{r}, \lambda) = \max\{|w| : w \in \Lambda_{\mathbf{r}}(\lambda)\},$$

we see that S^{m_1} is a father of $\Lambda_{\mathbf{r}}(\lambda)$, and $\Lambda_{\mathbf{r}}(\lambda)$ a father of S^{m_2}. Hence,

$$\mathcal{E}_{m_1}(f) \le \mathcal{E}_\lambda(f) \le \mathcal{E}_{m_2}(f) \quad (f \in \mathcal{D}).$$

Thus

$$\mathcal{E}(f) = \lim_{m_1\to\infty} \mathcal{E}_{m_1}(f) \le \lim_{\lambda\to 0} \mathcal{E}_\lambda(f) \le \lim_{m_2\to\infty} \mathcal{E}_{m_2}(f) = \mathcal{E}(f). \blacksquare$$

3. Effective resistance estimates. In this section we give two-sided estimates of the effective resistance R in terms of the Euclidean metric. The two exponents appearing in the two-sided estimates of R are calculated for some fractals, both nested and non-nested.

THEOREM 3.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ with a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$. Assume that $s_i < 1$ is the contraction ratio of F_i, that is,*

$$|F_i(x) - F_i(y)| \le s_i|x - y| \quad \textit{for } x, y \in \mathbb{R}^n.$$

Then there exists some $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x-y|^{\alpha_1} \le R(x,y) \quad \textit{where} \quad \alpha_1 = \max_{1\le i\le M} \frac{\log r_i}{\log s_i}. \tag{3.1}$$

Proof. Let $x_0 \ne y_0 \in K$. Without loss of generality, assume that

$$R(x_0, y_0) < (2c_1)^{-1}$$

where $c_1 > 0$ will be determined below. Set

$$\lambda = 2c_1 R(x_0, y_0) < 1.$$

Then $\Lambda_{\mathbf{r}}(\lambda)$ is a partition. There exist two words $w_1, w_2 \in \Lambda_{\mathbf{r}}(\lambda)$ such that $x_0 \in K_{w_1}$ and $y_0 \in K_{w_2}$. We claim that

$$K_{w_1} \cap K_{w_2} \ne \emptyset.$$

Otherwise there would exist a $\Lambda_{\mathbf{r}}(\lambda)$-harmonic function f satisfying

$$f|_{V_{w_1}} = 1 \quad \text{and} \quad f|_{V_\lambda \setminus V_{w_1}} = 0, \tag{3.2}$$

where $V_w := F_w(V_0)$ for $w \in W_*$, and $V_\lambda = \bigcup_{w\in\Lambda_{\mathbf{r}}(\lambda)} F_w(V_0)$. (We say that a function f on K is $\Lambda_{\boldsymbol{r}}(\lambda)$-*harmonic* if $\mathcal{E}(f,\varphi) = 0$ for any $\varphi \in \mathcal{D}$ with $\varphi|_{V_\lambda} = 0$.)

Note that $f(x_0) = 1$ and $f(y_0) = 0$. Since f is $\Lambda_{\mathbf{r}}(\lambda)$-harmonic, we have

$$\mathcal{E}(f) = \mathcal{E}_\lambda(f) = \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\mathcal{E}_0(f_w) \tag{3.3}$$

$$= \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}\Big(\frac{1}{2}\sum_{p,q\in V_0} H_{pq}(f(F_w(p)) - f(F_w(q)))^2\Big).$$

Using (3.2), we see that the right-hand side of (3.3) is actually equal to the sum of the terms

$$r_w^{-1}H_{pq}(f(F_w(p)) - f(F_w(q)))^2 = r_w^{-1}H_{pq},$$

with w ($\neq w_1$) running over all the words in $\Lambda_{\mathbf{r}}(\lambda)$ with $w \sim w_1$, and p and q running over V_0 such that $F_w(p) \in V_{w_1}$ (so that $f(F_w(p)) = 1$ and $f(F_w(q)) = 0$); all the other terms are equal to zero. Hence, noting that $r_w \geq \lambda r_{\min}$, the right-hand side of (3.3) is bounded by

$$H_{\max}M_0(M_0-1)r_w^{-1} \leq H_{\max}(r_{\min})^{-1}M_0(M_0-1)\lambda^{-1} =: c_1\lambda^{-1},$$

where $H_{\max} = \max_{p\neq q\in V_0} H_{pq}$, $M_0 = \sharp V_0$ and $r_{\min} = \min r_i$. Therefore, by (2.8), it follows that

$$R(x_0, y_0)^{-1} \leq c_1\lambda^{-1},$$

and so

$$R(x_0, y_0) \geq c_1^{-1}\lambda = 2R(x_0, y_0),$$

yielding a contradiction. So the claim holds.

Now let $z_0 \in K_{w_1} \cap K_{w_2}$. Since $x_0, z_0 \in K_{w_1}$, writing $x_0 = F_{w_1}(x_0')$ and $z_0 = F_{w_1}(z_0')$ for some $x_0', z_0' \in K$, we see that

$$|x_0 - z_0| = |F_{w_1}(x_0') - F_{w_1}(z_0')| \leq s_{w_1}\operatorname{diam}(K)$$

$$\leq (r_{w_1})^{1/\alpha_1}\operatorname{diam}(K) \leq \lambda^{1/\alpha_1}\operatorname{diam}(K).$$

Similarly, noting that $y_0, z_0 \in K_{w_2}$, we find that

$$|y_0 - z_0| \leq \lambda^{1/\alpha_1}\operatorname{diam}(K).$$

Therefore,

$$|x_0 - y_0| \leq |x_0 - z_0| + |z_0 - y_0| \leq 2\lambda^{1/\alpha_1}\operatorname{diam}(K) = cR(x_0, y_0)^{1/\alpha_1},$$

giving $R(x_0, y_0) \geq c^{-1}|x_0 - y_0|^{\alpha_1}$. ■

To bound R from above, we need the following *separation property*:

(C1) There exist a family $\mathbf{b} = \{b_i\}_{i=1}^M$ of numbers with $0 < b_i < 1$ for every i, and a constant $c_2 > 0$ such that, for any $0 < \lambda < 1$,

$$\operatorname{dist}(K_w, K_\tau) \geq c_2\lambda \quad \text{if } K_w \cap K_\tau = \emptyset$$

for $w, \tau \in \Lambda_{\mathbf{b}}(\lambda)$.

We remark that c_2 is independent of λ, but may depend on $\{b_i\}_{i=1}^M$. Condition (C1) says that any two disjoint components obtained from any partition $\Lambda_{\mathbf{b}}(\lambda)$ with $0 < \lambda < 1$ are a distance at least $c_2\lambda$ apart.

THEOREM 3.2. *Let* $(K, \{F_i\}_{i=1}^M)$ *be a p.c.f. fractal in* $\mathbb{R}^n$ $(n \geq 1)$ *with a regular harmonic structure* $(H, \{r_i\}_{i=1}^M)$. *Assume that condition* (C1) *holds for some* $\mathbf{b} = \{b_i\}_{i=1}^M$. *Then there exists* $c > 0$ *such that, for all* $x, y \in K$,

$$R(x,y) \leq c|x-y|^{\alpha_2} \quad \text{where} \quad \alpha_2 = \min_{1\leq i\leq M} \frac{\log r_i}{\log b_i}. \tag{3.4}$$

Proof. First note that $R(x,y) \leq c < \infty$ for all $x, y \in K$, since the harmonic structure is regular (see [10, Theorem 3.3.4, p. 85]). This implies that

$$|f(x) - f(y)|^2 \leq R(x,y)\mathcal{E}(f) \leq c\mathcal{E}(f)$$

for any $f \in \mathcal{D}$. In particular, for $x, y \in K_w$ $(w \in W_*)$, writing $x = F_w(x')$ and $y = F_w(y')$ for some $x', y' \in K$, we have

$$|f(x) - f(y)|^2 = |f_w(x') - f_w(y')|^2 \leq c\mathcal{E}(f_w). \tag{3.5}$$

Now let $x_0 \neq y_0 \in K$. Without loss of generality, we assume that

$$|x_0 - y_0| < c_2/2,$$

where c_2 is as in condition (C1). Let

$$\lambda = \frac{2}{c_2}|x_0 - y_0| < 1.$$

There are two words $w_1, w_2 \in \Lambda_{\mathbf{b}}(\lambda)$ such that $x_0 \in K_{w_1}$ and $y_0 \in K_{w_2}$. Then $K_{w_1} \cap K_{w_2} \neq \emptyset$; otherwise, condition (C1) would imply

$$|x_0 - y_0| \geq \operatorname{dist}(K_{w_1}, K_{w_2}) \geq c_2\lambda = 2|x_0 - y_0|,$$

a contradiction. Let $z_0 \in K_{w_1} \cap K_{w_2}$. For $f \in \mathcal{D}$, as $x_0, z_0 \in K_{w_1}$, we see from (3.5) and (2.11) that

$$\begin{aligned} |f(x_0) - f(z_0)|^2 &\leq c\mathcal{E}(f_{w_1}) = cr_{w_1}(r_{w_1})^{-1}\mathcal{E}(f_{w_1}) \\ &\leq cr_{w_1}\mathcal{E}(f) \leq c(b_{w_1})^{\alpha_2}\mathcal{E}(f) \leq c\lambda^{\alpha_2}\mathcal{E}(f). \end{aligned} \tag{3.6}$$

Similarly, since $z_0, y_0 \in K_{w_2}$, we have

$$|f(z_0) - f(y_0)|^2 \leq c\lambda^{\alpha_2}\mathcal{E}(f).$$

Therefore,

$$\begin{aligned} |f(x_0) - f(y_0)|^2 &\leq 2(|f(x_0) - f(z_0)|^2 + |f(z_0) - f(y_0)|^2) \\ &\leq c\lambda^{\alpha_2}\mathcal{E}(f) = c|x_0 - y_0|^{\alpha_2}\mathcal{E}(f), \end{aligned}$$

which gives

$$R(x_0, y_0) \leq c|x_0 - y_0|^{\alpha_2}.$$

Thus (3.4) follows. ∎

Condition (C1) may be replaced by the following *connectivity property*:

(C2) There exist $\widehat{\mathbf{b}} = \{\widehat{b}_i\}_{i=1}^M$ with $0 < \widehat{b}_i < 1$ for each i, a (small) constant $c_3 > 0$ and an integer M_1 such that, for any $0 < \lambda < 1$ and any $x_0 \in K$, each point $y \in B(x_0, c_3\lambda)$ can be connected to x_0 by a sequence $\{x_k\}_{k=0}^{n_0}$ of points in K with $1 \le n_0 \le M_1$, $x_{n_0} = y$ and $x_{k-1} \sim x_k$ for $1 \le k \le n_0$.

For a partition $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$ with $0 < \lambda < 1$, condition (C2) means that any point y in any ball $B(x_0, c_3\lambda)$ can be connected to its center x_0 by at most M_1 components obtained from the partition $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$.

THEOREM 3.3. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ $(n \ge 1)$ with a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$. Assume that condition* (C2) *holds for some $\widehat{\mathbf{b}} = \{\widehat{b}_i\}_{i=1}^M$. Then there exists $c > 0$ such that, for all $x, y \in K$,*

$$R(x,y) \le c|x-y|^{\alpha_3} \quad \textit{where} \quad \alpha_3 = \min_{1\le i\le M} \frac{\log r_i}{\log \widehat{b}_i}. \tag{3.7}$$

Proof. Let $x_0 \ne y_0 \in K$. Without loss of generality, we assume that $|x_0 - y_0| < c_3/2$ where c_3 is as in (C2). Set $\lambda := 2c_3^{-1}|x_0 - y_0|$. Let $\Lambda_{\widehat{\mathbf{b}}}(\lambda)$ be the partition with respect to $\widehat{\mathbf{b}}$ and λ. Note that $y_0 \in B(x_0, c_3\lambda)$. Then, by (C2), there exists a sequence $\{x_k\}_{k=0}^{n_0}$ of points with $1 \le n_0 \le M_1$, $x_{n_0} = y$, and

$$x_{k-1}, x_k \in K_{w_k} \quad \text{for some } w_k \in \Lambda_{\widehat{\mathbf{b}}}(\lambda)\ (k = 1, \ldots, n_0).$$

For $f \in \mathcal{D}$, as in (3.6), we have

$$|f(x_k) - f(x_{k-1})|^2 \le c r_{w_k} \mathcal{E}(f) \le c(\widehat{b}_{w_k})^{\alpha_3}\mathcal{E}(f) \le c\lambda^{\alpha_3}\mathcal{E}(f), \quad k = 1, \ldots, n_0.$$

Therefore,

$$\begin{aligned}|f(x_0) - f(y_0)|^2 &= \Big(\sum_{k=1}^{n_0}(f(x_k) - f(x_{k-1}))\Big)^2 \le n_0 \sum_{k=1}^{n_0}(f(x_k) - f(x_{k-1}))^2 \\ &\le cM_1^2\lambda^{\alpha_3}\mathcal{E}(f) = c|x_0 - y_0|^{\alpha_3}\mathcal{E}(f),\end{aligned}$$

which implies that

$$R(x_0, y_0) \le c|x_0 - y_0|^{\alpha_3}.$$

Thus (3.7) follows. ■

We remark that Theorem 3.2 or 3.3 implies the Morrey–Sobolev embedding of the function space $\mathcal{D}$:

$$|f(x) - f(y)| \le c|x-y|^{\beta}\sqrt{\mathcal{E}(f)} \tag{3.8}$$

for all $x, y \in K$ and all $f \in \mathcal{D}$, for some $c, \beta > 0$.

We now give some examples of p.c.f. fractals where condition (C1) or (C2) holds so that the conclusion of Theorem 3.2 or 3.3 is true.

• *Nested fractals.* The nested fractal $(K, \{F_i\}_{i=1}^M)$ was introduced by Lindstrøm [14]. It belongs to the class of p.c.f. fractals in $\mathbb{R}^n$ with the same contraction ratio $0 < \varrho < 1$, that is,

$$|F_i(x) - F_i(y)| = \varrho|x-y| \quad (x, y \in \mathbb{R}^n).$$

It is known that K has a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$ with $r_i := r < 1$ for $1 \le i \le M$ (see for example [15]). Moreover, condition (C2) holds for $\widehat{b}_i = \varrho$ for a certain class of nested fractals (see [12, Lemma 5.4]). Thus, by Theorems 3.1 and 3.3, there exists $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x-y|^{\log r/\log \varrho} \le R(x,y) \le c|x-y|^{\log r/\log \varrho}.$$

As a typical representative of nested fractals, the Sierpiński gasket K in $\mathbb{R}^n$ admits an effective resistance R satisfying

$$c^{-1}|x-y|^{\log(\frac{n+3}{n+1})/\log 2} \le R(x,y) \le c|x-y|^{\log(\frac{n+3}{n+1})/\log 2},$$

by taking $r_i = (n+1)/(n+3)$ when constructing the Dirichlet form. Note that the exponent

$$\frac{\log\left(\frac{n+3}{n+1}\right)}{\log 2} = \frac{\log(n+3)}{\log 2} - \frac{\log(n+1)}{\log 2} =: d_w - d_f$$

is the difference between the walk dimension d_w and the Hausdorff dimension d_f of the Sierpiński gasket (see also [18] for $n = 2$).

• *Vicsek sets.* Let

$$p_1 = (0,0), \quad p_2 = (1,0), \quad p_3 = (1,1), \quad p_4 = (0,1), \quad p_5 = (1/2, 1/2)$$

be the four corners and center of the unit square in the plane. Define

$$F_i(x) = \tfrac{1}{4}(x - p_i) + p_i \quad (1 \le i \le 4), \qquad F_5 = \tfrac{1}{2}(x - p_5) + p_5 \quad (x \in \mathbb{R}^2).$$

The *Vicsek set* is $K = \bigcup_{i=1}^5 F_i(K)$. It is a p.c.f. fractal but not a nested fractal, and the boundary is $V_0 = \{p_1, p_2, p_3, p_4\}$.

Let $\mathbf{b} = \{b_i\}_{i=1}^5$ where $b_i = 1/4$ for each i. Then the Vicsek set satisfies condition (C1). Indeed, for $0 < \lambda < 1$, let $m \ge 1$ be an integer such that $4^{-m} \le \lambda < 4^{-(m-1)}$. Then $\Lambda_{\mathbf{b}}(\lambda) = S^m$, and

$$\operatorname{dist}(K_w, K_\tau) \ge \tfrac{1}{2} \cdot 4^{-(m-1)} > \tfrac{1}{2}\lambda \quad \text{if } K_w \cap K_\tau = \emptyset$$

for $w, \tau \in S^m$. It is not hard to construct a regular harmonic structure $(H, \{r_i\}_{i=1}^5)$ on the Vicsek set. In fact, let

$$H = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix},$$

and $r_1 = r_2 = r_3 = r_4 = 2^{-1}(1-r)$ and $r_5 = r$ with $0 < r < 1$. One can verify that $(H, \{r_i\}_{i=1}^5)$ is a regular harmonic structure on K for any $0 < r < 1$. Thus we see from Theorems 3.1 and 3.2 that, for all $x, y \in K$,

$$c^{-1}|x-y|^{\alpha_1} \le R(x,y) \le c|x-y|^{\alpha_2}, \tag{3.9}$$

where

$$\alpha_1 = \max\left\{-\frac{\log(2^{-1}(1-r))}{\log 4}, -\frac{\log r}{\log 2}\right\},$$
$$\alpha_2 = \min\left\{-\frac{\log(2^{-1}(1-r))}{\log 4}, -\frac{\log r}{\log 4}\right\}.$$

Note that the effective resistance R here cannot be controlled by any powered Euclidean metric, that is,

$$R(x,y) \sim |x-y|^\theta \quad (\forall x, y \in K) \tag{3.10}$$

fails for any $\theta > 0$. In fact, let $m \ge 1$ be any integer and set $w_1 = 11\cdots 1 \in S^m$. Choose a family $\{(x_m, y_m)\}_{m\ge 1}$ of points in K, where

$$x_m = (0,0) = F_{w_1}(p_1), \quad y_m = (4^{-m}, 0) = F_{w_1}(p_2).$$

Clearly $x_m, y_m \in F_{w_1}(V_0)$ with $|x_m - y_m| = 4^{-m}$. Let f be the S^m-harmonic function on K satisfying $f(x_m) = 1$ and $f|_{V_m \setminus \{x_m\}} = 0$. Then we have

$$\mathcal{E}(f) = \mathcal{E}_m(f) = 3(2^{-1}(1-r))^{-m},$$

which gives

$$\begin{aligned} R(x_m, y_m)^{-1} &= \inf\{\mathcal{E}(u) : u \in \mathcal{D} \text{ and } u(x_m) = 1,\, u(y_m) = 0\} \\ &\le \mathcal{E}(f) = 3(2^{-1}(1-r))^{-m} = 3 \cdot 4^{m\theta_1} = 3|x_m - y_m|^{-\theta_1}, \end{aligned}$$

where $\theta_1 = -\log(2^{-1}(1-r))/\log 4$. Therefore, $R(x_m, y_m) \ge 3^{-1}|x_m - y_m|^{\theta_1}$. On the other hand, for any $u \in \mathcal{D}$, we see that

$$\begin{aligned} \mathcal{E}(u) &\ge \mathcal{E}_m(u) \ge r_{w_1}^{-1}(u(x_m) - u(y_m))^2 = (2^{-1}(1-r))^{-m}(u(x_m) - u(y_m))^2 \\ &= |x_m - y_m|^{-\theta_1}(u(x_m) - u(y_m))^2, \end{aligned}$$

which implies that $R(x_m, y_m) \le |x_m - y_m|^{\theta_1}$ by using (2.9). Therefore,

$$3^{-1}|x_m - y_m|^{\theta_1} \le R(x_m, y_m) \le |x_m - y_m|^{\theta_1}. \tag{3.11}$$

Similarly, let $w_2 = 55\cdots 5 \in S^m$, and take $x'_m = F_{w_2}(p_1)$ and $y'_m = F_{w_2}(p_2)$. Clearly $|x'_m - y'_m| = 2^{-m}$. By the same calculation as above, we can obtain

$$\frac{1-r}{3(1+r)} \cdot |x'_m - y'_m|^{\theta_2} \le R(x'_m, y'_m) \le |x'_m - y'_m|^{\theta_2}, \tag{3.12}$$

where $\theta_2 = -\log r/\log 2$. If $r \ne 1/2$, we see that $\theta_1 \ne \theta_2$. It follows from (3.11) and (3.12) that (3.10) cannot hold for any $\theta > 0$, provided that $r \ne 1/2$.

4. Domains of Dirichlet forms. Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$, and let $(\mathcal{E}, \mathcal{D})$ be the associated Dirichlet form defined as in (2.5) and (2.6). In this section, we give a characterization of the domain $\mathcal{D}$.

To this end, we need to introduce a measure μ. We choose μ to be the normalized self-similar measure with the standard weights $\{p_i\}_{i=1}^M$, that is,

$$\mu = \sum_{i=1}^{M} p_i \cdot \mu \circ F_i^{-1} \tag{4.1}$$

where $p_i = r_i^\alpha$, with α given by

$$\sum_{i=1}^{M} r_i^\alpha = 1. \tag{4.2}$$

For any $w \neq \tau \in W_*$, we have

$$\mu(K_w) = (r_w)^\alpha \quad \text{and} \quad \mu(K_w \cap K_\tau) = 0. \tag{4.3}$$

Observe that there exist constants $0 < c_4 \le c_5$ independent of x and λ such that for any $x \in K$ and $0 < \lambda < 1$,

$$B_R(x, c_4\lambda) \subset N_\lambda(x) \subset B_R(x, c_5\lambda), \tag{4.4}$$

where $B_R(x, \lambda) = \{y \in K : R(y, x) < \lambda\}$ is a ball in the metric R, and

$$N_\lambda(x) = \bigcup \{K_w : x \in K_w \text{ and } w \in \Lambda_{\mathbf{r}}(\lambda)\}$$

is the union of all components K_w ($w \in \Lambda_{\mathbf{r}}(\lambda)$) to which x belongs. Indeed, the first inclusion in (4.4) follows since, for $x \in K$ and $y \notin N_\lambda(x)$, one can find a function f such that $f(x) = 1$ and $f(y) = 0$, and $\mathcal{E}(f) \le (c_4\lambda)^{-1}$ for some $c_4 > 0$ (see the proof of Theorem 3.1). So $R(x, y) \ge c_4\lambda$ by using (2.8), and therefore $B_R(x, c_4\lambda) \subset N_\lambda(x)$. The second inclusion follows from [2, Prop. 8.9, p. 110].

THEOREM 4.1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal with a regular harmonic structure $(H, \{r_i\}_{i=1}^M)$, and let $(\mathcal{E}, \mathcal{D})$ be the associated Dirichlet form defined as in* (2.5) *and* (2.6). *Let μ be a self-similar measure with standard weights. Then there exists some $c > 0$ such that*

$$c^{-1} W_\alpha(f) \le \mathcal{E}(f) \le c W_\alpha(f) \tag{4.5}$$

for all $f \in C(K)$, where

$$W_\alpha(f) := \sup_{0<\lambda<1} \lambda^{-(2\alpha+1)} \int_K \int_{B_R(x, c_4\lambda)} |f(x) - f(y)|^2 \, d\mu(y) \, d\mu(x), \tag{4.6}$$

and the constants α and c_4 are as in (4.2) *and* (4.4), *respectively. In particular, $\mathcal{D} = \{f \in C(K) : W_\alpha(f) < \infty\}$.*

Note that (4.5) implies that $W_\alpha(f) < \infty$ if and only if $\mathcal{E}(f) < \infty$. We decompose Theorem 4.1 into Lemmas 4.3 and 4.4 below. In order to prove Lemma 4.3, we need the following proposition.

PROPOSITION 4.2. *Let* $(K, \{F_i\}_{i=1}^M)$, $(\mathcal{E}, \mathcal{D})$ *and* μ *be as in Theorem* 4.1. *Then*, *for* $0 < \lambda < 1$ *and* $f \in C(K)$,

$$\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} \int_{K_w} |f(x) - f_w(x_0)|^2 \, d\mu(x) \le c\lambda^{\alpha+1}\mathcal{E}(f) \tag{4.7}$$

for any x_0 *in* V_0, *with c independent of* λ, f *and* x_0.

Proof. The proof is motivated by [5]. Without loss of generality, assume that $f \in \mathcal{D}$ and $x_0 \in V_0$. Since $\Lambda_{\mathbf{r}}(\lambda)$ is a partition, so is

$$\{w\tau : w \in \Lambda_{\mathbf{r}}(\lambda) \text{ and } \tau \in S^k\}$$

for any $k \ge 1$. Therefore, for μ-almost all $x \in K$, there is exactly one $\tau \in S^k$ such that $x \in K_{w\tau}$. We define $f_k(x) := f_{w\tau}(x_0)$ if $x \in K_{w\tau}$. Obviously the function f_k is defined μ-almost everywhere on K, and is constant on each component of the form $K_{w\tau}$ where $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $\tau \in S^k$. Since f is continuous, we see that $f_k(x) \to f(x)$ for μ-almost all $x \in K$ as $k \to \infty$. In order to derive (4.7), it is enough to show that

$$\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} \int_{K_w} |f_k(x) - f_w(x_0)|^2 \, d\mu(x) \le c\lambda^{\alpha+1}\mathcal{E}(f). \tag{4.8}$$

In fact, if (4.8) holds, then letting $k \to \infty$ in (4.8) and using the dominated convergence theorem, we obtain (4.7).

Fix $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $\tau := i_1 \cdots i_k$ for $k \ge 1$ temporarily. Let

$$x_l = F_{wi_1\cdots i_l}(x_0), \quad 1 \le l \le k.$$

Note that

$$\begin{aligned}(f(x_k) - f(x_0))^2 &= \Big(\sum_{l=0}^{k-1} a_l^{-1/2} \cdot a_l^{1/2}(f(x_{l+1}) - f(x_l))\Big)^2 \\ &\le \Big(\sum_{l=0}^{k-1} a_l^{-1}\Big)\Big(\sum_{l=0}^{k-1} a_l(f(x_{l+1}) - f(x_l))^2\Big) \\ &\le c\sum_{l=0}^{k-1} a_l(f(x_{l+1}) - f(x_l))^2,\end{aligned} \tag{4.9}$$

where $\{a_l\}_{l=0}^\infty$ is a sequence of positive numbers satisfying $\sum_{l=0}^\infty a_l^{-1} < \infty$, which will be specified later on. Observing that

$$\begin{aligned}(f(x_{l+1}) - f(x_l))^2 &= (f_{wi_1\cdots i_l}(F_{i_{l+1}}(x_0)) - f_{wi_1\cdots i_l}(x_0))^2 \\ &\le c\mathcal{E}_1(f_{wi_1\cdots i_l}) \le c\mathcal{E}(f_{wi_1\cdots i_l}),\end{aligned}$$

we see from (4.9) that

$$(f(x_k) - f(x_0))^2 \le c \sum_{l=0}^{k-1} a_l \mathcal{E}(f_{wi_1\cdots i_l}).$$

Therefore, using the fact that $\mu(K_{w\tau}) = (r_w)^\alpha \mu(K_\tau) \le \lambda^\alpha \mu(K_\tau)$, we obtain

$$\int_{K_{w\tau}} (f_k(x) - f_w(x_0))^2 \, d\mu(x) = \mu(K_{w\tau})(f(x_k) - f(x_0))^2$$
$$\le c\lambda^\alpha \mu(K_{i_1\cdots i_k}) \sum_{l=0}^{k-1} a_l \mathcal{E}(f_{wi_1\cdots i_l})$$

for any $w \in \Lambda_{\mathbf{r}}(\lambda)$ and any $\tau := i_1 \cdots i_k \in S^k$ $(k \ge 1)$. Hence,

$$\int_{K_w} (f_k(x) - f_w(x_0))^2 \, d\mu(x) = \sum_{\tau \in S^k} \int_{K_{w\tau}} (f_k(x) - f_w(x_0))^2 \, d\mu(x) \tag{4.10}$$
$$\le c\lambda^\alpha \sum_{i_1,\dots,i_k} \mu(K_{i_1\cdots i_k}) \sum_{l=0}^{k-1} a_l \mathcal{E}(f_{wi_1\cdots i_l})$$
$$\le c\lambda^\alpha \sum_{l=0}^{k-1} a_l \sum_{i_1,\dots,i_l} \mathcal{E}(f_{wi_1\cdots i_l}).$$

In the last inequality above, we have exchanged the order of the summations, and then used the fact that $\sum_{i_{l+1},\dots,i_k \in S} \mu(K_{i_{l+1}\cdots i_k}) = 1$ and $\mu(K_{i_1\cdots i_l}) \le 1$ $(l \ge 1)$. On the other hand, for any $l \ge 0$,

$$\sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \sum_{i_1,\dots,i_l} \mathcal{E}(f_{wi_1\cdots i_l}) = \sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \sum_{i_1,\dots,i_l} (r_{wi_1\cdots i_l})(r_{wi_1\cdots i_l})^{-1} \mathcal{E}(f_{wi_1\cdots i_l}) \tag{4.11}$$
$$\le \lambda (r_{\max})^l \sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \sum_{i_1,\dots,i_l} (r_{wi_1\cdots i_l})^{-1} \mathcal{E}(f_{wi_1\cdots i_l})$$
$$= \lambda (r_{\max})^l \mathcal{E}(f),$$

since $r_{wi_1\cdots i_l} = r_w r_{i_1\cdots i_l} \le \lambda (r_{\max})^l$, where $r_{\max} := \max_i r_i < 1$. Therefore, from (4.10) and (4.11) we obtain

$$\sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \int_{K_w} (f_k(x) - f_w(x_0))^2 \, d\mu(x) \le c\lambda^\alpha \sum_{l=0}^{k-1} a_l \Big(\sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} \sum_{i_1,\dots,i_l} \mathcal{E}(f_{wi_1\cdots i_l}) \Big)$$
$$\le c\lambda^{\alpha+1} \mathcal{E}(f) \sum_{l=0}^{\infty} a_l (r_{\max})^l \le c\lambda^{\alpha+1} \mathcal{E}(f),$$

where we have chosen $a_l := (r_{\max})^{-l/2}$ that satisfies $\sum_{l=0}^{\infty} a_l^{-1} < \infty$. Thus (4.8) follows. This finishes the proof. ∎

LEMMA 4.3. *Let* $(K,\{F_i\}_{i=1}^M)$, $(\mathcal{E},\mathcal{D})$ *and* μ *be as in Theorem* 4.1. *Then there exists some* $c>0$ *such that*, *for all* $f\in C(K)$,

$$W_\alpha(f)\le c\mathcal{E}(f), \tag{4.12}$$

where $W_\alpha(f)$ *is defined as in* (4.6).

Proof. Assume that $f\in\mathcal{D}$; otherwise (4.12) holds automatically. Let $0<\lambda<1$. Note that $\mu(K_w\cap K_\tau)=0$ for any distinct $w,\tau\in\Lambda_{\mathbf{r}}(\lambda)$. Set

$$I_\lambda(f):=\int_K\int_{B_R(x,c_4\lambda)}|f(x)-f(y)|^2\,d\mu(y)\,d\mu(x).$$

Then, by (4.4),

$$\begin{aligned}I_\lambda(f)&=\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\int_{K_w}\int_{B_R(x,c_4\lambda)}|f(x)-f(y)|^2\,d\mu(y)\,d\mu(x)\\&\le\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}\sum_{\tau\sim w}\int_{K_w}\int_{K_\tau}|f(x)-f(y)|^2\,d\mu(y)\,d\mu(x).\end{aligned}$$

For $x\in K_w$, $y\in K_\tau$, let $z_0\in K_w\cap K_\tau=F_w(V_0)\cap F_\tau(V_0)$ (if $w=\tau$, we simply take any point $z_0\in F_w(V_0)$ and run the same proof as below; so we only consider the case $w\ne\tau$). Using the elementary inequality

$$|f(x)-f(y)|^2\le 2(|f(x)-f(z_0)|^2+|f(z_0)-f(y)|^2),$$

and the fact that $\sharp\{\tau:\tau\sim w\}\le M_2$ for an integer M_2 independent of w and λ (cf. [10, Lemma 4.2.3, p. 139]), we obtain

$$I_\lambda(f)\le c\lambda^\alpha\sum_{\substack{w\in\Lambda_{\mathbf{r}}(\lambda)\\ z_0\in F_w(V_0)}}\int_{K_w}|f(x)-f(z_0)|^2\,d\mu(x).$$

Let $z_0=F_w(x_0)$ for some $x_0\in V_0$. By (4.7) and the fact that $\sharp V_0<\infty$, it follows immediately that

$$I_\lambda(f)\le c\lambda^{2\alpha+1}\mathcal{E}(f),$$

proving (4.12). ∎

LEMMA 4.4. *Let* $(K,\{F_i\}_{i=1}^M)$, $(\mathcal{E},\mathcal{D})$ *and* μ *be as in Theorem* 4.1. *Then*

$$\mathcal{E}(f)\le cW_\alpha(f) \tag{4.13}$$

for all $f\in C(K)$, *where* $c>0$.

Proof. Let $0<\lambda<c_4/c_5\le 1$. Let $f\in C(K)$. Without loss of generality, we assume that $W_\alpha(f)<\infty$. We have

$$\mathcal{E}_\lambda(f)=\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}r_w^{-1}\mathcal{E}_0(f_w)\le c\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)}r_w^{-1}\Big(\sum_{p,q\in F_w(V_0)}(f(p)-f(q))^2\Big). \tag{4.14}$$

Noting that, for any $x_0\in K_w$,

$$|f(p)-f(q)|^2\le 2((f(p)-f(x_0))^2+(f(x_0)-f(q))^2),$$

we see that

$$|f(p) - f(q)|^2 \le \frac{2}{\mu(K_w)}\Big(\int\limits_{K_w} (f(p) - f(x_0))^2\, d\mu(x_0) + \int\limits_{K_w} (f(x_0) - f(q))^2\, d\mu(x_0)\Big).$$

Hence,

$$\sum_{p,q\in F_w(V_0)} (f(p) - f(q))^2 \le M_3 \sum_{p\in F_w(V_0)} \frac{1}{\mu(K_w)} \int\limits_{K_w} (f(p) - f(x_0))^2\, d\mu(x_0), \tag{4.15}$$

where $M_3 = 4\sharp(F_w(V_0)) = 4\sharp V_0$. Let $w \in \Lambda_{\mathbf{r}}(\lambda)$ and $p \in F_w(V_0)$ be fixed.

We now estimate the last integral. Let $0 < a < 1$ be any fixed number (for example, $a = 1/2$). For each integer $l \ge 0$, let $\Lambda(\lambda a^l) := \Lambda_{\mathbf{r}}(\lambda a^l)$ be a partition. We choose a sequence of subsets of K_w:

$$K_w,\ K_{w\tau_1},\ K_{w\tau_2},\ \ldots,$$

such that $w\tau_l \in \Lambda(\lambda a^l)$ and $p \in K_{w\tau_l}$ for each $l \ge 0$. Note that $K_{w\tau_l} \subset K_{w\tau_i}$ for any $l > i \ge 0$, because $\lambda a^l < \lambda a^i$, and so the partition $\Lambda(\lambda a^i)$ is a father of $\Lambda(\lambda a^l)$. For simplicity, we write

$$K'_0 = K_w, \qquad K'_l = K_{w\tau_l} \qquad (l \ge 1).$$

Note that, for any $x_l \in K'_l$ ($l \ge 0$),

$$\begin{aligned}(f(p) - f(x_0))^2 &= \Big((f(p) - f(x_k)) + \sum_{l=0}^{k-1} a_l^{-1/2} \cdot a_l^{1/2}(f(x_{l+1}) - f(x_l))\Big)^2 \\ &\le 2(f(p)-f(x_k))^2 + 2\Big(\sum_{l=0}^{\infty} a_l^{-1}\Big)\Big(\sum_{l=0}^{k-1} a_l(f(x_{l+1}) - f(x_l))^2\Big),\end{aligned}$$

where $\{a_l\}_{l=0}^{\infty}$ is a sequence of positive numbers satisfying $\sum_{l=0}^{\infty} a_l^{-1} < \infty$, which will be determined below. Integrating the above inequality with respect to each $x_l \in K'_l$ for $0 \le l \le k$, and then dividing by $\mu(K'_0)\cdots\mu(K'_k)$, we obtain

$$\begin{aligned}\frac{1}{\mu(K_w)} \int\limits_{K_w} (f(p) - f(x_0))^2\, d\mu(x_0) &\le \frac{2}{\mu(K'_k)} \int\limits_{K'_k} (f(p) - f(x_k))^2\, d\mu(x_k) \\ &\quad + c\sum_{l=0}^{k-1} \frac{a_l}{\mu(K'_{l+1})\mu(K'_l)} \int\limits_{K'_l}\int\limits_{K'_{l+1}} (f(x_{l+1}) - f(x_l))^2\, d\mu(x_{l+1})\, d\mu(x_l).\end{aligned} \tag{4.16}$$

Note that the first term on the right-hand side of (4.16) tends to zero as $k \to \infty$, since $\{K'_k\}$ shrinks to p as $k \to \infty$ and f is continuous. In order to estimate the second term, we set

$$A_{w,k}(f) := \sum_{l=0}^{k-1} \frac{a_l}{\mu(K'_{l+1})\mu(K'_l)} \times \int_{K'_l} \int_{K'_{l+1}} (f(x_{l+1}) - f(x_l))^2 \, d\mu(x_{l+1}) \, d\mu(x_l). \tag{4.17}$$

By (4.4), we have

$$K'_l \subset N_{\lambda a^l}(x_l) \subset B_R(x_l, c_5 \lambda a^l)$$

for any $x_l \in K'_l$ and $l \geq 0$. Using the fact that

$$K'_{l+1} \subset K'_l \subset K_w,$$

we obtain

$$\int_{K'_l} \int_{K'_{l+1}} (f(x_{l+1}) - f(x_l))^2 \, d\mu(x_{l+1}) \, d\mu(x_l) \leq \int_{K_w} \int_{B_R(x_l, c_5 \lambda a^l)} (f(x_{l+1}) - f(x_l))^2 \, d\mu(x_{l+1}) \, d\mu(x_l).$$

Note that, using (4.3) and the fact that $w\tau_l \in \Lambda_{\mathbf{r}}(\lambda a^l)$, we get

$$\mu(K'_l) = (r_{w\tau_l})^\alpha \sim (\lambda a^l)^\alpha \quad \text{for any } l \geq 0.$$

Therefore, it follows from (4.17) that

$$A_{w,k}(f) \leq c \sum_{l=0}^{k-1} a_l (\lambda a^l)^{-2\alpha} \times \int_{K_w} \int_{B_R(x, c_5 \lambda a^l)} (f(y) - f(x))^2 \, d\mu(y) \, d\mu(x). \tag{4.18}$$

Hence, combining (4.14)–(4.16) shows that, for any $k \geq 0$,

$$\mathcal{E}_\lambda(f) \leq c \sum_{w \in \Lambda_{\mathbf{r}}(\lambda)} r_w^{-1} A_{w,k}(f) + \sum_{\substack{w \in \Lambda_{\mathbf{r}}(\lambda) \\ p \in F_w(V_0)}} r_w^{-1} \frac{c}{\mu(K'_k)} \int_{K'_k} (f(p) - f(z))^2 \, d\mu(z). \tag{4.19}$$

On the other hand, noting that $r_w \sim \lambda$ for $w \in \Lambda_{\mathbf{r}}(\lambda)$, it follows from (4.18) that

$$(4.20)\quad \sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} r_w^{-1}A_{w,k}(f) \le c\lambda^{-1}\sum_{w\in\Lambda_{\mathbf{r}}(\lambda)} A_{w,k}(f)$$

$$\le c\lambda^{-1}\sum_{l=0}^{k-1} a_l(\lambda a^l)^{-2\alpha}\int_K\int_{B_R(x,c_5\lambda a^l)} (f(y)-f(x))^2\,d\mu(y)\,d\mu(x)$$

$$= c\sum_{l=0}^{k-1} a_l\,a^l\Big\{(c_5\lambda a^l)^{-(2\alpha+1)}\int_K\int_{B_R(x,c_5\lambda a^l)} (f(y)-f(x))^2\,d\mu(y)\,d\mu(x)\Big\}$$

$$\le cW_\alpha(f)\sum_{l=0}^{k-1} a_l a^l \le cW_\alpha(f).$$

Here we have chosen $a_l := a^{-l/2}$ that satisfies $\sum_{l\ge 0} a_l^{-1} < \infty$. Therefore, by (4.19) and (4.20),

$$\mathcal{E}_\lambda(f) \le cW_\alpha(f) + \sum_{\substack{w\in\Lambda_{\mathbf{r}}(\lambda)\\ p\in F_w(V_0)}} r_w^{-1}\frac{c}{\mu(K_k')}\int_{K_k'} (f(p)-f(z))^2 d\mu(z) \qquad (k\ge 0),$$

where c is independent of k and λ. Letting $k\to\infty$, we see that

$$\mathcal{E}_\lambda(f) \le cW_\alpha(f).$$

This gives (4.13) by letting $\lambda\to 0$ and using (2.13). ∎

Finally, we remark that Theorem 4.1 follows directly from Lemmas 4.3 and 4.4.

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