STUDIA MATHEMATICA 179 (3) (2007)

Around the Kato generation theorem for semigroups

by

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Abstract. We show that the result of Kato on the existence of a semigroup solving the Kolmogorov system of equations in l_1 can be generalized to a larger class of the so-called Kantorovich–Banach spaces. We also present a number of related generation results that can be proved using positivity methods, as well as some examples.

1. Introduction. In his seminal paper [18], Kato pioneered the use of what later became known as positivity techniques to prove the existence of a semigroup solving the Kolmogorov system of equations in the space l_1, and provided some characterization of its generator. This result can also be found in several monographs such as, e.g., [11, 12, 16]. It seemed that a particular lattice structure of l_1 was essential for the proof, and a few generalizations of Kato's result which appeared later, [26, 27, 4, 5, 7], were all confined to the so-called AL-spaces which include l_1 and L_1 spaces. Precisely speaking, an AL-space is a Banach lattice whose norm is additive on the positive cone (as are the norms of l_1 and L_1 spaces) and, in fact, it can be proved that every AL-space is lattice isometric to an L_1 space (see e.g. [2, Theorem 12.26]).

However, as we will see in this article, what Kato's proof really requires is not the AL structure of the underlying space but the property that any non-negative increasing and norm bounded sequence is norm convergent. The latter is the defining property of the so-called Kantorovich–Banach spaces (KB-spaces) that include, among others, the AL-spaces and reflexive spaces. Precisely, it can be proved that X is a KB-space if and only if the space c_0 is not lattice embeddable in X [2, Theorem 14.12].

2000 *Mathematics Subject Classification*: 47D06, 47B60, 34G10, 92D25.

Key words and phrases: KB-spaces, positive semigroups, birth-and-death problems, perturbations of semigroups.

Work of J. Banasiak supported by the National Research Foundation of South Africa under the grant GUN 2053716 and the University of KwaZulu-Natal Research Fund.

Research of M. Lachowicz supported by Grant UE Contract no. MRTN-CT-2004-503661.

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In this paper we shall view Kato's result as a perturbation theorem providing conditions under which a positive perturbation of a generator of a positive semigroup is still a generator of a semigroup and, using the above observation, we extend it to KB-spaces. Moreover, we provide several other generation results which utilize similar techniques and, in particular, we generalize a theorem of Desch [14, 27] which allows us to deal with not necessarily positive perturbations.

We also relate our results to the existing perturbation theorems and provide examples of applications to birth-and-death type problems.

Acknowledgements. The authors express sincere thanks to the anonymous referee for numerous comments and suggestions which substantially helped to improve the paper.

2. Mathematical preliminaries. In this section we shall introduce the notation used in the paper and collect most of the definitions and results on which the paper is based for easy reference.

In many natural sciences' applications the quantities described by the model should be real and nonnegative, e.g., probability, particle or mass density, absolute temperature. To cater for this, the original setting for abstract models should be a real Banach space with a notion of positivity compatible with the original linear structure, that is, a real Banach lattice. However, when we discuss spectral properties of the model, we shall move to a complex space through the procedure called complexification which is described in detail below.

Let X be a real Banach lattice. We denote by X^* its topological dual which is also a Banach lattice; the duality pairing is denoted by $\langle\cdot,\cdot\rangle$. The order in any Banach lattice will be denoted by $\geq$. For any subset Z of X or X^*, we denote by Z_+ the nonnegative part of Z, that is, the set of all $z \in Z$ satisfying $z \geq 0$.

For a given linear operator A on X, $\varrho(A)$ and $\sigma(A)$ denote, respectively, the resolvent set and the spectrum of A. The spectrum $\sigma(A)$ is subdivided into the point spectrum (eigenvalues) $\sigma_{\mathrm{p}}(A)$, the continuous spectrum $\sigma_{\mathrm{c}}(A)$ and the residual spectrum $\sigma_{\mathrm{r}}(A)$. Let $R(\lambda, A)$, $\lambda \in \varrho(A)$, denote the resolvent of A. If A is bounded, then $r_\sigma(A)$ denotes its spectral radius.

We will be working with positive operators. Let us recall that a linear operator A on a Banach lattice X is said to be *positive* if $Ax \geq 0$ for all $x \in D(A)_+ := D(A) \cap X_+$. A positive operator defined on the whole space is bounded. Furthermore, an additive positive operator is fully determined by its restriction to the positive cone, that is, if A is an additive positive operator on X_+, then it extends to a unique positive linear operator on X. Also the norm of A is determined by the values of $\|Ax\|$ on $\{x \in X_+;\ \|x\| \leq 1\}$. In particular, if $0 \leq A \leq B$, then $\|A\| \leq \|B\|$. The above results are well-known and the proofs can be found in, e.g., [8, Section 2.2].

An operator A is called *resolvent positive* if there exists $\omega \in \mathbb{R}$ such that $R(\lambda, A)$ is positive for all $\lambda > \omega$. A semigroup $(G(t))_{t\geq 0}$ is said to be *positive* if the operators $G(t)$ are positive for all $t \geq 0$. It turns out that $(G(t))_{t\geq 0}$ is positive if and only if its generator is resolvent positive.

An important class of Banach lattices, which will play a significant rôle later, are AL-spaces [1, 2]. We say that a Banach lattice is an *AL-space* if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$. A standard (and, up to a lattice isometry, generic) example of an AL-space is $L_1(\Omega, d\mu)$, where (Ω, μ) is a measure space [2, Theorem 12.26].

Another important class of Banach lattices used in this paper are KB-spaces. We say that a Banach lattice X is a *KB-space* (*Kantorovich–Banach space*) whenever every increasing norm bounded sequence of elements of X_+ is norm convergent. The following statements are equivalent [2, Theorem 14.12]:

1. X is a KB-space;
2. X is weakly sequentially complete;
3. c_0 is not (lattice) embeddable in X.

In particular, AL-spaces and reflexive Banach lattices are KB-spaces.

Even if we are working in a real setting, to apply, for instance, spectral theory we need to move to a complex setting. Let X be a real space. Its complexification is defined as $X_C = X \times X$ where, following the scalar convention, we shall write $(x, y) = x + iy$. Vector operations are defined as in the scalar case, and the real vector space X is identified with the real subspace $X + i0 \subset X_C$. The norm

$$\|x + iy\|_C := \| \, |x + iy| \, \|, \tag{1}$$

with modulus defined as $|x + iy| = \sup_{\theta\in[0,2\pi]}(x \cos\theta + y \sin\theta)$ (see, e.g., [1, p. 104]), is a lattice norm on X_C which coincides with the standard one on $l_p, L_p(\Omega)$, $1 \leq p \leq \infty$, and $C(\Omega)$. Moreover, $\| \cdot \|_C$ is equivalent to any usual product norm in $X \times X$. Note, however, that standard product norms on $X \times X$ may fail to preserve the homogeneity of the norm (see [8, Example 2.88]).

An element $x \in X_C$ is said to be *positive* ($x \geq 0$) if $x = |x|$ (see e.g. [20, p. 244]). In particular, only real elements of X_C can be positive.

If A is a linear operator on X with domain $D(A)$, then it can be extended to X_C according to the formula

$$A_C(x + iy) = Ax + iAy, \quad D(A_C) = D(A) + iD(A). \tag{2}$$

It follows that if A is bounded and positive, then

$$\|A_C\|_C = \|A\|, \tag{3}$$

but, in general, (3) fails for nonpositive operators. However, it is interesting to note that in $l_p, L_p(\Omega)$ and $C(\Omega)$, (3) holds for arbitrary operators [13, pp. 175–176].

Thus, when dealing with real positive operators, we can confine ourselves to real Banach spaces. In fact, fundamental theorems of semigroup theory such as the Hille–Yosida or Lumer–Phillips theorems, various perturbation theorems and Trotter–Kato type results are valid in both real and complex setting. Hence, for instance, if an operator A generates a positive semigroup of contractions in a real Banach lattice X, then the complexification (2) of this semigroup is a semigroup of positive contractions on X_C. In particular, the complexification A_C of A is also a dissipative operator in X_C.

The following, frequently used lattice versions of the dominated and monotone convergence theorems for series are relatively straightforward to prove [8, Theorem 2.91].

THEOREM 2.1. *Let $(x_n(t))_{n\in\mathbb{N}}$ be family of nonnegative sequences in a Banach lattice X, parameterized by a parameter $t \in T \subset \mathbb{R}$, and let $t_0 \in \overline{T}$.*

(i) *If for each $n \in \mathbb{N}$ the function $t \mapsto x_n(t)$ is nondecreasing and $\lim_{t\nearrow t_0} x_n(t) = x_n$ in norm, then*

$$\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t) = \sum_{n=0}^{\infty} x_n, \tag{4}$$

irrespective of whether the right-hand side exists in X or $\|\sum_{n=0}^{\infty} x_n\| := \sup\{\|\sum_{n=0}^{N} x_n\|;\ N \in \mathbb{N}\} = \infty$. In the latter case the equality should be understood as the norms of both sides being infinite.

(ii) *If $\lim_{t\to t_0} x_n(t) = x_n$ in norm for each $n \in \mathbb{N}$ and there exists $(a_n)_{n\in\mathbb{N}}$ with $\sum_{n=0}^{\infty} \|a_n\| < \infty$ such that $x_n(t) \le a_n$ for any $t \in T$ and $n \in \mathbb{N}$, then* (4) *holds as well.*

REMARK 2.2. Note that if X is a KB-space, then $\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t) \in X$ implies the convergence of $\sum_{n=0}^{\infty} x_n$. In fact, since $x_n \ge 0$ (by closedness of the positive cone), $N \mapsto \sum_{n=0}^{N} x_n$ is nondecreasing, and hence either $\sum_{n=0}^{\infty} x_n \in X$, or $\|\sum_{n=0}^{\infty} x_n\| = \infty$, and in the latter case we would have $\|\lim_{t\nearrow t_0} \sum_{n=0}^{\infty} x_n(t)\| = \infty$.

The following formula will be frequently used. If A is the generator of a C_0-semigroup $(G(t))_{t\ge0}$, then for any $x \in X$,

$$G(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^{n} x, \tag{5}$$

and the limit is uniform in t on bounded intervals. In particular, (5) shows that if $R(\lambda, A) \geq 0$ for sufficiently large λ, then $(G(t))_{t\geq 0}$ is a positive semigroup.

The notation $A \in \mathcal{G}(M, \omega)$ means that the operator A is the infinitesimal generator of the semigroup $(G(t))_{t\geq 0}$ satisfying the estimate $\|G(t)\| \leq Me^{\omega t}$ for some constants $M \geq 0$ and $\omega \in \mathbb{R}$.

For the reader's convenience we also recall the Trotter–Kato theorem [21, Theorem 3.4.3] and some of its consequences which play an important rôle in this paper.

THEOREM 2.3. *Assume $A_n \in \mathcal{G}(M, \omega)$. If there exists λ_0 with $\Re\lambda_0 > \omega$ such that*

(a) $\lim_{n\to\infty} R(\lambda_0, A_n)x = R(\lambda_0)x$ *exists for every $x \in X$,*
(b) *the range of $R(\lambda_0)$ is dense in X,*

then there exists a unique operator $A \in G(M, \omega)$ such that $R(\lambda_0) = R(\lambda_0, A)$. Moreover, if $(G_n(t))_{t\geq 0}$ is the semigroup generated by A_n and $(G(t))_{t\geq 0}$ is generated by A, then for any $x \in X$,

$$\lim_{n\to\infty} G_n(t)x = G(t)x \tag{6}$$

uniformly in t on bounded intervals.

Assumption (b) can be verified by applying the following result [17, Theorem IX.2.17]:

COROLLARY 2.4. *If the limit*

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A_n)x = x \tag{7}$$

is uniform in n, then $R(\lambda)$ is the resolvent of a densely defined closed operator in X.

3. Generalized Kato perturbation theorem. In this section we shall discuss a generalization of Kato's perturbation theorem ([18]) and of some related results to KB-spaces. A more exhaustive discussion of this topic can be found in [8].

LEMMA 3.1. *Let $0 \neq x \in X_+$. Then there is $x^* \in X_+^*$ satisfying $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$.*

Proof. We have $\|x\| = \sup_{\|y^*\|\leq 1}\langle y^*, x\rangle = \langle x^*, x\rangle$ for some $x^* \in X^*$ with $\|x^*\| = 1$, by the Hahn–Banach theorem. If $0 \neq x^* \notin X_+^*$, then

$$0 < \|x\| = \langle x^*, x\rangle = \langle x_+^*, x\rangle - \langle x_-^*, x\rangle \leq \langle x_+^*, x\rangle$$

and $\|x_+^*\| \leq \|x^*\| \leq 1$ as $x_+^* \leq |x^*|$. Thus, $\langle x_+^*, x\rangle = \langle x^*, x\rangle = \|x\|$. If $\|x_+^*\| < 1$, then for $\widetilde{x}^* = \|x_+^*\|^{-1}x_+^*$ we would have $\|\widetilde{x}^*\| = 1$ and $\langle \widetilde{x}^*, x\rangle > \langle x^*, x\rangle$, which is impossible. Thus, x_+^* satisfies the conditions of the lemma. ∎

Theorem 3.2. *Let X be a real KB-space. Assume that the operators $(A, D(A))$ and $(B, D(B))$ with $D(A) \subset D(B)$ satisfy*:

(A1) *A generates a positive semigroup of contractions $(G_A(t))_{t\geq 0}$*,
(A2) *$r_\sigma(BR(\lambda, A)) \leq 1$ for some $\lambda > 0$*,
(A3) *$Bx \geq 0$ for $x \in D(A)_+$*,
(A4) *for any $x \in D(A)_+$ there is $x^* \geq 0$ such that $\langle x^*, x\rangle = \|x\|$ and $\langle x^*, (A + B)x\rangle \leq 0$.*

Then there is an extension $(K, D(K))$ of $(A+B, D(A))$ generating a strongly continuous semigroup of positive contractions, *denoted by $(G_K(t))_{t\geq 0}$. The generator K satisfies*, *for all $\lambda > 0$*,

$$R(\lambda, K)x = \lim_{n\to\infty} R(\lambda, A)\sum_{k=0}^{n}(BR(\lambda, A))^k x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x. \tag{8}$$

Remark 3.3. If $-A$ is a positive operator (which was the case in the situation dealt with by Kato), then assumption (A2) can be replaced by the simpler one

(A2′) $\|Bx\| \leq \|Ax\|$, $x \in D(A)_+$.

In fact, we then have

$$0 \leq -AR(\lambda, A) = I - \lambda R(\lambda, A) \leq I$$

so that $\|AR(\lambda, A)y\| \leq \|y\|$ for all $y \in X_+$, and by positivity, for any $y \in X$. Thus, $\|Ax\| \leq \|(\lambda I - A)x\|$ for all $x \in D(A)$. Hence, for any $x \in D(A)_+$,

$$\|Bx\| \leq \|Ax\| \leq \|(\lambda I - A)x\|,$$

which, upon substituting $x = R(\lambda, A)y$, yields $\|BR(\lambda, A)y\| \leq \|y\|$ for y in X_+. Thus $\|BR(\lambda, A)\| \leq 1$ and (A2) is satisfied.

Remark 3.4. If assumption (A2) is satisfied for some $\lambda_0 > 0$, then it is satisfied for all $\lambda > \lambda_0$. In fact, writing the resolvent equation

$$BR(\lambda, A) - BR(\lambda_0, A) = (\lambda_0 - \lambda)BR(\lambda_0, A)R(\lambda, A)$$

we see from the positivity that $BR(\lambda_0, A) \geq BR(\lambda, A)$, and the norm estimate follows.

Proof of Theorem 3.2. We define operators K_r, $0 \leq r < 1$, by $K_r = A + rB$, $D(K_r) = D(A)$. By writing

$$R(\lambda, A + rB) = (I - rBR(\lambda, A))(\lambda I - A),$$

we see that as $r_\sigma(rBR(\lambda,A)) \le r < 1$, the resolvent $R(\lambda, A+rB)^{-1}$ exists and is given by

$$R(\lambda,K_r) = R(\lambda,A)\sum_{n=0}^{\infty} r^n (BR(\lambda,A))^n \tag{9}$$

with the series converging absolutely and each term being positive. Let $x^* \ge 0$ be such that $\langle x^*, x\rangle = \|x\|$ (see Lemma 3.1). For $x \in D(A)_+$ and $r < 1$ we have

$$\langle x^*, (A+rB)x\rangle = \langle x^*, (A+B)x\rangle + (r-1)\langle x^*, Bx\rangle \le 0 \tag{10}$$

on account of (A4) and $Bx, x^* \ge 0$. Thus, following the argument of [21, p. 14], we obtain, by the above,

$$\|(\lambda I - K_r)x\|\,\|x\| \ge \langle x^*, (\lambda I - K_r)x\rangle = \lambda\langle x^*, x\rangle - \langle x^*, K_r x\rangle \ge \lambda\|x\|$$

for all $x \in D(A)_+$. Taking $y \in X_+$, we have $R(\lambda, K_r)y = x \in D(A)_+$ so that we can rewrite this as

$$\|R(\lambda,K_r)y\| \le \lambda^{-1}\|y\| \tag{11}$$

for all $y \in X_+$ and, since $R(\lambda, K_r)$ is positive, this can be extended to the whole space X. Therefore, by the Hille–Yosida theorem, for each $0 \le r < 1$ the operator $(K_r, D(A))$ generates a contraction semigroup which is positive. This semigroup will be denoted by $(G_r(t))_{t\ge 0}$.

From (9) we see that the net $(R(\lambda,K_r)x)_{0\le r<1}$ is increasing as $r \nearrow 1$ for each $x \in X_+$ and $(\|R(\lambda,K_r)x\|)_{0\le r<1}$ is bounded, by (11). As we assumed that X is a KB-space, there is an element $y_{\lambda,x} \in X_+$ such that

$$\lim_{r\nearrow 1} R(\lambda,K_r)x = y_{\lambda,x}$$

in X. This convergence can then be extended onto the whole space by linearity and, by (11), we obtain the existence of a bounded positive operator on X which we shall denote by $R(\lambda)$. To be able to use the Trotter–Kato theorem, it is now enough to prove that for any $x \in X$ the limit

$$\lim_{\lambda\to\infty} \lambda R(\lambda,K_r)x = x$$

is uniform in r so that the assumptions of Corollary 2.4 are satisfied. Let $x \in D(A)$. Then, as

$$K_r R(\lambda,K_r) = I - \lambda R(\lambda,K_r),$$

we have, by (11),

$$\begin{aligned}\|\lambda R(\lambda,K_r)x - x\| = \|K_r R(\lambda,K_r)x\| = \|R(\lambda,K_r)K_r x\| &\le \lambda^{-1}\|(A+rB)x\|\\ &\le \lambda^{-1}(\|Ax\| + \|Bx\|),\end{aligned}$$

and the limit is indeed uniform in r. Since $D(A)$ is dense in X, for $y \in X$

we take $x \in D(A)$ with $\|y - x\| < \varepsilon$ to obtain, again by (11),

$$\begin{aligned}\|\lambda R(\lambda, K_r)y - y\| &\le \lambda\|R(\lambda, K_r)(y - x)\| + \|y - x\| + \|\lambda R(\lambda, K_r)x - x\| \\ &\le 2\varepsilon + \lambda^{-1}(\|Ax\| + \|Bx\|),\end{aligned}$$

which gives uniform convergence. The Trotter–Kato theorem shows that $R(\lambda)$ is defined for all $\lambda > 0$ and it is the resolvent of a densely defined closed operator K which generates a semigroup of contractions $(G_K(t))_{t\ge0}$; moreover, for any $x \in X$,

$$\lim_{r\nearrow1} G_r(t)x = G_K(t)x, \tag{12}$$

and the limit is uniform in t on bounded intervals and monotone as $r \nearrow 1$ and $x \ge 0$ (the monotonicity follows from the monotonicity of resolvents in r and the representation formula (5) for semigroups).

Furthermore, from Theorem 2.1(i) we have

$$\begin{aligned}R(\lambda, K)x &= \lim_{r\nearrow1} \sum_{k=0}^{\infty} r^k R(\lambda, A)(BR(\lambda, A))^k x \\ &= \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x, \qquad x \in X_+,\end{aligned} \tag{13}$$

where, in particular, the last series converges by Remark 2.2. Extension to X is done by linearity and clearly

$$\sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x = \lim_{n\to\infty} R(\lambda, A) \sum_{k=0}^{n} (BR(\lambda, A))^k x, \qquad x \in X,$$

which completes the proof of (8).

The proof that K is an extension of $A + B$ is done exactly as in [18], by noting that the nth partial sum $R^{(n)}(\lambda)$ of the series in (13) satisfies, for $x \in D(A)$,

$$R^{(n)}(\lambda)(\lambda I - A)x = x + R^{(n-1)}(\lambda)Bx.$$

Hence, letting $n \to \infty$ and rearranging we obtain $R(\lambda, K)(\lambda I - (A + B))x = x$, which shows that $K \supseteq A + B$. ∎

We explicitly state the version of Theorem 3.2 for complex spaces.

Corollary 3.5. *Let X_C be the complexification of X with the norm* (1). *If the assumptions of Theorem* 3.2 *are satisfied, then the complexification K_C of K defined by* (2) *is the generator of a positive semigroup of contractions on X_C, which is the complexification of $(G_K(t))_{t\ge0}$. In particular, this is true in $l_p, L_p(\Omega)$, $p \in [1,\infty)$, and in $C(\Omega)$ spaces, with the usual norms.*

Proof. The fact that the complexification of $(G_K(t))_{t\ge0}$ is a positive semigroup of contractions in X_C follows from the properties of complexification discussed in Section 2. Since the norm in X_C is equivalent to any standard product norm in $X \times X$, this semigroup is differentiable at $t = 0$ only on elements of the form $x + iy$ where $x, y \in D(K)$. The final statement follows from the fact that the usual norm on these spaces coincides with the complexification norm (1). ∎

One should compare this theorem with Theorems 3.3.2, 3.3.4 and Corollaries 3.3.3 and 3.3.5 of [21]. Firstly, we observe that Theorem 3.3.2 implies that Corollary 3.3.3, Theorem 3.3.4 and Corollary 3.3.5 of [21] can be phrased in the following, more general form (see also [9]).

THEOREM 3.6. *Let A and B be linear operators in a Banach space X with $D(A) \subset D(B)$ and*

$$\|Bx\| \le \alpha \|Ax\| + \beta \|x\|, \qquad x \in D(A), \tag{14}$$

where $0 \le \alpha \le 1$ and $\beta \ge 0$. Assume further that A is the generator of a semigroup of contractions and $A + tB$ is dissipative for any $t \in [0, 1]$. Then:

- (i) *if $\alpha < 1$, then $A + B$ is the generator of a contractive semigroup*;
- (ii) *if $\alpha = 1$ and additionally B^*, the adjoint of B, is densely defined, then $\overline{A + B}$ is the generator of a contractive semigroup.*

In particular, *if X is reflexive and B is closable*, *then the assumption of* (ii) *is satisfied.*

The difference between [21] and this formulation is that in the former the author assumes that B is dissipative, which is a stronger assumption used only to prove that $A + tB$ is dissipative for all $t \in [0, 1]$ (see the comment [21, p. 83]). Having noted this, the proof of the above theorem follows as the respective proofs of [21]. In many applications, such as discussed in this paper, B is not dissipative while the $A + tB$ are.

It is possible to strengthen Theorem 3.6 to bring it closer to Theorem 3.2 (see [9]), as follows:

THEOREM 3.7. *Let X be a Banach lattice and let $(G(t))_{t \ge 0}$ be the semigroup generated by $A+B$ or $\overline{A + B}$ under the conditions of Theorem* 3.6. *If A is a resolvent positive operator and B is positive*, *then $(G(t))_{t \ge 0}$ is positive.*

Thus, if X is reflexive and B is closable, then Theorem 3.6 is evidently stronger than Theorem 3.2 as it requires positivity of neither $(G_A(t))_{t \ge 0}$ nor B. Moreover, A-boundedness of B (requirement (14)) is weaker than assumption (A2) and, finally, in Theorem 3.6 we obtain the full characterization of the generator as $\overline{A + B}$. However, checking the closability of the operator B in particular applications may be difficult while positivity is often obvious. Also, there are a large class of nonclosable operators which can be nevertheless positive, e.g. finite-rank operators (in particular, functionals) are closable if and only if they are bounded [17, p. 166]. An example of this kind is presented below. Moreover, Theorem 3.2 gives a constructive formula

(8) for the resolvent of the generator and allows other representation results (see [8]). Finally, what is probably most important, in nonreflexive spaces Theorem 3.2 covers a substantially different class of phenomena as in many cases the generator does not coincide with $\overline{A+B}$.

EXAMPLE 3.8. We provide an example of the Cauchy problem for which the results of [21, Section 3.3] are not immediately applicable but which can be easily solved using Theorem 3.2. Consider the problem

$$\frac{\partial x}{\partial t}(t,s) = -\nu(s)x(t,s) + \mu(s)\int_0^1 \nu(r)x(t,r)\,dr, \qquad 0 \le s \le 1,\ t > 0,$$
$$x(0,s) = x_0(s) \tag{15}$$

in $X = L_2([0,1])$ (over the real numbers). Assume that $0 \le \nu \in L_1([0,1]) \setminus L_2([0,1])$ satisfies $\int_0^1 \nu(s)\,ds = 1$ and that $0 \le \mu \le \nu$ satisfies $\int_0^1 \mu^2(s)\,ds = 1$. These assumptions are satisfied, e.g., by $\nu(s) = 2s^{-1/2}$ and $\mu = \nu$ on $[e^{-4},1]$ and $\mu = 0$ elsewhere. Defining $Ax = -\nu x$ on $D(A) = \{x \in X;\ \nu x \in X\}$ we see that A generates a positive semigroup of contractions. By the Schwarz inequality we have

$$\int_0^1 \nu(s)x(s)ds \le \sqrt{\int_0^1 \nu^2(s)x^2(s)\,ds},$$

hence $Bx := \mu\int_0^1 \nu(s)x(s)\,ds$ is well-defined and positive on $D(A)$. It is a rank-one operator and thus it is not closable (see [17, p. 166]). Even more, simple calculation shows that, since $\nu \notin X$, we have $D(B^*) = \{0\}$. On the other hand,

$$\begin{aligned}\|Bx\|^2 &= \int_0^1 \mu^2(s)\Big(\int_0^1 \nu(r)x(r)\,dr\Big)^2 ds = \Big(\int_0^1 \nu(r)x(r)\,dr\Big)^2\\ &\le \int_0^1 \nu^2(r)x^2(r)\,dr = \|Ax\|^2,\end{aligned}$$

and so assumption (A2′) of Remark 3.3 is satisfied. Furthermore, taking $x \ge 0$ and using $\mu \le \nu$, we obtain

$$\begin{aligned}(x, Ax+Bx) &= -\int_0^1 \nu(s)x^2(s)\,ds + \Big(\int_0^1 \mu(s)x(s)\,ds\Big)\Big(\int_0^1 \nu(s)x(s)\,ds\Big)\\ &\le -\int_0^1 \nu(s)x^2(s)\,ds + \Big(\int_0^1 \nu(s)x(s)\,ds\Big)^2\end{aligned}$$

$$\le -\int_0^1 \nu(s)x^2(s)\,ds + \Big(\int_0^1 \nu(s)\,ds\Big)\Big(\int_0^1 \nu(s)x^2(s)\,ds\Big)$$
$$= \int_0^1 \nu(s)x^2(s)\,ds\Big(-1 + \int_0^1 \nu(s)\,ds\Big) = 0,$$

where in the last line we used the remaining assumption on ν. Here $(\cdot,\cdot)$ denotes the standard scalar product in $L_2([0,1])$ and, to simplify notation, we wrote x instead of $x/\|x\|$ which does not affect assumption (A4). This shows that all assumptions of Theorem 3.2 are satisfied and there is an extension of $A + B$ which generates a positive semigroup of contractions.

REMARK 3.9. Yet another look at the relation between K and $A+B$ in L_p spaces is offered by the result of [24] that states that if T is a positive operator on L_p satisfying $\|T\| \le 1$ and $p \in (1,\infty)$, then there exists a primitive nth root of unity in $\sigma_{\mathrm{p}}(T)$ if and only if every nth root of unity is in $\sigma_{\mathrm{p}}(T)$ if and only if the same holds true for T^*. Setting $T = BR(\lambda, A)$ and invoking [15, Theorem 3.2 and the preceding considerations], we see that as $1 \notin \sigma_{\mathrm{p}}(BR(\lambda, A))$, we have $1 \notin \sigma_{\mathrm{p}}(BR(\lambda, A))^*$, so that $1 \notin \sigma_{\mathrm{r}}BR(\lambda, A)$ and consequently $K = \overline{A+B}$.

A crucial property that allows for the proof of the above result of [24] for $p > 1$ but not for $p = 1$ is that $x \in X^*_+$ and $x \le T^*x$ implies $x = T^*x$. Clearly, it is satisfied for any $p \in (1,\infty)$ but, in general, fails in $X^* = L_\infty$.

An important property of the semigroup constructed in [18] (see also [16]) is that it is a smallest substochastic semigroup whose generator is an extension of $A + B$. In the present setting we can even prove a slightly stronger result.

PROPOSITION 3.10. *Let D be a core of A. If $(G(t))_{t\ge 0}$ is another positive semigroup generated by an extension of $(A + B, D)$, then $G(t) \ge G_K(t)$.*

Proof. Let K' be the generator of $(G(t))_{t\ge 0}$. First, we show that K' is an extension of $A+B$. If $x \in D(A)$, then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = Ax$. Assumption (A2) ensures, in particular, that $D(A) \subset D(B)$ and B is continuous on $D(A)$ in the graph norm of A, hence Bx_n converges to Bx and therefore $(A + B)x_n$ converges to $(A + B)x$. Since K' is closed (as a generator) and since for $x_n \in D$ we have $K'x_n = (A + B)x_n$, x belongs to $D(K')$ and $K'x = (A + B)x$.

Since K' generates a positive semigroup, the resolvent $R(\lambda, K')$ exists and is positive for sufficiently large λ. As $D(K') \supset D(A)$ and $K'x = (A + B)x$ on $D(A)$, we have

$$\begin{aligned} R(\lambda, K') - R(\lambda, K_r) &= (R(\lambda, K')(\lambda I - K_r) - I)R(\lambda, K_r) \\ &= R(\lambda, K')(\lambda I - K_r - \lambda I + K')R(\lambda, K_r) = R(\lambda, K')(K' - K_r)R(\lambda, K_r) \\ &= R(\lambda, K')(A + B - A - rB)R(\lambda, K_r) = (1 - r)R(\lambda, K')BR(\lambda, K_r). \end{aligned}$$

Since $r < 1$ and all the operators are positive, we obtain

$$R(\lambda, K') \geq R(\lambda, K_r).$$

Since $R(\lambda, K_r) \nearrow R(\lambda, K)$, we have $R(\lambda, K') \geq R(\lambda, K)$ and by the representation formula (5) we conclude that this inequality holds for semigroups. ■

Applicability of Theorem 3.2 depends on whether we can prove that the operator A is the generator of a positive semigroup of contractions. In many cases of practical importance: birth-and-death problems, pure fragmentation or spatially homogeneous linear Boltzmann equation [8], this is obvious as A is a diagonal (multiplication) operator. If, however, we allow spatial dependence and/or external field, then A is given by

$$A = A_0 - N,$$

where A_0 can be a first order streaming operator, a diffusion operator or possibly the sum of both, and N is the multiplication by a positive, but often very singular, function. Then determining whether A is a generator becomes a nontrivial problem. The following theorem, the proof of which uses ideas of Theorem 3.2, provides a partial solution to this problem.

THEOREM 3.11. *Let $(A_0, D(A_0))$ be the generator of a positive semigroup of contractions on a KB-space X and $(N, D(N))$ be a positive operator. Assume that there exists an increasing sequence $((N_n, D(N_n)))_{n\in\mathbb{N}}$ of positive operators satisfying*:

1. *$D(A_0) \cap D(N)$ is dense in X.*
2. *$D(N_n) \supset D(N)$.*
3. *There is a dense set $D \subset D(A_0) \cap D(N)$ such that $\lim_{n\to\infty} N_n y = Ny$ for $y \in D$.*
4. *$(A_0 - N_n, D(A_0) \cap D(N_n))$ generates a positive semigroup of contractions for $n = 1, 2, \ldots$.*

Then there is an extension $(\mathcal{A}, D(\mathcal{A}))$ of $(A_0 - N, D)$ which generates a semigroup of contractions.

Proof. Fix n for the time being. $A_0 - N_n$ generates a positive semigroup of contractions denoted by $(G_n(t))_{t\geq 0}$. Denote by $x_i \in D(A_0) \cap D(N_i)$ the solution to the equation

$$\lambda x_i - A_0 x_i + N_i x_i = y$$

where $\lambda > 0$ and $y \in X_+$. The resolvent of $A_0 - N_i$, say $R_i(\lambda)$, is positive for $\lambda > 0$ and therefore $x_i \geq 0$. Fix λ and $m > n$. Then

$$y = \lambda x_m - A_0 x_m + N_m x_m = \lambda x_m - A_0 x_m + N_n x_m + (N_m - N_n) x_m$$

so that

$$x_m = R_n(\lambda)(y - (N_m - N_n)x_m) \leq R_n(\lambda) y = x_n$$

as $(N_m - N_n)x_m \geq 0$ by monotonicity of $(N_n)_{n\in\mathbb{N}}$. Hence, the resolvents $R_n(\lambda)y$ form a decreasing sequence of nonnegative elements. Since clearly, for any fixed n_0 and $n \geq n_0$, $x_{n_0} - x_n$ is nonnegative and increasing with $\|x_{n_0} - x_n\| \leq 2/\lambda$ and we are in a KB-space, $R_n(\lambda)$ strongly converges to a (positive) operator $R(\lambda)$. To show that this is the resolvent of a densely defined operator we use Corollary 2.4. This requires showing that the limit

$$\lim_{\lambda\to\infty} \lambda R_n(\lambda) y = y$$

is uniform in n for any $y \in X$. Let $x \in D$. Then for any δ there is n_0 such that for any $n > n_0$ we have $\|N_n x - Nx\| < \delta$. Taking $n > n_0$ we have

$$\begin{aligned} \|\lambda R_n(\lambda)x - x\| &= \|R_n(\lambda)(A_0 - N_n)x\| \leq \lambda^{-1}(\|A_0 x\| + \|N_n x\|) \\ &\leq \lambda^{-1}(\|A_0 x\| + \|Nx\| + \|Nx - N_n x\|) \leq \lambda^{-1}(\|A_0 x\| + \|Nx\| + \delta) \end{aligned} \tag{16}$$

so that the convergence is indeed uniform in n. Let $y \in X$ and fix $\varepsilon > 0$. Then, by density of D, there exists $x \in D$ satisfying $\|y - x\| < \varepsilon$ and for this x we can write the estimate above with n_0 depending only on x and δ. Hence

$$\begin{aligned} \|\lambda R_n(\lambda)y - y\| &\leq \|\lambda R_n(\lambda)(y - x)\| + \|\lambda R_n(\lambda)x - x\| + \|y - x\| \\ &\leq 2\varepsilon + \lambda^{-1}(\|A_0 x\| + \|Nx\| + \delta) \end{aligned}$$

and the last term can be made smaller than ε by taking λ large enough, independently of n. Hence also here the convergence is uniform in n and we can use the Trotter–Kato theorem, Theorem 2.3, to deduce that $R(\lambda) = R(\lambda, \mathcal{A})$ where $\mathcal{A}$ is a densely defined operator generating a positive semigroup of contractions. To show that this is an extension of $(A_0 - N, D)$, let $y \in D$. Then

$$R(\lambda, \mathcal{A})(\lambda I - (A_0 - N))y = \lim_{n\to\infty} R_n(\lambda)(\lambda I - (A_0 - N))y,$$

and, on the other hand,

$$\begin{aligned} R_n(\lambda)(\lambda I - (A_0 - N))y &= R_n(\lambda)(\lambda I - (A_0 - N_n))y + R_n(\lambda)(N - N_n)y \\ &= y + R_n(\lambda)(N - N_n)y \to y \end{aligned}$$

by uniform boundedness of $R_n(\lambda)$. Thus, $\mathcal{A}|_D = (A_0 - N)|_D$. ∎

REMARK 3.12. A closer inspection of the proof shows that assumption 3 was used twice: first, to show the convergence of the resolvents to the resolvent of a densely defined operator, and then to show that the generator is an extension of $A_0 - N$. However, the first part has not utilized the full assumption and can be proved under either of the following assumptions:

(a) There is a set $D \subset D(A_0) \cap D(N)$ such that $D_+ - D_+$ is dense in X and $N_n y \le Ny$ for all $y \in D_+$.
(b) There is a dense set $D \subset D(A_0) \cap D(N)$ and a number M such that $\|N_n y - Ny\| \le M$ for all $y \in D$ and n.

The proof for (b) is identical to the above if one realizes that δ in (16) can be any number, not necessarily small. The proof for (a) is as follows. Let $x = x_+ - x_- \in D_+ - D_+$. Then

$$\begin{aligned}
&\|\lambda R_n(\lambda)x - x\| \\
&\quad = \|(A_0 - N_n)R_n(\lambda)x\| = \|R_n(\lambda)(A_0 - N_n)x\| \le \lambda^{-1}(\|A_0 x\| + \|N_n x\|) \\
&\quad \le \lambda^{-1}(\|A_0 x\| + \|N_n x_+\| + \|N_n x_-\|) \le \lambda^{-1}(\|A_0 x\| + \|N x_+\| + \|N x_-\|),
\end{aligned}$$

with the rest as above. Note that we used $D_+ = D \cap X_+ \subset D \subset D(A_0) \cap D(N_n)$.

In both cases, however, we need the convergence $N_n y \to Ny$ on some set to get an extension property. If this set is $\{0\}$, then this property becomes trivial.

Assumptions (A1)–(A4) look considerably more involved than Kato's original assumptions but they have to cater for a possibly more complicated structure of the underlying space and of the operators A and B. If X is an L_1 space (or, in general, an AL-space), then these assumptions can be significantly simplified, leading to results already obtained in [26, 4, 5, 7].

Corollary 3.13. *Suppose that the operators* $(A, D(A))$ *and* $(B, D(B))$ *in* $X = L_1(\Omega, d\mu)$ *satisfy*:

1. $(A, D(A))$ *generates a positive semigroup of contractions* $(G_A(t))_{t\ge 0}$,
2. $D(B) \supset D(A)$ *and* $Bx \ge 0$ *for* $x \in D(A)_+$,
3. *for all* $x \in D(A)_+$,

$$\int_\Omega (Ax + Bx)\, d\mu \le 0. \tag{17}$$

Then the assumptions of Theorem 3.2 *are satisfied.*

Proof. First, assumption 3 immediately gives assumption (A4), that is, dissipativity on the positive cone. Next, take $x = R(\lambda, A)y$ for $y \in X_+$ so that $x \in D(A)_+$. Since $R(\lambda, A)$ is a surjection from X onto $D(A)$, from

$$(A + B)x = (A + B)R(\lambda, A)y = -y + BR(\lambda, A)y + \lambda R(\lambda, A)y$$

we obtain

$$-\int_\Omega y\, d\mu + \int_\Omega BR(\lambda, A)y\, d\mu + \lambda \int_\Omega R(\lambda, A)y\, d\mu \le 0. \tag{18}$$

Since in AL-spaces the norm of a nonnegative element is given by the integral, we obtain

$$\lambda\|R(\lambda,A)y\| + \|BR(\lambda,A)y\| - \|y\| \le 0, \qquad y \in X_+, \tag{19}$$

which immediately yields $\|BR(\lambda,A)\| \le 1$, hence (A2) is satisfied. ∎

The following result is similar to Desch's perturbation theorem [14] but can be applied to possibly nonpositive perturbations.

COROLLARY 3.14. *Assume that A is the generator of a positive C_0-semigroup of contractions in $X = L_1(\Omega, d\mu)$ and let $B = B_+ - B_-$ be such that $B_\pm \ge 0$, $D(B_\pm) \supset D(A)$ and there exists $C \ge 0$ with $D(A) \subset D(C)$ such that $B_+ + B_- \le C$ on $D(A)_+$ and, for all $x \in D(A)_+$,*

$$\int_\Omega (Ax + Cx)\,d\mu \le 0. \tag{20}$$

Then there is an extension K_B of $A + B$ that generates a semigroup of contractions.

Proof. Define $|B| = B_+ + B_-$. Clearly, for $x \in D(A)_+$,

$$\int_\Omega (Ax + |B|x)\,d\mu = \int_\Omega (Ax + Cx)\,d\mu + \int_\Omega (|B|x - Cx)\,d\mu \le 0$$

so that, by Corollary 3.13, $|B|$ satisfies all assumptions of Theorem 3.2, and we have $\|BR(\lambda,A)\| \le \|\,|B|R(\lambda,A)\| \le 1$. Hence, as in the proof of that theorem, $(A + r|B|, D(A))$ generates a positive semigroup of contractions, and an extension of $A + |B|$, denoted by $K_{|B|}$, with resolvent given by (8) with B replaced by $|B|$, generates a positive semigroup of contractions. Also, for $y \in X_+$, $r \le 1$, and $\lambda > 0$,

$$\sum_{j=0}^{n} |R(\lambda,A)r^j(BR(\lambda,A))^j y| \le \sum_{j=0}^{\infty} R(\lambda,A)(|B|R(\lambda,A))^j y.$$

Using additivity of the norm on the positive cone, we obtain

$$\sum_{j=0}^{n} r^j\|R(\lambda,A)(BR(\lambda,A))^j y\| \le \|R(\lambda,K_{|B|})y\|, \tag{21}$$

so the series $\mathcal{R}_r(\lambda)y := \sum_{j=0}^{\infty} r^j R(\lambda,A)(BR(\lambda,A))^j y$ is absolutely convergent for any $y \in X$ and $0 \le r \le 1$. For $0 \le r < 1$, $\mathcal{R}_r(\lambda)y$ is dominated by a geometric series and, by standard calculations, equals the resolvent $R(\lambda, A + rB)$ of the operator $A + rB$.

From (21),

$$\|R(\lambda, A + rB)\| \le \lambda^{-1}$$

as $K_{|B|}$ is dissipative. Hence $(A + rB, D(A))$ generates a semigroup of contractions for each $r < 1$. From the dominated convergence theorem, Theorem

2.1(ii), we infer that for each $y \in X$,

$$\lim_{r\to 1} R(\lambda, A + rB)y = \mathcal{R}_1(\lambda)y.$$

We now use the Trotter–Kato theorem exactly as in the proof of Theorem 3.2. Thus, we have to prove that for any $f \in X$ the limit

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A + rB)y = y$$

is uniform in r. Let $y \in D(A)$. Then as

$$(A + rB)R(\lambda, A + rB) = I - \lambda R(\lambda, A + rB)$$

we have, by dissipativity,

$$\|\lambda R(\lambda, A + rB)y - y\| \le \lambda^{-1}(\|Ay\| + \|By\|)$$

so that the limit is uniform in r. Since $D(A)$ is dense in X, for $z \in X$ we take $x \in D(A)$ with $\|z - x\| < \varepsilon$ to obtain, again by dissipativity,

$$\|\lambda R(\lambda, A + rB)z - z\| \le 2\varepsilon + \lambda^{-1}(\|Ax\| + \|Bx\|),$$

which gives uniform convergence. The Trotter–Kato theorem shows that $\mathcal{R}_1(\lambda)$ is the resolvent of a densely defined closed operator K_B which generates a semigroup of contractions $(G_{K_B}(t))_{t\ge 0}$. To show that K_B is an extension of $A + B$, we simply repeat the argument from the proof of Theorem 3.2. ∎

4. Applications to birth-and-death problems. We shall consider a particular case of the Kolmogorov system, called the *birth-and-death system*:

$$\begin{aligned} x_0' &= -a_0x_0 + d_1x_1,\\ &\ \vdots\\ x_n' &= -a_nx_n + d_{n+1}x_{n+1} + b_{n-1}x_{n-1},\\ &\ \vdots \end{aligned} \tag{22}$$

The classical applications of this system are in population theory. In this case x_n is the probability that the population considered consists of n individuals and its state can change by either death or birth of an individual, moving the population to state $n-1$ or $n+1$, respectively, hence the name of birth-and-death system. The classical birth-and-death system is formally conservative, which requires $a_n = d_n + b_n$. However, recently a number of other important applications emerged. For example ([19, 25]), we can consider an ensemble of cancer cells structured by the number of copies of a drug-resistant gene they contain. Here, the number of cells with n copies of the gene can change due to mutations but the cells also undergo division without changing the number of genes in their offspring, which is modelled by a nonzero sequence $(c_n)_{n\in\mathbb{N}}$, defined by $c_n = b_n + d_n - a_n$. Furthermore, system (22) can be thought of as a simplified kinetic system consisting of particles labelled by their internal energy n and interacting inelastically with the surrounding matter where, in each interaction, they can either gain or lose a unit of energy. Some particles can decay without a trace or be removed from the system leading again to a nonzero $(c_n)_{n\in\mathbb{N}}$.

The solvability of (22) has been studied by various methods for several decades, with first definitive results obtained in [18, 23, 22]; see also a modern account in [3]. However, motivated by the probabilistic interpretation, these works were confined to the conservative case $c_n = 0$ and to the spaces l_1 and c_0. The methods employed to prove the existence of solutions utilized this probabilistic structure quite extensively and did not seem to admit an easy extension to other spaces and nonconservative systems.

Let boldface letters denote sequences, e.g. $\mathbf{x} = (x_0, x_1, \ldots, x_n, \ldots)$. We also assume that the sequences $\mathbf{d}$, $\mathbf{b}$ and $\mathbf{a}$ are nonnegative with $b_{-1} = d_0 = 0$.

We denote by $\mathcal{K}$ the matrix of coefficients of the right-hand side of (22) and at the same time, without causing any misunderstanding, the formal operator in the space l of all sequences, acting as $(\mathcal{K}\mathbf{x})_n = b_{n-1}x_{n-1} - a_nx_n + d_{n+1}x_{n+1}$. In the same way, we define $\mathcal{A}$ and $\mathcal{B}$ as $(\mathcal{A}\mathbf{x})_n = -a_nx_n$ and $(\mathcal{B}\mathbf{x})_n = b_{n-1}x_{n-1} + d_{n+1}x_{n+1}$, respectively. Let $\mathcal{K}_p$ denote the maximal realization of $\mathcal{K}$ in l_p, $p \in [1,\infty)$, that is,

$$\mathcal{K}_p\mathbf{x} = \mathcal{K}\mathbf{x}$$

on

$$D(\mathcal{K}_p) = \{\mathbf{x} \in l_p;\ \mathcal{K}\mathbf{x} \in l_p\}. \tag{23}$$

Lemma 4.1. *The maximal operator $\mathcal{K}_p$ is closed for any $p \in [1,\infty)$.*

Proof. Let $\mathbf{x}^{(n)} \to \mathbf{x}$ and $\mathcal{K}_p\mathbf{x}^{(n)} \to \mathbf{y}$ in l_p as $n \to \infty$. From this it follows that for any k, $x_k^{(n)} \to x_k$ and, from the definition of $\mathcal{K}_p$, $y_k = b_{k-1}x_{k-1} + a_kx_k + d_{k+1}x_{k+1}$, that is, $\mathcal{K}_p\mathbf{x} = \mathbf{y}$. ∎

Next, define the operator A_p by restricting $\mathcal{A}$ to

$$D(A_p) = \{\mathbf{x} \in l_p;\ \mathcal{A}_p\mathbf{x} \in l_p\} = \Big\{\mathbf{x} \in l_p;\ \sum_{n=0}^{\infty} a_n^p|x_n|^p < \infty\Big\}.$$

Lemma 4.2. *$(A_p, D(A_p))$ is the generator of a semigroup of contractions in l_p.*

Proof. A_p is clearly densely defined with resolvent $R(\lambda, A_p)$ for $\lambda > 0$ given by

$$(R(\lambda, A_p)\mathbf{y})_n = \frac{y_n}{\lambda + a_n}$$

(recall $a_n \geq 0$). Thus,

$$\|R(\lambda, A_p)\mathbf{y}\|_p^p = \sum_{n=0}^{\infty} \frac{1}{(\lambda + a_n)^p} |y_n|^p \leq \frac{1}{\lambda^p} \|\mathbf{y}\|_p^p,$$

so the lemma follows by the Hille–Yosida theorem. ■

THEOREM 4.3. *Assume that sequences* $\mathbf{b}$ *and* $\mathbf{d}$ *are nondecreasing and there is* $\alpha \in [0,1]$ *such that for all* n,

$$0 \leq b_n \leq \alpha a_n, \qquad 0 \leq d_{n+1} \leq (1-\alpha) a_n. \tag{24}$$

Then there is an extension K_p *of the operator* $(A_p + B_p, D(A_p))$, *where* $B_p = \mathcal{B}|_{D(A_p)}$, *that generates a positive semigroup of contractions in* l_p, $p \in (1, \infty)$.

Proof. The operator B_p is clearly positive; we must show that it maps $D(A_p)$ into l_p. For $\mathbf{x} \in D(A_p)$ we have, with $b_{-1} = d_0 = 0$,

$$\begin{aligned} \|B_p\mathbf{x}\|_p &= \Big(\sum_{n=0}^{\infty} |b_{n-1}x_{n-1} + d_{n+1}x_{n+1}|^p\Big)^{1/p} \\ &\leq \Big(\sum_{n=0}^{\infty} b_{n-1}^p |x_{n-1}|^p\Big)^{1/p} + \Big(\sum_{n=0}^{\infty} d_{n+1}^p |x_{n+1}|^p\Big)^{1/p} \\ &\leq \Big(\sum_{n=0}^{\infty} b_n^p |x_n|^p\Big)^{1/p} + \Big(\sum_{n=0}^{\infty} d_n^p |x_n|^p\Big)^{1/p}. \end{aligned}$$

By monotonicity of $\mathbf{d}$ we have $d_n \leq d_{n+1}$ so that by (24) we obtain

$$\|B_p\mathbf{x}\|_p \leq \Big(\sum_{n=0}^{\infty} a_n^p |x_n|^p\Big)^{1/p} = \|A_p\mathbf{x}\|_p.$$

Thus, $B_p D(A_p) \subset l_p$. Moreover, since $-A_p$ is a positive operator, we see, by Remark 3.3, that the assumptions (A2)–(A3) of Theorem 3.2 are satisfied.

To prove (A4) we take $\mathbf{x} \in D(A_p)_+$ and the corresponding element $\widetilde{\mathbf{x}} = (\widetilde{x}_n)_{n\in\mathbb{N}}$,

$$\widetilde{x}_n = \begin{cases} 0 & \text{if } x_n = 0, \\ x_n^{p-1} & \text{if } x_n \neq 0. \end{cases}$$

Hence $\widetilde{\mathbf{x}} \in l_q$, where $1/p + 1/q = 1$. Note that since, clearly, assumption (A4) is not affected by multiplying x^* by a positive factor, in the definition of $\widetilde{\mathbf{x}}$ we dropped the factor $\|\mathbf{x}\|_p^{1-p}$ to simplify notation. For simplicity we assume $x_n \neq 0$ for any $n \in \mathbb{N}$. From (24) we have $a_n \geq b_n + d_{n+1}$, so that

$$\begin{aligned} \langle K_p\mathbf{x}, \widetilde{\mathbf{x}}\rangle &= \sum_{n=0}^{\infty} (K_p\mathbf{x})_n x_n^{p-1} \\ &= -\sum_{n=0}^{\infty} a_n x_n^p + \sum_{n=0}^{\infty} b_{n-1}x_{n-1}x_n^{p-1} + \sum_{n=0}^{\infty} d_{n+1}x_{n+1}x_n^{p-1} \end{aligned}$$

$$\leq -\sum_{n=0}^{\infty} b_n x_n^p - \sum_{n=0}^{\infty} d_{n+1} x_n^p + \sum_{n=0}^{\infty} b_{n-1} x_{n-1} x_n^{p-1} + \sum_{n=0}^{\infty} d_{n+1} x_{n+1} x_n^{p-1},$$

where the calculations above are justified by the convergence of all series (see e.g. [9]). Thus, by the Hölder inequality, we obtain

$$\begin{aligned}\langle K_p \mathbf{x}, \widetilde{\mathbf{x}}\rangle \leq \Big(\sum_{n=0}^{\infty} b_n x_n^p\Big)^{1/p} \Big(\sum_{n=0}^{\infty} b_n x_{n+1}^p\Big)^{1/q} - \sum_{n=0}^{\infty} b_n x_n^p \\ + \Big(\sum_{n=0}^{\infty} d_n x_n^p\Big)^{1/p} \Big(\sum_{n=0}^{\infty} d_{n+1} x_n^p\Big)^{1/q} - \sum_{n=0}^{\infty} d_{n+1} x_n^p,\end{aligned}$$

and, using $b_n \leq b_{n+1}$ and $d_n \leq d_{n+1}$, we obtain $\langle K_p \mathbf{x}, \widetilde{\mathbf{x}}\rangle \leq 0$. ∎

COROLLARY 4.4. *Let $p \in (1, \infty)$. Then $K_p = \overline{A_p + B_p}$.*

Proof. As in Lemma 4.1, we can prove that $\mathcal{B}$ is closed and thus B_p is closable. Hence the statement follows from Theorem 3.6. Alternatively, the statement follows directly from Remark 3.9. ∎

COROLLARY 4.5. *Let $p = 1$. Assume that sequences* $\mathbf{b}$ *and* $\mathbf{d}$ *are nonnegative and*

$$a_n \geq b_n + d_n. \tag{25}$$

Then there is an extension K_1 of the operator $(A_1 + B_1, D(A_1))$, where $B_1 = \mathcal{B}|_{D(A_1)}$, that generates a positive semigroup of contractions on l_1.

Proof. We have

$$D(A_1) = \Big\{\mathbf{x} \in l_1;\ \sum_{n=0}^{\infty} a_n |x_n| < \infty\Big\}$$

and, from (25), $0 \leq b_n \leq a_n$ and $0 \leq d_n \leq a_n$ for $n \in \mathbb{N}$. Hence, B_1 is well-defined and condition (17) takes the form

$$\begin{aligned}\sum_{n=0}^{\infty} ((A_1 + B_1)\mathbf{x})_n &= -\sum_{n=0}^{\infty} a_n x_n + \sum_{n=0}^{\infty} b_{n-1} x_{n-1} + \sum_{n=0}^{\infty} d_{n+1} x_{n+1} \\ &= -\sum_{n=0}^{\infty} a_n x_n + \sum_{n=0}^{\infty} b_n x_n + \sum_{n=0}^{\infty} d_n x_n \leq 0,\end{aligned}$$

where we used the convention $b_{-1} = d_0 = 0$. The statement now follows by Corollary 3.13. ∎

REMARK 4.6. The above theorem was also proved in [9] but using Theorem 3.7.

Corollary 4.7. *Let $p = 1$. Assume that sequences* $\mathbf{b}$ *and* $\mathbf{d}$ *satisfy*

$$a_n \geq |b_n| + |d_n|. \tag{26}$$

Then there is an extension K_1 of the operator $(A_1 + B_1, D(A_1))$, where $B_1 = \mathcal{B}|_{D(A_1)}$, that generates a semigroup of contractions on l_1.

Proof. This follows immediately from Corollaries 3.14 and 4.5. ∎

Remark 4.8. In contradistinction to the case $p > 1$, for $p = 1$ in general $K_1 \neq \overline{A_1 + B_1}$, see [6].

Remark 4.9. There is a difference in conditions ensuring dissipativity in l_p for $p > 1$ and in l_1. In the first case we require $a_n \geq b_n + d_{n+1}$, and in the second $a_n \geq b_n + d_n$. Since $(d_n)_{n\in\mathbb{N}}$ is assumed to be increasing, the condition for $p > 1$ is stronger. However, if for $p > 1$ the coefficient a_n satisfies the condition for l_1, we can redefine $\widetilde{a}_n = a_n + d_n - d_{n+1}$ so that $\widetilde{a}_n$ satisfies the proper l_p-condition. Now, if $d_{n+1} - d_n$ is bounded (e.g. for affine coefficients), then the existence of the semigroup with the original coefficients can be established by the bounded perturbation theorem. The resulting semigroup, however, may not be contractive.

Theorem 4.10. *For any $p \in [1, \infty)$ we have $K_p \subset \mathcal{K}_p$.*

Proof. First we note that if $\mathbf{x}^r \to \mathbf{x}$ as $r \to 1$ in l_p, then for any n,

$$\begin{aligned}\lim_{r\to 1}((I - \mathcal{K}_p)\mathbf{x}^r)_n &= \lim_{r\to 1} x_n^r + a_n x_n^r - b_{n-1}x_{n-1}^r - d_{n+1}x_{n+1}^r \\ &= x_n + a_n x_n - b_{n-1}x_{n-1} - d_{n+1}x_{n+1} \\ &= ((I - \mathcal{K}_p)\mathbf{x})_n.\end{aligned} \tag{27}$$

Set $\mathbf{x}^r = R(1, A + rB)\mathbf{y}$ for $\mathbf{y} \in l_p$. We know that $\mathbf{x}^r \to R(1, K_p)\mathbf{y}$ as $r \to 1$. Since $R(1, A + rB)$ is the resolvent of $(A + rB, D(A))$ which is a restriction of the maximal realization of $-\mathcal{A} + r\mathcal{B}$, we have

$$\begin{aligned}((I - \mathcal{K}_p)\mathbf{x}^r)_n &= x_n^r + a_n x_n^r - rb_{n-1}x_{n-1}^r - rd_{n+1}x_{n+1}^r \\ &\quad - (1 - r)(b_{n-1}x_{n-1}^r + d_{n+1}x_{n+1}^r) \\ &= y_n - (1 - r)(b_{n-1}x_{n-1}^r + d_{n+1}x_{n+1}^r).\end{aligned}$$

Since n is fixed, we see that the last term tends to zero and by (27) we obtain $((I - \mathcal{K}_p)\mathbf{x})_n = y_n$, that is,

$$(I - \mathcal{K}_p)R(1, K_p)\mathbf{y} = \mathbf{y}. ∎$$

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Received May 11, 2005
Revised version February 9, 2007 (5649)