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Weak-L^p solutions for a model of self-gravitating particles with an external potential

by

ANDRZEJ RACZYŃSKI (Wrocław)

Abstract. The existence of solutions to a nonlinear parabolic equation describing the temporal evolution of a cloud of self-gravitating particles with a given external potential is studied in weak-L^p spaces (i.e. Marcinkiewicz spaces). The main goal is to prove the existence of global solutions and to study their large time behaviour.

1. Introduction. We consider the Cauchy problem for the equation

$$u_t = \Delta u + \nabla \cdot (u\nabla\phi) + \nabla \cdot (u\nabla\Phi), \tag{1}$$

coupled with the Poisson equation $\Delta\phi = u$ written in the form

$$\nabla\phi = \nabla E_n * u, \tag{2}$$

where $E_n(z) = -((n-2)\sigma_n)^{-1}|z|^{2-n}$, $n \geq 3$, is the fundamental solution of the Laplacian in $\mathbb{R}^n$ and σ_n is the area of the unit sphere in $\mathbb{R}^n$. We supplement the system (1)–(2) with the initial condition

$$u(x,0) = u_0(x). \tag{3}$$

The system above describes the temporal evolution of the density $u(x,t)$ of a cloud of self-gravitating particles and the potential $\phi(x,t)$ generated by gravitational interaction between them. The function $\Phi(x)$ in the third term on the right-hand side of (1) represents the given external potential.

The model (1)–(3) can also be considered with electric interactions replacing the gravitational ones. In this case the equation (2) should be rewritten as

$$\nabla\phi = -\nabla E_n * u.$$

Usually, the results for this model are "better" than for the gravitational one, but the methods used here for the construction of solutions do not allow us to obtain qualitatively different results. Thus, the results we obtain for this

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model are the same (with similar proofs) as for the model considered in this paper.

Let us review briefly the basic literature and previous results concerning the model in question.

The physical interpretation of the system has a long history and goes back to Nernst and Planck (see [7] and references therein). In the original problem electric interactions were assumed and the problem was considered in a bounded smooth domain in $\mathbb{R}^n$.

A very good introduction to mathematical aspects of the physical problems which are modelled by our problem and its generalizations can be found in [18] (also [19] is useful).

A good background and physical motivation for the gravitational interpretation of the system (1)–(3) can be found in [22].

The majority of the relevant papers are devoted to the problem in bounded domains of $\mathbb{R}^n$ with appropriate boundary conditions (usually *no-flux* boundary condition and Dirichlet condition for the potential ϕ). For such a problem the external potential Φ is usually M^*E_n (interpreted as putting an additional mass or charge M^* at the origin) and the results strongly depend on the dimension of the space and the type of interactions.

Let us briefly overview the results obtained (they come mainly from the papers [6], [7], [14], [22]).

For the Coulomb case (with electrical interactions assumed) the main results have been obtained for radially symmetric solutions with integrated density u as a main tool. With $M_0 = \int_\Omega u$ the existence of stationary solutions for $n = 2$, $M^* < 4\pi$ and any M_0 has been established. In the two-dimensional case for $M^* \geq 4\pi$ and any M_0, and for arbitrary values of M_0 and M^* in three- or more dimensions, the nonexistence of steady states has been proved.

Similar results have been obtained for evolution solutions (together with their convergence to stationary solutions for $n = 2, 3$).

The same results have been obtained for the problem in the whole space.

For $n = 2$ also self-similar solutions have been considered (by the definition, such solutions are defined in the whole space $\mathbb{R}^2$). Just as for radially symmetric solutions, the existence of solutions for $M^* < 4\pi$ and arbitrary M_0 has been proved.

For more general assumptions on Φ and without radial symmetry assumed, the existence of local-in-time weak solutions has also been proved, but only in bounded domains.

In the gravitational case the existence of solutions depends on the dimension and the values of M^* and M_0, but in a more sophisticated way than in the Coulomb case.

To be more precise, for radially symmetric solutions in two dimensions the existence of solutions (stationary and global-in-time) depends on the quantity M_0+2M^*. If this quantity is less than 8π, the existence of solutions is guaranteed, while $M_0+2M^*>8\pi$ implies the nonexistence.

In the three-dimensional case the nonexistence of stationary and local-in-time solutions has been proved for $M^*>0$.

For the problem in the whole space $\mathbb{R}^n$, the nonexistence of local-in-time solutions has been proved for $n=2$ and $M^*\geq 4\pi$, and for $n\geq 3$ and $M^*>0$.

For self-similar solutions in $\mathbb{R}^2$ the condition $M_0+2M^*<8\pi$ guarantees the existence of such solutions.

The existence of weak local-in-time solutions has been established (for bounded domains) also for more general assumptions on Φ.

In [13] the assumption $\Phi\equiv 0$ is made but the paper contains a good introduction to the problem in the whole $\mathbb{R}^n$.

Most of the results mentioned above concern the case $\Phi=M^*E_n$. The ideas used in those papers have been developed in [5] where the general form of the flux (in our paper given by $\nabla(\phi+\Phi)$) has been considered (in a bounded domain). Under some hypotheses on L^p estimates of this flow the existence and uniqueness of local-in-time as well as stationary solutions have been proved.

R. F. Streater extended the classical Nernst–Planck–Debye–Hückel drift-diffusion system for charged but noninteracting particles (see [20]) by introducing a new variable (temperature). The augmented model with an additional equation for the heat flow has been considered in e.g. [4], [8]. Since Streater's model in the general setting is difficult, in the present paper the external potential is assumed to be 0. The first results concerning nonuniqueness of steady states with $\Phi\not\equiv 0$ can be found in, for example, [15].

As stated in [7], introducing an additional potential (even in the form $\Phi=M^*E_n$) implies that the problem is more difficult mainly because of singular terms containing the derivatives of the potential. This as well as external potentials of general form raise the question about the functional setting in which the existence of solutions should be considered.

Results for the problem without external potentials have been given in [2] and [3].

For the problem with external potential, suitable assumptions on Φ and the proof of existence have been given in spaces of pseudomeasures ([17]).

The main aim of this paper is to point out another example of spaces in which global singular solutions can exist as well as to specify requirements on external potentials to obtain such solutions.

The importance of such a choice of space will be discussed below.

Additionally, our knowledge of spaces in which singular solutions for the problem without external potential can exist is also enriched (recall that for such problems self-similar solutions can also be considered).

Finally, in the last section a result concerning asymptotic behaviour of the solutions obtained will be given.

In this paper we look for solutions of the problem (1)–(3) in the Marcinkiewicz space $L^{p,\infty}(\mathbb{R}^n)$. Let us recall the definition:

$$L^{p,\infty}(\mathbb{R}^n)=\Big\{v\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\|v\|_{p,\infty}\equiv\sup_{E\subset\mathbb{R}^n}|E|^{-1+1/p}\int_E|v(x)|\,dx<\infty\Big\},$$

where $p>1$, and E runs through Borel sets with finite and positive measure $|E|$. Sometimes, to show that $f\in L^{p,\infty}(\mathbb{R}^n)$ it is better to use the quantity

$$\|f\|^*_{p,\infty}=\sup_{s>0}s|\{x:|f(x)|>s\}|^{1/p}$$

(which is not a norm), for which the following inequalities hold:

$$\|f\|^*_{p,\infty}\le\|f\|_{p,\infty}\le\frac{p}{p-1}\,\|f\|^*_{p,\infty}.$$

The reader can find more properties of the spaces $L^{p,\infty}(\mathbb{R}^n)$ in, e.g., [9, Sec. 2].

Notations. We denote by $\|f\|_p$ the norm in the usual Lebesgue space $L^p(\mathbb{R}^n)$. The symbol C denotes various inessential constants which may vary from line to line.

Let

$$\mathcal{X}_p=\mathcal{C}_w([0,\infty);L^{p,\infty}(\mathbb{R}^n))$$

be the space of vector-valued functions $u=u(x,t)$ such that

- $\int_{\mathbb{R}^n}u(x,t)\phi(x)\,dx\to\int_{\mathbb{R}^n}u(x,0)\phi(x)\,dx$ as $t\searrow 0$ for each test function $\phi\in\mathcal{S}(\mathbb{R}^n)$, the Schwartz class,
- $u(t)$ is a bounded and continuous function from $(0,T]$ to $L^{p,\infty}(\mathbb{R}^n)$ in the norm topology of $L^{p,\infty}(\mathbb{R}^n)$.

The necessity of considering $\mathcal{C}_w$ instead of the space of strongly continuous functions $\mathcal{C}([0,\infty);L^{p,\infty}(\mathbb{R}^n))$ is caused by the fact that the heat semigroup is not strongly continuous on $L^{p,\infty}(\mathbb{R}^n)$ but only weakly continuous. To see this, it is enough to check that $\|e^{t\Delta}|x|^{-n/p}-|x|^{-n/p}\|_{p,\infty}=\mathrm{const}$.

$L^{p,\infty}(\mathbb{R}^n)$ is an example of a space which is not separable but is dual to a separable space. In fact, for the Lorentz space $L^{p',1}(\mathbb{R}^n)$ (p' is the conjugate index to p) we have $L^{p,\infty}(\mathbb{R}^n)=(L^{p',1})^*(\mathbb{R}^n)$ and $L^{p',1}(\mathbb{R}^n)$ is separable with dense subset $\mathcal{S}(\mathbb{R}^n)$. Examples of applications of such spaces to the Navier–Stokes or nonlinear heat problems can be found in [10], [11].

By a *mild solution* of the problem (1)–(3), we understand a solution $u \in \mathcal{C}_w([0,\infty); L^{p,\infty}(\mathbb{R}^n))$ of the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\phi(s))\,ds + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\Phi(s))\,ds, \tag{4}$$

where $\nabla\phi(s) = \nabla E_n * u(s)$, and the integral is the Bochner integral. The first term is well defined (in the paper [1, Lemmas 2&3] the reader can find a detailed proof that $e^{t\Delta}u_0 \in \mathcal{X}_p$ for every $u_0 \in L^{p,\infty}(\mathbb{R}^n)$ and $t \ge 0$) but such a meaning of a solution is not suitable for our construction of solutions of the Cauchy problem. The difficulty is caused by the fact that for $p \ge n/2$ the term $e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\phi(s))$ is not Bochner integrable on $[0,T]$ with values in $L^{p,\infty}(\mathbb{R}^n)$. To see this, it is enough to observe that for stationary solutions which are homogeneous of degree -2 (see a note about the Chandrasekhar solution below), the term in question corresponds to a tempered homogeneous distribution of degree -4. Thus, there exists a distribution H such that

$$e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\phi(s)) = (t-s)^{-2}H\left(\frac{\cdot}{\sqrt{t-s}}\right).$$

Since

$$\|f(\lambda\,\cdot)\|_{p,\infty} = \lambda^{-n/p}\|f\|_{p,\infty}, \tag{5}$$

we have

$$\|e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\phi(s))\|_{p,\infty} = (t-s)^{-2+n/2p}\|H\|_{p,\infty},$$

which implies that this term is not Bochner integrable for $p \ge n/2$. To remove this difficulty, the integrals with respect to s in equations (4) should be defined in the weak sense (as, for example, in [3] and [23, Def. 2]). For more explanations, we refer the reader to [3] and references therein. Nevertheless, a distributional solution of (1)–(3) which belongs to $\mathcal{X}_p$ is a solution of the integral equation (4) and vice versa. This equivalence can be proved following the computations for the Navier–Stokes equations in [23, Th. 5.2].

The importance of the space $L^{n/2,\infty}(\mathbb{R}^n)$ comes from the fact that for $n \ge 3$ there exists a stationary singular solution to the problem with no external potential. It is called the *Chandrasekhar solution* and has the form

$$u_{\mathrm{C}}(x) = 2(n-2)|x|^{-2}.$$

It is easy to check that $u_{\mathrm{C}}(x)$ belongs to $L^{n/2,\infty}(\mathbb{R}^n)$ (as well as to $\mathcal{X}_{n/2}$, if u_{C} is interpreted as a constant function of t). Indeed, we have $\|u_{\mathrm{C}}\|_{n/2,\infty} = 2(n-2)\|\,|x|^{-2}\|_{n/2,\infty} \le 2n\sigma_n^{n/2}$.

As stated in [3], it is expected that u_C is a solution with critical singularity of the initial data in the sense that for small initial data $u_0 \le \varepsilon u_C$, $0 < \varepsilon \ll 1$, the solution exists, and for initial conditions u_0 such that $u_0(x) > u_C$ there is no solution to the problem (1)–(3).

Another advantage of the use of the spaces $L^{p,\infty}(\mathbb{R}^n)$ is that this natural extension of the space L^p contains homogeneous functions of degree $-n/p$ (which, of course, do not belong to $L^p(\mathbb{R}^n)$). For our model this gives us an opportunity to consider self-similar solutions to the problem (1)–(3) (with $\Phi \equiv 0$), i.e. functions $u(t)$ with the scaling property $u(t) \equiv u_\lambda(t) = \lambda^2 u(\lambda x, \lambda^2 t)$ for all $\lambda > 0$, in the space $L^{n/2,\infty}(\mathbb{R}^n)$.

To simplify the notation, we will denote the quadratic term in (4) by $B(u,u)$, with the bilinear form B defined by

$$B(u,v) = \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(s) \nabla \phi(s))\, ds,$$

where $\phi(s)$ is obtained from v by putting $\nabla\phi(s) = \nabla E_n * v(s)$.

Analogously, denote the linear term in (4) by Lu with L defined by

$$Lu = \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(s) \nabla \Phi(s))\, ds,$$

where Φ is a given external potential. In this way our problem can be rewritten as follows:

$$u(t) = e^{t\Delta} u_0 + B(u,u) + Lu.$$

2. Main tools. A modification of a well-known theorem by Y. Meyer ([12]) is the main tool which will be used in this paper. The theorem below gives the existence and uniqueness of the solution via a contraction mapping argument.

THEOREM 2.1. *Let $\mathcal{X}$ be a Banach space. Assume that $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a bilinear form such that*

$$\|B(y,z)\|_{\mathcal{X}} \le K \|y\|_{\mathcal{X}} \|z\|_{\mathcal{X}}$$

for some $K > 0$ and all $y, z \in \mathcal{X}$. Let $L : \mathcal{X} \to \mathcal{X}$ be a continuous linear operator with

$$\|Ly\|_{\mathcal{X}} \le \ell \|y\|_{\mathcal{X}}$$

with some $\ell < 1$.

(i) *For every $a \in \mathcal{X}$ such that $\|a\|_{\mathcal{X}} < (1-\ell)^2/4K$, there exists a solution $x \in \mathcal{X}$ to the equation*

$$x = a + Lx + B(x,x). \tag{6}$$

This solution satisfies the estimate

$$\|x\|_{\mathcal{X}} \le \frac{1-\ell-\sqrt{(1-\ell)^2-4K\|a\|_{\mathcal{X}}}}{2K} \le \frac{1-\ell}{2K}.$$

Moreover, for $\|a\|_{\mathcal{X}} < (1-\ell)^2/4K$, *this solution is unique in the open ball in* $\mathcal{X}$ *of radius* $(1-\ell)/2K$.

(ii) *The solution obtained depends continuously on* a *in the following way. For* $b \in \mathcal{X}$ *such that* $\|b\|_{\mathcal{X}} \le \varepsilon < (1-\ell)^2/4K$ *and for* v *being the solution of the equation* $v = b + Lv + B(v,v)$ *we have*

$$\|u-v\|_{\mathcal{X}} \le ((1-\ell)^2 - 4K\varepsilon)^{-1/2}\|a-b\|_{\mathcal{X}}.$$

The choice of the function space $\mathcal{X}$ is of crucial importance in the application of the above theorem.

We recall a slight modification of a theorem by E. Terraneo ([21, Prop. 1.5]) which will be used to estimate the bilinear form $B(u,v)$.

THEOREM 2.2. *Let* $n > 1$, $1 < p < n$, *and* $1/r = 1/p - 1/n$. *Let* $f \in L^\infty((0,T); L^{p,\infty}(\mathbb{R}^n))$ *for any* $0 < T \le \infty$. *Then there exists* $C = C(r,p) > 0$ *such that for every* $a \in [0,t)$ *with* $t \le T$,

$$\Big\| \int_a^t e^{(t-s)\Delta} \nabla f(s)\, ds \Big\|_{r,\infty} \le C \sup_{a<s<t} \|f(s)\|_{p,\infty}. \ \blacksquare \tag{7}$$

Let us also recall two estimates for the norm of the product and convolution of two functions in Marcinkiewicz spaces. The proofs of the following lemmas (even in a more general setting) can be found, e.g., in [16, Theorems 3.4 and 2.5]).

LEMMA 2.3 (weak Hölder inequality). *For* $1 < p \le \infty$ *and* $1 < q, r < \infty$ *such that* $1/r = 1/p + 1/q$ *and for all* $f \in L^{p,\infty}(\mathbb{R}^n)$ *and* $g \in L^{q,\infty}(\mathbb{R}^n)$, *the product* fg *belongs to* $L^{r,\infty}(\mathbb{R}^n)$ *and*

$$\|fg\|_{r,\infty} \le C\|f\|_{p,\infty}\|g\|_{q,\infty},$$

with a constant $C = C(p,q)$. $\blacksquare$

LEMMA 2.4 (weak Young inequality). *For* $1 < p, q < \infty$ *and* $1 < r < \infty$ *such that* $1 + 1/r = 1/p + 1/q$ *and for all* $f \in L^{p,\infty}(\mathbb{R}^n)$ *and* $g \in L^{q,\infty}(\mathbb{R}^n)$, *the convolution* $f * g$ *belongs to* $L^{r,\infty}(\mathbb{R}^n)$ *and*

$$\|f * g\|_{r,\infty} \le C\|f\|_{p,\infty}\|g\|_{q,\infty},$$

with a constant $C = C(p,q,n)$. $\blacksquare$

3. Global solutions in the space $\mathcal{X}_{n/2}$, $n > 3$. In this section we prove the existence of global solutions in the space $\mathcal{X}_{n/2}$, $n > 3$. Since the three-dimensional case needs a restriction to a subspace, a counterpart of the result below for that case will be proved in the next section.

First observe that the function $|x|^{-\gamma}$ for $0<\gamma<n$ belongs to $L^{n/\gamma,\infty}(\mathbb{R}^n)$ with

$$\| \, |x|^{-\gamma}\|_{n/\gamma,\infty} \le \frac{n}{n-\gamma}\, \sigma_n^{\gamma/n},$$

so that the Chandrasekhar solution $u_{\mathrm{C}}(x)$ belongs to $L^{n/2,\infty}(\mathbb{R}^n), n \ge 3$.

Next, we consider some properties of the heat semigrup $e^{t\Delta}u_0$ in the scale of Marcinkiewicz spaces.

LEMMA 3.1. *For any $u_0 \in L^{p,\infty}(\mathbb{R}^n)$, we have $e^{t\Delta}u_0 \in \mathcal{X}_p$ with the estimate*

$$\|e^{t\Delta}u_0\|_{\mathcal{X}_p} \le C(n,p)\|u_0\|_{p,\infty}.$$

Proof. This is obvious due to the following inequality ([1, Lemma 1]) for $1<r\le p<\infty$:

$$\|e^{t\Delta}f\|_{p,\infty} \le C(n,p,r)t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\|f\|_{r,\infty} \tag{8}$$

with $r=p$. ∎

Combining Theorem 2.2 with the weak Hölder and Young inequalities we obtain the crucial estimate of the bilinear form B.

LEMMA 3.2. *Let $n>3$. There exists a constant $K=K(n)>0$ such that for any $u,v \in \mathcal{X}_{n/2}$ we have*

$$\|B(u,v)\|_{\mathcal{X}_{n/2}} \le K\|u\|_{\mathcal{X}_{n/2}}\|v\|_{\mathcal{X}_{n/2}}.$$

Proof. Applying Theorem 2.2 we get

$$\|B(u,v)(t)\|_{r,\infty} \le C(r,n) \sup_{0<s<t} \|uw\|_{rn/(r+n),\infty}$$

with an arbitrary $r>n/(n-1)$. Estimating the last norm we use the weak Hölder inequality to obtain

$$\|uw\|_{rn/(r+n),\infty} \le C(r,n)\|u\|_{r,\infty}\|w\|_{n,\infty}.$$

Since $w=\nabla E_n * v$, due to the weak Young inequality we have

$$\|w\|_{n,\infty} = \|\nabla E_n * v\|_{n,\infty} \le C(r,n)\|\nabla E_n\|_{nr/(nr+r-n),\infty}\|v\|_{r,\infty}.$$

The gradient $|\nabla E_n(x)| = \sigma_n^{-1}|x|^{1-n}$ belongs to $L^{n/(n-1),\infty}(\mathbb{R}^n)$, thus $nr/(nr+r-n)$ must be equal to $n/(n-1)$, which implies that $r=n/2$. This choice of r also implies that n must be greater than 3. So we have

$$\|\nabla E_n\|_{nr/(nr+r-n),\infty} = \sigma_n^{-1}\| \, |x|^{1-n}\|_{n/(n-1),\infty} \le n\sigma_n^{-1/n}.$$

Summing the inequalities above we arrive at

$$\|B(u,v)(t)\|_{r,\infty} \le C(n) \sup_{0<s<t} \|u(s)\|_{n/2,\infty}\|v(s)\|_{n/2,\infty} \le C(n)\|u\|_{\mathcal{X}_{n/2}}\|v\|_{\mathcal{X}_{n/2}}.$$

Taking the supremum with respect to $t>0$ completes the proof. ∎

The next step is to prove a similar estimate for $\|Lu\|_{\mathcal{X}_p}$.

LEMMA 3.3. *For $n > 3$, any function $u \in L^{r,\infty}(\mathbb{R}^n)$ and any external potential Φ such that $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ we have*

$$\|Lu\|_{\mathcal{X}_r} \le \ell \|u\|_{\mathcal{X}_r}$$

with a constant $\ell = \ell(n, r, \Phi)$ independent of u.

Proof. Due to Theorem 2.2 we have, for any $r > n/(n-1)$,

$$\|Lu(t)\|_{r,\infty} \le C(n,r) \sup_{0<s<t} \|u\nabla\Phi\|_{rn/(r+n),\infty}.$$

As in the previous lemma we estimate

$$\|u\nabla\Phi\|_{rn/(r+n),\infty} \le C(r,n)\|u\|_{\mathcal{X}_r}\|\nabla\Phi\|_{n,\infty},$$

which completes the proof. ■

REMARK. In Lemma 3.3 the function u belongs to $\mathcal{X}_r(\mathbb{R}^n)$ for any $r > n/(n-1)$ but in the following theorem we restrict ourselves to $r = n/2$ since the estimate for $B(u,v)$ holds true only for that r.

To apply Theorem 2.1 it remains to prove the weak continuity in t of the quantities $B(u,v)(t)$ and $Lu(t)$. This can be done by rewriting them in a similar way to what was done for the estimates above, so we leave this to the reader.

Applying Theorem 2.1 and Lemmas 3.2 and 3.3 we arrive at the following theorem:

THEOREM 3.4. *Let $n > 3$ and $u_0 \in L^{n/2,\infty}(\mathbb{R}^n)$. For Φ such that $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ and $\|\nabla\Phi\|_{n,\infty}$ is small enough, namely $\ell = \ell(n,\Phi)$ defined in Lemma 3.3 is smaller than 1, for $K = K(n)$ (defined in Lemma 3.2) and u_0 such that*

$$\|e^{t\Delta}u_0\|_{\mathcal{X}_{n/2}} < \frac{(1-\ell)^2}{4K},$$

there exists a solution $u \in \mathcal{X}_{n/2}$ of the equation

$$u = e^{t\Delta}u(0) + Lu + B(u,u). \tag{9}$$

The solution is unique among those satisfying the condition $\|u\|_{\mathcal{X}_{n/2}} < \frac{1-\ell}{2K}$.

4. Existence of a solution in a subspace of $\mathcal{X}_{n/2}$, $n \ge 3$. In this section we prove the existence of solutions in a subspace of $\mathcal{X}_{n/2}$ with a control of the decay of the $L^{r,\infty}(\mathbb{R}^n)$ norm ($r > n/2$) of the solution. Such a restriction allows us not only to prove the existence of global solutions in that subspace for $n > 3$ but also obtain global solutions in the three-dimensional case.

Define

$$\mathcal{Y}_\alpha = \{v \in L^\infty_{\mathrm{loc}}((0,\infty); L^{\alpha,\infty}(\mathbb{R}^n)) : \|v\|_{\mathcal{Y}_\alpha} \equiv \sup_{t>0} t^{1-n/2\alpha}\|v(t)\|_{\alpha,\infty} < \infty\},$$

where $\alpha > n/2$.

Let us begin with u_0.

The regularizing effect of the heat semigroup in the scale of Marcinkiewicz spaces is expressed in

LEMMA 4.1. *For any $u_0 \in L^{n/2,\infty}(\mathbb{R}^n)$ the solution of the heat equation $e^{t\Delta}u_0$ belongs to $\mathcal{Y}_\alpha$ for all $\alpha \geq n/2$.*

Proof. This is a simple consequence of (8). Indeed,

$$\|e^{t\Delta}u_0\|_{\mathcal{Y}_\alpha} = \sup_{t>0} t^{1-n/2\alpha}\|e^{t\Delta}u_0\|_{\alpha,\infty} \leq C(n,\alpha)\|u_0\|_{n/2,\infty}$$

for $\alpha > n/2$ and $u_0 \in L^{n/2,\infty}$. ∎

REMARK. Note that $u_0 \in L^{n/2,\infty}(\mathbb{R}^n)$ implies $e^{t\Delta}u_0 \in \mathcal{X}_{n/2} \cap \mathcal{Y}_\alpha$ for all $\alpha > n/2$, $n \geq 3$.

The following lemma establishes the required estimates for the bilinear form B in the space $\mathcal{Y}_\alpha$.

LEMMA 4.2. *Let $n \geq 3$. For any $n/2 < \alpha < n$ there exists a constant $K = K(n,\alpha)$ such that for all $u, v \in \mathcal{Y}_\alpha$,*

$$\|B(u,v)\|_{\mathcal{Y}_\alpha} \leq K\|u\|_{\mathcal{Y}_\alpha}\|v\|_{\mathcal{Y}_\alpha}.$$

Proof. Since the semigroup $e^{t\Delta}$ acts by convolution with the heat kernel, we have

$$\|B(u,v)(t)\|_{\alpha,\infty} \leq C(n,\alpha)\int_0^t (t-s)^{-n/2-1/2}\bigg\|\mathcal{P}\bigg(\frac{\cdot}{\sqrt{t-s}}\bigg) * (u\nabla\phi)\bigg\|_{\alpha,\infty} ds,$$

where $\mathcal{P}(x) = e^{-|x|^2}x$, with $\|\mathcal{P}(\cdot/\sqrt{t-s})\|_{p,\infty} = C(n,\alpha)(t-s)^{n/2p}\|\mathcal{P}\|_{p,\infty}$ (by (5)). Applying the weak Young and Hölder inequalities for $1/p = 1 + 1/n - 1/\alpha$ and $1/q = 2/\alpha - 1/n$ we get

$$\begin{aligned}
&\|B(u,v)(t)\|_{\alpha,\infty} \\
&\quad \leq C(n,\alpha)\int_0^t (t-s)^{-n/2\alpha}\|\mathcal{P}\|_{p,\infty}\|(u\cdot\nabla\phi)(s)\|_{q,\infty}\,ds \\
&\quad \leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}\|u(s)\|_{\alpha,\infty}\|\nabla\phi(s)\|_{n\alpha/(n-\alpha)}\,ds\Big)\|\mathcal{P}\|_{p,\infty} \\
&\quad \leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}\|u(s)\|_{\alpha,\infty}\|v(s)\|_{\alpha,\infty}\,ds\Big)\|\nabla E_n\|_{n/(n-1),\infty}\|\mathcal{P}\|_{p,\infty} \\
&\quad \leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}s^{-2+n/\alpha}\,ds\Big)\|\mathcal{P}\|_{p,\infty}\|u\|_{\mathcal{Y}_\alpha}\|v\|_{\mathcal{Y}_\alpha}\|\nabla E_n\|_{n/(n-1),\infty}.
\end{aligned}$$

The last integral converges if and only if $\alpha \in (n/2, n)$, and is equal to

$$t^{-1+n/2\alpha}\int_0^1 (1-x)^{-n/2\alpha}x^{-2+n/\alpha}\,dx = t^{-1+n/2\alpha}\mathbf{B}\left(-\frac{n}{2\alpha}+1, -1+\frac{n}{\alpha}\right),$$

where $\mathbf{B}$ is the Euler Beta function. The condition $\alpha \in (n/2, n)$ implies that the required conditions $p, q > 1$ are also fulfilled.

Multiplying both sides of the inequality by $t^{1-n/2\alpha}$ and taking supremum over $t > 0$ we arrive at the required estimate of the bilinear form $B(u, v)$. ∎

Now, we prove a similar estimate for the operator L.

LEMMA 4.3. *Let $n \geq 3$. For any $n/2 < \alpha < n$, $u \in \mathcal{Y}_\alpha$ and any external potential Φ such that $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ we have*

$$\|Lu\|_{\mathcal{Y}_\alpha} \leq \ell\|u\|_{\mathcal{Y}_\alpha},$$

where $\ell = \ell(n, \alpha, \Phi)$.

Proof. We split the integral into two terms:

$$\begin{aligned} Lu &= \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\Phi(s))\,ds \\ &= \int_0^{t/2} e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\Phi(s))\,ds + \int_{t/2}^{t} e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\Phi(s))\,ds \\ &\equiv L_1u + L_2u. \end{aligned}$$

For L_1 we have

$$\|L_1u(t)\|_{\alpha,\infty} \leq C(n,\alpha)\int_0^{t/2}(t-s)^{-n/2-1/2}\left\|\mathcal{P}\left(\frac{\cdot}{\sqrt{t-s}}\right)*(u\nabla\Phi)\right\|_{\alpha,\infty}ds.$$

Analogously as in Lemma 4.2 we estimate

$$\begin{aligned} &\|L_1u(t)\|_{\alpha,\infty} \\ &\quad\leq C(n,\alpha)\int_0^{t/2}(t-s)^{-1}\|\mathcal{P}\|_{n/(n-1),\infty}\|(u\cdot\nabla\Phi)(s)\|_{n\alpha/(n+\alpha),\infty}\,ds \\ &\quad\leq C(n,\alpha)\Big(\int_0^{t/2}(t-s)^{-1}\|u(s)\|_{\alpha,\infty}\,ds\Big)\|\nabla\Phi\|_{n,\infty}\|\mathcal{P}\|_{n/(n-1),\infty} \\ &\quad\leq C(n,\alpha)\Big(\int_0^{t/2}(t-s)^{-1}s^{-1+n/2\alpha}\,ds\Big)\|\mathcal{P}\|_{n/(n-1),\infty}\|u\|_{\mathcal{Y}_\alpha}\|\nabla\Phi\|_{n,\infty}. \end{aligned}$$

The last integral converges, and equals $t^{-1+n/2\alpha}\int_0^{1/2}(1-\xi)^{-1}\xi^{-1+n/2\alpha}\,d\xi$.

Thus we have

$$\|L_1 u(t)\|_{\alpha,\infty} \le C(n,\alpha,\Phi)t^{-1+n/2\alpha}\|u\|_{\mathcal{Y}_\alpha}. \tag{10}$$

Here, the singularity of the function $(t-s)^{-1}$ causes that the above estimate cannot be true in the whole interval $(0,t)$. This is the reason we split the operator L into L_1 and L_2. We circumvent this difficulty in the integral L_2 by using Theorem 2.2. Applying the inequality (7) with $r = \alpha, p = n\alpha/(n+\alpha)$, $a = t/2$, and then using the weak Hölder inequality we have

$$\begin{aligned}
\|L_2 u(t)\|_{\alpha,\infty} &\le \Big\| \int_{t/2}^{t} e^{(t-s)\Delta}\nabla(u\cdot\nabla\Phi)(s)\,ds\Big\|_{\alpha,\infty} \\
&\le C(n,\alpha) \sup_{t/2<s<t} \|(u\cdot\nabla\Phi)(s)\|_{n\alpha/(n+\alpha),\infty} \\
&\le C(n,\alpha) \sup_{t/2<s<t} \|u(s)\|_{\alpha,\infty}\|\nabla\Phi\|_{n,\infty} \\
&\le C(n,\alpha)\left(\frac{t}{2}\right)^{-(1-n/2\alpha)} \sup_{t/2<s<t} s^{1-n/2\alpha}\|u(s)\|_{\alpha,\infty}\|\nabla\Phi\|_{n,\infty} \\
&\le C(n,\alpha)t^{-(1-n/2\alpha)}\|u\|_{\mathcal{Y}_\alpha}\|\nabla\Phi\|_{n,\infty}.
\end{aligned}$$

Thus, we arrive at

$$\|L_2 u(t)\|_{\alpha,\infty} \le C(n,\alpha,\Phi)t^{-1+n/2\alpha}\|u\|_{\mathcal{Y}_\alpha}. \tag{11}$$

Summing up (10) and (11), multiplying the result by $t^{1-n/2\alpha}$, and finally taking supremum over $t>0$, we arrive at the required estimate for the operator L. ■

Until now we used only the space $\mathcal{Y}_\alpha$, $\alpha \in (n/2, n)$, but we may have difficulty in defining the convergence to initial data u_0 (the norms $\|u(t)\|_{\alpha,\infty}$ may tend to ∞ as $t \to 0$). Therefore we restrict our considerations to a subspace of $\mathcal{X}_{n/2}$, namely $\mathcal{X}_{n/2} \cap \mathcal{Y}_\alpha$, i.e. we consider two Banach space norms for the construction of solutions.

The norm in this space is given by

$$\|v\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} = \|v\|_{\mathcal{X}_{n/2}} + \|v\|_{\mathcal{Y}_\alpha}.$$

To prove the existence of a global solution for $n>3$ we can apply Lemmas 3.2 and 4.2 to get the estimate

$$\|B(u,v)\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} \le K\|u\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha}\|v\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha}.$$

Then, applying Lemmas 3.3 and 4.3, we get

$$\|Lu\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} \le \ell\|u\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha}.$$

These inequalities together with the weak continuity allow us to apply Theorem 2.1 to obtain the existence and uniqueness of global solutions for $n>3$.

But replacing $\mathcal{X}_{n/2}$ by its subspace gives us also the opportunity to prove global existence in the three-dimensional case.

To do this we have to replace Lemmas 3.2 and 3.3 by their more sophisticated version.

LEMMA 4.4. *For $n \geq 3$ and $\alpha \in (n/2, n)$, there exists a constant $K = K(n,\alpha)$ such that for all $u \in \mathcal{X}_{n/2}$ and $v \in \mathcal{Y}_\alpha$,*

$$\|B(u,v)\|_{\mathcal{X}_{n/2}} \leq K\|u\|_{\mathcal{X}_{n/2}}\|v\|_{\mathcal{Y}_\alpha}.$$

Proof. Similarly to the proof of Lemma 4.2 we begin with the estimate

$$\|B(u,v)(t)\|_{n/2,\infty} \leq C(n)\int_0^t (t-s)^{-n/2-1/2}\left\|\mathcal{P}\left(\frac{\cdot}{\sqrt{t-s}}\right) * (u\nabla\phi)\right\|_{n/2,\infty} ds,$$

where $\mathcal{P}(x)$ is as in the above mentioned proof. Applying once again the weak Young and Hölder inequalities for $1/p = 1+1/n-1/\alpha$ and $1/q = 1/n+1/\alpha$, we get

$$\begin{aligned}
\|B(u,v)(t)\|_{n/2,\infty} &\leq C(n,\alpha)\int_0^t (t-s)^{-n/2\alpha}\|\mathcal{P}\|_{p,\infty}\|(u\cdot\nabla\phi)(s)\|_{q,\infty}\,ds \\
&\leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}\|u(s)\|_{n/2,\infty}\|\nabla\phi(s)\|_{n\alpha/(n-\alpha)}\,ds\Big)\|\mathcal{P}\|_{p,\infty} \\
&\leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}\|u(s)\|_{n/2,\infty}\|v(s)\|_{\alpha,\infty}\,ds\Big)\|\nabla E_n\|_{n/(n-1),\infty}\|\mathcal{P}\|_{p,\infty} \\
&\leq C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}s^{-1+n/2\alpha}\,ds\Big)\|\mathcal{P}\|_{p,\infty}\|u\|_{\mathcal{X}_{n/2}}\|v\|_{\mathcal{Y}_\alpha}\|\nabla E_n\|_{n/(n-1),\infty}.
\end{aligned}$$

The last integral converges if and only if $\alpha > n/2$ and equals

$$\int_0^1 (1-x)^{-n/2\alpha}x^{-1+n/2\alpha}\,dx = \mathbf{B}\left(-\frac{n}{2\alpha}+1, \frac{n}{\alpha}\right),$$

where $\mathbf{B}$ is the Euler Beta function. The conditions $\alpha \in (n/2, n)$ and $n \geq 3$ imply that the required conditions $p, q > 1$ are also fullfilled. Taking the supremum over $t > 0$ we arrive at the required estimate of the bilinear form $B(u,v)$. ■

To prove the estimate for L for $n = 3$ (the proofs remain valid also for $n > 3$) we modify the proof of Lemma 4.3 to get the following

LEMMA 4.5. *Let $n \geq 3$. For $n/2 < \alpha < n$, $u \in \mathcal{Y}_\alpha$ and any external potential Φ such that $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ we have*

$$\|Lu\|_{\mathcal{X}_{n/2}} \leq \ell\|u\|_{\mathcal{Y}_\alpha},$$

where $\ell = \ell(n,\alpha,\Phi)$.

Proof. To prove the lemma we do not need to split the integral into two terms as in Lemma 4.3.

In fact, for $1/p = 1 + 1/n - 1/\alpha$ and any $\alpha \in (n/2, n)$ we have

$$\begin{aligned}
&\|Lu(t)\|_{n/2,\infty} \\
&\quad \le C(n,\alpha)\int_0^t (t-s)^{-n/2-1/2}\Big\|\mathcal{P}\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * (u\nabla\Phi)\Big\|_{n/2,\infty}\,ds \\
&\quad \le C(n,\alpha)\int_0^t (t-s)^{-n/2-1/2+n/2p}\|\mathcal{P}\|_{p,\infty}\|(u\cdot\nabla\Phi)(s)\|_{n\alpha/(n+\alpha),\infty}\,ds \\
&\quad \le C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}\|u(s)\|_{\alpha,\infty}\,ds\Big)\|\nabla\Phi\|_{n,\infty}\|\mathcal{P}\|_{p,\infty} \\
&\quad \le C(n,\alpha)\Big(\int_0^t (t-s)^{-n/2\alpha}s^{-1+n/2\alpha}\,ds\Big)\|\mathcal{P}\|_{p,\infty}\|u\|_{\mathcal{Y}_\alpha}\|\nabla\Phi\|_{n,\infty}.
\end{aligned}$$

The last integral is equal to $\int_0^1 (1-\xi)^{-n/2\alpha}\xi^{-1+n/2\alpha}\,d\xi < \infty$. Thus we have

$$\|Lu(t)\|_{n/2,\infty} \le C(n,\alpha,\Phi)\|u\|_{\mathcal{Y}_\alpha}.$$

Taking the supremum over $t > 0$ leads to the required estimate for L. ■

Thus taking into account Lemmas 4.2–4.5 we get

$$\|B(u,v)\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} \le K\|u\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha}\|v\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} \tag{12}$$

and

$$\|Lu\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} \le \ell\|u\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha}. \tag{13}$$

These inequalities and the weak continuity property (which can be proved in a similar way) applied to Theorem 2.1 give the existence of global solutions in the subspace $\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha$ for $n = 3$ as well as for $n > 3$.

We have

THEOREM 4.6. *Let $n \ge 3$ and $n/2 < \alpha < n$. For Φ with $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ and $u_0 \in L^{n/2,\infty}$ such that*

$$\ell = \ell(n,\alpha,\|\nabla\Phi\|_{n,\infty}) < 1, \qquad \|e^{t\Delta}u_0\|_{\mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha} < \frac{(1-\ell)^2}{4K},$$

where the constants $K = K(n,\alpha)$ and ℓ come from (12) *and* (13), *there exists a solution $u \in \mathcal{X}_{n/2}\cap\mathcal{Y}_\alpha$ to the problem* (9). *The solution is unique among those in the open ball centered at the origin with radius equal to $(1-\ell)/2K$.*

5. Conclusions and remarks. Since for $n > 3$ we proved the existence of solutions in both the whole space $\mathcal{X}_{n/2}$ and its subspace we obtain a simple consequence of the theorems above:

PROPOSITION 5.1. *Let $n > 3$ and $n/2 < \alpha < n$. For Φ such that $\nabla\Phi \in L^{n,\infty}(\mathbb{R}^n)$ with $\|\nabla\Phi\|_{n,\infty}$ small enough that $\ell = \ell(n,\alpha,\|\nabla\Phi\|_{n,\infty}) < 1$, there is no stationary solution $U = U(x)$ with small norm $\|U\|_{n/2,\infty}$.*

Proof. Assume that such a stationary solution U exists. Since its norm $\|U\|_{n/2,\infty}$ is small enough, we can obtain the unique global solution belonging to $\mathcal{X}_{n/2}$ (due to Theorem 3.4) and to its subspace $\mathcal{X}_{n/2} \cap \mathcal{Y}_\alpha$, $n/2 < \alpha < n$ (due to Theorem 4.6). From Theorem 3.4 we conclude that U must be that global solution (but independent of time). Due to the uniqueness and Theorem 4.6, U must also belong to $\mathcal{Y}_\alpha$. This implies that $t^{1-n/2\alpha}\|U\|_{n/2,\infty}$ has to be bounded, which is impossible. Thus, a stationary solution with small $L^{n/2,\infty}$ norm cannot exist. ■

The space of pseudomeasures. Another possible functional setting is the space of pseudomeasures

$$\mathcal{PM}^\alpha = \{v \in \mathcal{S}'(\mathbb{R}^n) : \widehat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ \|v\|_{\mathcal{PM}^\alpha} \equiv \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^n} |\xi|^\alpha |\widehat{v}(\xi)| < \infty\}.$$

The existence and uniqueness of the global ($\alpha = n-2$) and local ($\alpha \in (n-1, n-2]$) solutions in these spaces have been considered in [17].

We are mainly interested in the space to which the Chandrasekhar solution belongs, i.e. $\mathcal{PM}^{n-2}$. The intersection of $L^{n/2,\infty}(\mathbb{R}^n)$ and $\mathcal{PM}^{n-2}(\mathbb{R}^n)$ is not empty (e.g., the Chandrasekhar and homogeneous functions are there). However, it is not obvious how to characterize the sets $L^{n/2,\infty} \setminus \mathcal{PM}^{n-2}$ (or $\mathcal{PM}^{n-2} \setminus L^{n/2,\infty}$).

6. Asymptotic stability of solutions. In this section we describe the asymptotics of solutions to the evolution problem. The first observation is that if we let $u(t)$ be the solution with $u_0 \in L^{n/2,\infty}$, and $v(t)$ be the solution of the heat equation with the same initial condition, then, due to the estimates for $\|B(u,v)\|_{n/2,\infty}$ and $\|Lu\|_{n/2,\infty}$,

$$\begin{aligned}\|u(t) - v(t)\|_{n/2,\infty} \\ \le \|B(u,u)\|_{n/2,\infty} + \|Lu\|_{n/2,\infty} \le K\|u\|^2_{\mathcal{X}_{n/2}} + \ell\|u\|_{\mathcal{X}_{n/2}} \le \mathrm{const.}\end{aligned}$$

This means that the solutions $u(t)$ stay in a neighbourhood of $v(t)$, i.e. u is a perturbation of v.

THEOREM 6.1. *Let $u(t)$ and $v(t)$ be the solutions of the problem with the same external potential Φ and initial data $u(0)$ and $v(0)$ respectively. Choose $u(0)$ and $v(0)$ such that $\|e^{t\Delta}u(0)\|_{\mathcal{X}_{n/2}} \le \varepsilon < (1-\ell)^2/4K$ and $\|e^{t\Delta}v(0)\|_{\mathcal{X}_{n/2}} \le \varepsilon < (1-\ell)^2/4K$. If, additionally, the solutions of the heat equation with the same initial conditions $u(0), v(0)$ approach each other, i.e.*

$$\lim_{t\to\infty} \|e^{t\Delta}(u(0)-v(0))\|_{n/2,\infty} = 0, \tag{14}$$

then for ε small enough we also have

$$\lim_{t\to\infty} \|u(t)-v(t)\|_{n/2,\infty} = 0.$$

Proof. Before we prove the main conclusion, let us estimate the norm $\| \int_0^t e^{(t-s)\Delta}\nabla(fg)(s)\,ds\|_{n/2,\infty}$. Using similar calculations to those in Section 4 we have

$$\begin{aligned}
&\Big\| \int_0^t e^{(t-s)\Delta}\nabla(fg)(s)\,ds\Big\|_{n/2,\infty} \\
&\quad\le \Big\| \int_0^{t/p} e^{(t-s)\Delta}\nabla(fg)(s)\,ds\Big\|_{n/2,\infty} + \Big\| \int_{t/p}^{t} e^{(t-s)\Delta}\nabla(fg)(s)\,ds\Big\|_{n/2,\infty} \\
&\quad\le C_1(n)\|\mathcal{P}\|_{n/(n-1),\infty}\|g\|_{\mathcal{X}_n} \int_0^{t/p} (t-s)^{-1}\|f(s)\|_{n/2,\infty}\,ds \\
&\qquad + C_2(n)\|g\|_{\mathcal{X}_n} \sup_{t/p\le s\le t} \|f(s)\|_{n/2,\infty} \\
&\quad\le C_1(n)\,\frac{p}{p-1}\,\|\mathcal{P}\|_{n/(n-1),\infty}\|g\|_{\mathcal{X}_n} \int_0^{1/p} \|f(ts)\|_{n/2,\infty}\,ds \\
&\qquad + C_2(n)\|g\|_{\mathcal{X}_n} \sup_{t/p\le s\le t} \|f(s)\|_{n/2,\infty}.
\end{aligned}$$

Applying the above estimate for quadratic and linear terms we get

$$\begin{aligned}
\|u(t)-v(t)\|_{n/2,\infty} &\le \|e^{t\Delta}(u(0)-v(0))\|_{n/2,\infty} + \|B(u-v,u)\|_{n/2,\infty} \\
&\quad + \|B(v,u-v)\|_{n/2,\infty} + \|L(u-v)\|_{n/2,\infty} \\
&\le \|e^{t\Delta}(u(0)-v(0))\|_{n/2,\infty} \\
&\quad + (2K\max\{\|u\|_{\mathcal{X}_{n/2}}, \|v\|_{\mathcal{X}_{n/2}}\} + \ell) \\
&\quad \times \Big(C_1(n)\,\frac{p}{p-1}\,\|\mathcal{P}\|_{n/(n-1),\infty} \int_0^{1/p} \|(u-v)(ts)\|_{n/2,\infty}\,ds \\
&\quad + C_2(n) \sup_{t/p\le s\le t} \|(u-v)(s)\|_{n/2,\infty}\Big)
\end{aligned}$$

where K and ℓ are the constants obtained in Section 3 (we also use the estimate for $\|\nabla\phi\|_{n,\infty}$). Thus we arrive at

$$\|u(t)-v(t)\|_{n/2,\infty} \le \|e^{t\Delta}(u(0)-v(0))\|_{n/2,\infty} + (1-\sqrt{(\ell-1)^2-4K\varepsilon}) \times \Big(C_1(n)\|\mathcal{P}\|_{n/(n-1),\infty}\frac{p}{p-1}\int_0^{1/p}\|(u-v)(ts)\|_{n/2,\infty}\,ds + C_2(n)\sup_{t/p\le s\le t}\|(u-v)(s)\|_{n/2,\infty}\Big).$$

Since we want to find $\lim_{t\to\infty}\|u(t)-v(t)\|_{n/2,\infty}$, let us define

$$A = \limsup_{t\to\infty}\|u(t)-v(t)\|_{n/2,\infty} = \lim_{k\to\infty}\sup_{t\ge k\in\mathbb{N}}\|u(t)-v(t)\|_{n/2,\infty}.$$

Observe that, due to assumption (14) and the Lebesgue dominated convergence theorem, the inequality above can be rewritten as

$$A \le \Big(C_1(n)\|\mathcal{P}\|_{n/(n-1),\infty}\frac{1}{p-1} + C_2(n)\Big)(1-\sqrt{(\ell-1)^2-4K\varepsilon})A.$$

Since the second term of the right-hand side is less than 1, and $C_1(n)\|\mathcal{P}\|_{n/(n-1),\infty}\frac{1}{p-1} \to 0$ as $p\to\infty$, the asymptotic stability depends on the value of $C_2(n)$ only. If $C_2(n)<1$ then $A=0$. Even for $C_2(n)\ge 1$ we can also get $A=0$ but only for ε small enough. This implies that $A=0$, so

$$\lim_{t\to\infty}\|u(t)-v(t)\|_{n/2,\infty} = 0.$$

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Instytut Matematyczny
Uniwersytet Wrocławski
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: Andrzej.Raczynski@math.uni.wroc.pl

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