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Besov algebras on Lie groups of polynomial growth

by

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Abstract. We prove an algebra property under pointwise multiplication for Besov spaces defined on Lie groups of polynomial growth. When the setting is restricted to H-type groups, this algebra property is generalized to paraproduct estimates.

1. Introduction

1.1. Lie groups of polynomial growth. In this paper $\mathbb{G}$ is a unimodular connected Lie group endowed with the Haar measure. By "unimodular" we mean that the Haar measure is left- and right-invariant. Denoting by $\mathcal{G}$ the Lie algebra of $\mathbb{G}$, we consider a family $\mathbb{X} = \{X_1, \ldots, X_k\}$ of left-invariant vector fields on $\mathbb{G}$ satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the X_i's. In the following, although not stated, all the function spaces depend on $\mathbb{X}$.

A standard metric on $\mathbb{G}$, called the *Carnot–Carathéodory* metric, is naturally associated with $\mathbb{X}$ and is defined as follows: Let $\ell : [0,1] \to \mathbb{G}$ be an absolutely continuous path. We say that ℓ is *admissible* if there exist measurable functions $c_1, \ldots, c_k : [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$, one has $\ell'(t) = \sum_{i=1}^k c_i(t) X_i(\ell(t))$. If ℓ is admissible, its length is defined by $|\ell| = \int_0^1 (\sum_{i=1}^k |c_i(t)|^2\, dt)^{1/2}$. For all $x, y \in \mathbb{G}$, define $d(x,y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition).

This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group, and x so that the distance from x to y is equal to $|y^{-1}x|$. For all $\rho > 0$, denote by $B(x,\rho)$ the open ball in $\mathbb{G}$ with respect to the Carnot–Carathéodory distance and by $V(\rho)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the *local dimension* of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $\rho \in]0,1[$,

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$$c\rho^d \le V(\rho) \le C\rho^d$$

(see [NSW]). When $\rho > 1$, two situations may occur (see [G]):

- Either there exist $c, C, D > 0$ such that $c\rho^D \le V(\rho) \le C\rho^D$ for all $\rho > 1$ where D is called the *dimension at infinity* of the group (note that, unlike d, D does not depend on $\mathbb{X}$). The group is then said to have *polynomial growth.*
- Or there exist $c_1, c_2, C_1, C_2 > 0$ such that $c_1 e^{c_2\rho} \le V(\rho) \le C_1 e^{C_2\rho}$ for all $\rho > 1$, and the group is said to have *exponential growth.*

When $\mathbb{G}$ has polynomial growth, it is plain that there exists a constant $C > 0$ such that $V(2\rho) \le CV(\rho)$ for all $\rho > 0$. This implies in turn that there exist $C > 0$ and $\kappa > 0$ such that $V(\theta\rho) \le C\theta^\kappa V(\rho)$ for all $\rho > 0$ and $\theta > 1$.

We denote by $\Delta_{\mathbb{G}} = \sum_{i=1}^k X_i^2$ the sublaplacian on $\mathbb{G}$.

1.2. Nilpotent Lie groups. A Lie group is said to be *nilpotent* if its Lie algebra $\mathcal{G}$ is nilpotent; more precisely, writing $\mathcal{G}^1 = \mathcal{G}$ and defining inductively$\mathcal{G}^{k+1} = [\mathcal{G}^k, \mathcal{G}^k]$, there is n such that $\mathcal{G}^n = \{0\}$. It can be shown that such groups are always of polynomial growth (see for instance [Du]).

1.3. Stratified (Carnot) and H-type groups. Stratified groups are a particular version of nilpotent groups, which admit a *stratified* structure and for which $V(\rho) \sim \rho^Q$ for some positive Q, for all $\rho > 0$. One advantage of this additional structure is that such groups admit dilations. Important examples of such groups are H-type groups, a particular example being the Heisenberg group.

More precisely, a *stratified* (or *Carnot*) *Lie group* $\mathbb{G}$ is simply connected and its Lie algebra admits a stratification, i.e. there exist linear subspaces $V_1, \dots, V_r$ of $\mathcal{G}$ such that $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$, $[V_1, V_i] = V_{i+1}$ for $i = 1, \dots, r-1$ and $[V_1, V_r] = 0$. Here $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ where $X \in V_1$ and $Y \in V_i$. We say that $\mathbb{G}$ has step r.

Carnot groups are nilpotent. Furthermore, via the exponential map, $\mathbb{G}$ and $\mathcal{G}$ can be identified as manifolds. The dilations γ_δ ($\delta > 0$) are then defined (on the Lie algebra level) by

$$\gamma_\delta(x_1 + \cdots + x_r) = \delta x_1 + \delta^2 x_2 + \cdots + \delta^r x_r, \qquad x_i \in V_i.$$

We define the *homogeneous dimension* of $\mathbb{G}$ to be

$$Q = \dim V_1 + 2\dim V_2 + \cdots + r\dim V_r.$$

If $\mathbb{G}$ is a Carnot group, we have $V(\rho) \sim \rho^Q$ for all $\rho > 0$ (see [FS]). For instance the Heisenberg group $\mathcal{H}^d$ is a Carnot group and $Q = 2d + 2$.

The previous abstract definition of Carnot groups is not always very practical. It is however possible to prove (see [BU]) that any N-dimensional Carnot group of step 2 with m generators is isomorphic to $(\mathbb{R}^N, \circ)$ with the law given by ($N = m+n$, $x^{(1)} \in \mathbb{R}^m, x^{(2)} \in \mathbb{R}^n$)

$$(x^{(1)}, x^{(2)}) \circ (y^{(1)}, y^{(2)}) = \begin{pmatrix} x_j^{(1)} + y_j^{(1)},\ j = 1, \dots, m \\ x_j^{(2)} + y_j^{(2)} + \frac{1}{2}\langle x^{(1)}, U^{(j)} y^{(1)}\rangle,\ j = 1, \dots, n \end{pmatrix},$$

where $U^{(j)}$ are $m \times m$ linearly independent skew-symmetric matrices.

With this at hand, one can give the definition of a group of Heisenberg-type (*H-type* henceforth). These groups are two-step stratified nilpotent Lie groups whose Lie algebra carries a suitably compatible inner product (see [Ka]). One of these groups is the nilpotent Iwasawa subgroup of semisimple Lie groups of split rank one (see [Ko]).

More precisely, an *H-type group* is a Carnot group of step 2 with the following property: the Lie algebra $\mathcal{G}$ of $\mathbb{G}$ is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that if $\mathcal{Z}$ is the center of $\mathcal{G}$, then $[\mathcal{Z}^\perp, \mathcal{Z}^\perp] = \mathcal{Z}$ and moreover for every $z \in \mathcal{Z}$, the map $J_z : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ defined by $\langle J_z(v), w\rangle = \langle z, [v, w]\rangle$ for every $w \in \mathcal{Z}^\perp$ is an orthogonal map whenever $\langle z, z\rangle = 1$.

If $m = \dim \mathcal{Z}^\perp$ and $n = \dim \mathcal{Z}$, then any H-type group is canonically isomorphic to $\mathbb{R}^{m+n}$ with the above group law, where the matrices $U^{(j)}$ satisfy the additional condition $U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$ for every $r, s \in \{1, \dots, n\}$ with $r \neq s$. Whenever the center of the group is one-dimensional, the group is canonically isomorphic to the Heisenberg group on $\mathbb{R}^{m+1}$.

We shall always identify $\mathcal{Z}^\perp$ with $\mathbb{C}^\ell$ with $2\ell = m$ and $\mathcal{Z}$ to $\mathbb{R}^n$ thanks to the discussion above. Note that the homogeneous dimension of an H-type group so defined is $Q = 2\ell + n$. On an H-type group $\mathbb{G}$, the vector fields given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{2\ell} z_l U_{l,j}^{(k)} \frac{\partial}{\partial t_k} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{2\ell} z_l U_{l,j+\ell}^{(k)} \frac{\partial}{\partial t_k}$$

for $j = 1, \dots, \ell$, $z = (x, y) \in \mathbb{R}^{2\ell}$ and $t \in \mathbb{R}^n$ are in the algebra $\mathcal{G}$.

In the following we shall denote by $\mathcal{X}$ any element of the family $(X_1, \dots, X_\ell, Y_1, \dots, Y_\ell)$. The hypoelliptic Kohn Laplacian on an H-type group is

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} + \frac{1}{4}|x|^2 \sum_{s=1}^{n} \frac{\partial^2}{\partial t_s^2} + \sum_{s=1}^{n}\sum_{i,j=1}^{m} x_i U_{ij}^{(s)} \frac{\partial^2}{\partial t_s \partial x_j}.$$

1.4. Main results and structure of the paper. In [CRT], the authors investigate the algebra properties of the Bessel space

$$L_\alpha^p(\mathbb{G}) = \{f \in L^p(\mathbb{G}) : (-\Delta_{\mathbb{G}})^{\alpha/2} f \in L^p(\mathbb{G})\}$$

and its homogeneous counterpart, where $\mathbb{G}$ is any unimodular Lie group.

Our first theorem concerns Besov spaces in the general setting of groups with polynomial growth. The case $s \in (0, 1)$ is obtained, for both inhomogeneous and homogeneous spaces, by using an equivalent definition in terms of differences (see [S]). The general case is only proved in the case of inhomogeneous spaces and uses the fact that local Riesz transforms are continuous in L^p for $1 < p < \infty$ (whence the restriction on p below), along with an interpolation argument to obtain all values of s.

Theorem 1.1. *Let $\mathbb{G}$ be a Lie group of polynomial growth. For every $s \in (0,1)$ and $1 \le p,q \le \infty$, the spaces $B^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $\dot{B}^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ are algebras under pointwise multiplication. So is the space $B^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ for $s \ge 1$ and $1 < p < \infty$.*

Remark 1.2. In Propositions 3.3 and 3.4, we shall give a generalization of Theorem 1.1 to the case when $L^\infty(\mathbb{G})$ is replaced by $L^r(\mathbb{G})$.

One can obtain the full range of indices p, as well as homogeneous Besov spaces, in the context of H-type groups thanks to the paraproduct algorithm. Before stating the result let us give an intermediate statement in the case of nilpotent groups. Its proof requires the continuity of Riesz transforms, as well as a result which, to our knowledge, is new even for the Heisenberg group (see Proposition 4.1) and which links $\dot{B}^s_{p,q}(\mathbb{G})$ and $\dot{B}^{s+1}_{p,q}(\mathbb{G})$ in terms of the action of X_i and not only powers of the sublaplacian.

Theorem 1.3. *Let $\mathbb{G}$ be a nilpotent Lie group. Then for every $1 \le s < d$, the space $\dot{B}^s_{d/s,1}(\mathbb{G})$ is embedded in $L^\infty(\mathbb{G})$ and is an algebra. Moreover for every $s > 1$ and $1 < p < \infty$, if $f, g \in \dot{B}^s_{p,(s-1)/s}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ then $fg \in \dot{B}^s_{p,1}(\mathbb{G}) \cap L^\infty(\mathbb{G})$.*

Finally if $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, $1 \le q \le \infty$, $f \in \dot{B}^s_{p_1,q}(\mathbb{G}) \cap L^{p_1}(\mathbb{G})$ and $g \in \dot{B}^s_{p_2,q}(\mathbb{G}) \cap L^{p_2}(\mathbb{G})$ then $fg \in \dot{B}^s_{p,q}(\mathbb{G}) \cap L^p(\mathbb{G})$, for any $s > 0$.

Remark 1.4. Unfortunately we are unable to obtain, in the case of nilpotent groups, the full algebra property due to the (technical) fact that Besov spaces do not interpolate well when the integrability indices are different. The second property in Theorem 1.3 is almost an algebra property, except for a loss in the third (summation) index. As to the last property this time the integrability index is changed in the product. The reason for those losses will appear clearly in the proof of the theorem.

Finally, in the context of H-type groups, thanks to paraproduct techniques, one can enlarge the range of admissible spaces and prove the following result.

Theorem 1.5. *Let $\mathbb{G}$ be an H-type group. For every $s > 0$ and every p and q such that $1 \le p,q \le \infty$, the spaces $B^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ and $\dot{B}^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ are algebras under pointwise multiplication.*

Besov spaces are defined in the next section, and Theorems 1.1 and 1.3 are proved in Sections 3 and 4 respectively. We present the proof of Theorem 1.5 in Section 5.

We shall write $A \lesssim B$ if there is a universal constant C such that $A \leq CB$. Similarly we shall write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Littlewood–Paley decomposition on groups of polynomial growth, and Besov spaces. This section is devoted to a presentation of the Littlewood–Paley decomposition on groups of polynomial growth, together with some standard applications. A general approach to the Littlewood–Paley decomposition on Lie groups of polynomial growth is investigated in [FMV]. We also refer to [BGX] or [BG] for the case of the Heisenberg group. We recall here the construction of the homogeneous and inhomogeneous decompositions. For details and proofs of the results presented in this section we refer to [C2], [FMV] and [H].

2.1. Littlewood–Paley decomposition. We first review the dyadic decomposition constructed in [FMV]. Let $\chi \in C^\infty(\mathbb{R})$ be an even function such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $[0, 1/4]$, $\chi = 0$ on $[1, \infty[$. Define $\psi(x) = \chi(x/4) - \chi(x)$, so that the support of ψ is included in $[-4, -1/4] \cup [1/4, 4]$. Then

$$\forall \tau \in \mathbb{R}^*, \quad \sum_{j \in \mathbb{Z}} \psi(2^{-2j}\tau) = 1 \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \psi(2^{-2j}\tau) = 1, \quad \forall \tau \in \mathbb{R}.$$

Introduce the spectral decomposition of the hypoelliptic Laplacian

$$-\Delta_{\mathbb{G}} = \int_0^\infty \lambda \, dE_\lambda.$$

Then we have

$$\chi(-\Delta_{\mathbb{G}}) = \int_0^\infty \chi(\lambda) \, dE_\lambda \quad \text{and} \quad \psi(-2^{-2j}\Delta_{\mathbb{G}}) = \int_0^\infty \psi(2^{-2j}\lambda) \, dE_\lambda.$$

For $j \in \mathbb{N}$ we define

$$S_0 f = \chi(-\Delta_{\mathbb{G}}) f \quad \text{and} \quad \Delta_j f = \psi(-2^{-2j}\Delta_{\mathbb{G}}) f.$$

The *homogeneous Littlewood–Paley decomposition* of $f \in \mathcal{S}'(\mathbb{G})$ is $f = \sum_{j \in \mathbb{Z}} \Delta_j f$, while the *inhomogeneous* one is $f = S_0 f + \sum_{j=0}^\infty \Delta_j f$.

THEOREM 2.1 ([FMV]). *Let $\mathbb{G}$ be a Lie group of polynomial growth and $p \in (1, \infty)$. Then $u \in L^p(\mathbb{G})$ if and only if $S_0 u, \sqrt{\sum_{j=0}^\infty |\Delta_j u|^2} \in L^p(\mathbb{G})$. Moreover,*

$$\|u\|_{L^p(\mathbb{G})} \sim \|S_0 u\|_{L^p(\mathbb{G})} + \Big\| \Big(\sum_{j=0}^\infty |\Delta_j u|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{G})}.$$

We shall denote by Ψ_j the kernel of the operator $\psi(-2^{-2j}\Delta_{\mathbb{G}})$. One can show that Ψ_j is mean free (see Corollary 5.1 of [C1] for Carnot groups, and Theorem 7.1.2 of [C2] for an extension to groups of polynomial growth). On Carnot groups, Ψ_j has the dilation property

$$\Psi_j(x) = 2^{Qj}\Psi_0(2^j x).$$

In the more general context of groups of polynomial growth, this does not hold but one has nevertheless the following important estimates: Let $\alpha \in \mathbb{N}$, $I \in \bigcup_{\beta\in\mathbb{N}}\{1,\dots,k\}^\beta$, $p \in [1,\infty]$. Then (see [FMV])

$$\forall j \geq 0, \quad \|(1+|\cdot|)^\alpha X^I \Psi_j\|_{L^p(\mathbb{G})} \lesssim 2^{j(d/p'+|I|)}, \tag{2.1}$$

where $1/p + 1/p' = 1$. Here $X^I = X_{i_1}\dots X_{i_\beta}$ and $|I| = \beta$. Moreover as proved in [C2, Theorem 7.1.2],

$$\forall j \in \mathbb{Z}, \quad \|X_i \Psi_j\|_{L^1(\mathbb{G})} \lesssim 2^j. \tag{2.2}$$

Finally, putting together the classical estimates on the heat kernel (see [CRT] or [VSC] for instance) and the methods of [FMV] yields, for any $\alpha \geq 0$,

$$\forall j \in \mathbb{Z}, \quad \big\||\cdot|^\alpha \Psi_j\big\|_{L^1(\mathbb{G})} \lesssim 2^{j\alpha}. \tag{2.3}$$

2.2. Besov spaces. As a standard application of the Littlewood–Paley decomposition, one can define (inhomogeneous) Besov spaces on Lie groups with polynomial growth in the following way: Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. Then $B^s_{p,q}(\mathbb{G})$ is the space

$$\Big\{ f \in \mathcal{S}'(\mathbb{G}) : \|f\|_{B^s_{p,q}(\mathbb{G})} = \|S_0 f\|_{L^p(\mathbb{G})} + \Big(\sum_{j=0}^{\infty}(2^{js}\|\Delta_j f\|_{L^p(\mathbb{G})})^q\Big)^{1/q} < \infty \Big\},$$

with the obvious modification if $q = \infty$. When $p = q = 2$ one recovers the usual Sobolev spaces (see for instance [BG] for the case of the Heisenberg group). Note that when $s > 0$ one sees easily that $\|S_0 f\|_{L^p(\mathbb{G})}$ may be replaced by $\|f\|_{L^p(\mathbb{G})}$. Using the Bernstein inequalities (Proposition 4.2 of [FMV]) one finds immediately that if $s > 0$ then

$$p_1 \leq p_2 \;\Rightarrow\; B^{s+d/p_1-d/p_2}_{p_1,q} \cap L^{p_2} \hookrightarrow B^s_{p_2,q} \cap L^{p_1} \tag{2.4}$$

where recall that d is the local dimension of $\mathbb{G}$.

One can also define the homogeneous counterpart of the above norm:

$$\|f\|_{\dot B^s_{p,q}(\mathbb{G})} = \Big(\sum_{j\in\mathbb{Z}}(2^{js}\|\Delta_j f\|_{L^p(\mathbb{G})})^q\Big)^{1/q}$$

but proving that this does provide a (quasi)-Banach space is not an easy matter, and this is actually not true in general, even in the euclidean case (see for instance [BCD, Chapter 2] for comments on that subject). To obtain a Banach space in the context of Carnot groups, the homogeneous space $\dot B^s_{p,q}(\mathbb{G})$ can be defined as the set of functions in $\mathcal{S}'(\mathbb{G})$ modulo polynomials, such that the above norm is finite (see [FM]). In the present study however this will not be an issue, even if the group is not stratified: we define $\dot{B}^s_{p,q}(\mathbb{G})$ as the completion for the above norm of the set of smooth functions such that $\Delta_j f \to 0$ as $j \to -\infty$, and we shall always be considering the intersection of $\dot{B}^s_{p,q}(\mathbb{G})$ with a Banach space (such as L^∞).

Note that the Bernstein inequalities imply as in (2.4) that

$$p_1 \le p_2 \;\Rightarrow\; \dot{B}^{s+d/p_1-d/p_2}_{p_1,q} \hookrightarrow \dot{B}^s_{p_2,q}.$$

Besov spaces are often defined using the heat flow (the advantage being that this does not require the Littlewood–Paley machinery). In [FMV], the authors prove that if $s \in \mathbb{R}$, then $f \in B^s_{p,q}(\mathbb{G})$ is equivalent to: for all $t > 0$, the function $e^{t\Delta_{\mathbb{G}}} f$ belongs to $L^p(\mathbb{G})$ and

$$\Big(\int_0^1 t^{-sq/2} \|(t(-\Delta_{\mathbb{G}}))^{m/2} e^{t\Delta_{\mathbb{G}}} f\|^q_{L^p(\mathbb{G})} \frac{dt}{t}\Big)^{1/q} < \infty \tag{2.5}$$

for $m \ge 0$ greater than s. We shall not use this characterization here.

3. Proof of Theorem 1.1

3.1. The case $s \in (0,1)$. In the case $s \in (0,1)$, we use an idea of [CRT] which consists in representing the Besov norm by suitable functionals. More precisely, writing $\tau_w f(w') = f(w'w)$ we introduce the following functional (note that it differs slightly from that used in [CRT]):

$$\mathcal{S}_{s,p} f(w) = \frac{\|\tau_w f - f\|_{L^p(\mathbb{G})}}{|w|^s}.$$

Proposition 3.1. *Let $\mathbb{G}$ be a Lie group of polynomial growth. Then for any $s \in (0,1)$ and $p, q \in [1,\infty]$, we have*

$$\|f\|_{B^s_{p,q}(\mathbb{G})} \sim \|f\|_{L^p(\mathbb{G})} + \|\mathcal{S}_{s,p} f\|_{L^q(\mathbb{G}, \mathbf{1}_{|y|\le 1} dy/V(|y|))},$$
$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{G})} \sim \|\mathcal{S}_{s,p} f\|_{L^q(\mathbb{G}, dy/V(|y|))}.$$

Once Proposition 3.1 is proved, the algebra property follows immediately in the case when $s \in (0,1)$. Indeed, let $f, g \in B^s_{p,q}(\mathbb{G}) \cap L^\infty(\mathbb{G})$ for $s \in (0,1)$. It is easy to see that

$$\mathcal{S}_{s,p}(fg) \le \|f\|_{L^\infty} \mathcal{S}_{s,p} g + \|g\|_{L^\infty} \mathcal{S}_{s,p} f, \tag{3.1}$$

which gives the result after using the equivalence of Proposition 3.1. The same holds in the homogeneous case.

Remark 3.2. One can extend (3.1) to the following, with $1/a_i + 1/b_i = 1/p$:

$$\mathcal{S}_{s,p}(fg) \le \|f\|_{L^{a_1}} \mathcal{S}_{s,b_1} g + \|g\|_{L^{a_2}} \mathcal{S}_{s,b_2} f\ .$$

We now prove Proposition 3.1. Note that this result was already proved in [S] using the characterization (2.5). We present a proof using the Littlewood–Paley definition here, which is inspired by the proof of the euclidean case in [BCD] for instance. We need to prove that for $s \in (0,1)$,

$$\sum_{j\in\mathbb{Z}} (2^{js}\|\Delta_j f\|_{L^p})^q \sim \|f\|_{L^p}^q + \int_{\mathbb{G}} \mathbf{1}_{|w|\le 1} \frac{\|\tau_w f - f\|_{L^p}^q}{V(|w|)|w|^{sq}}\, dw$$

with the obvious modification if $q = \infty$. Compared to the euclidean case, we miss the usual dilation property, which will be replaced by estimate (2.1). The classical proof also uses a Taylor expansion at order one, which we must adapt to our context in order to use only horizontal vector fields (which alone appear in (2.1)). Let us start by noticing that

$$\|f\|_{L^p}^q \le \sum_{j\le 0} (2^{js}\|\Delta_j f\|_{L^p})^q.$$

Next let us bound $\|\tau_w \Delta_j f - \Delta_j f\|_{L^p}$. Recalling that $\Delta_j = \sum_{|j'-j|\le 1} \Delta_j \Delta_{j'}$, we have

$$\tau_w \Delta_j f - \Delta_j f = \sum_{|j'-j|\le 1} \Delta_{j'} f \star (\tau_w \Psi_j - \Psi_j),$$

where Ψ_j is the kernel associated with $\psi(2^{-2j}\Delta_{\mathbb{G}})$. It follows by Young's inequality that

$$\|\tau_w \Delta_j f - \Delta_j f\|_{L^p} \le \sum_{|j'-j|\le 1} \|\Delta_{j'} f\|_{L^p} \|\tau_w \Psi_j - \Psi_j\|_{L^1}.$$

Now we estimate $\|\tau_w \Psi_j - \Psi_j\|_{L^1}$. We have

$$\begin{aligned}(\tau_w \Psi_j - \Psi_j)(x) &= \int_0^1 \frac{d}{ds} \Psi_j(x\varphi(s))\, ds \\ &= \sum_{\ell=1}^k \int_0^1 c_\ell(s) (X_\ell(x\varphi(s))\Psi_j)(x\varphi(s))\, ds,\end{aligned}$$

where φ is an admissible path linking e to w. It follows that

$$\begin{aligned}\|\tau_w \Psi_j - \Psi_j\|_{L^1} &\le \int_{\mathbb{G}} \sum_{\ell=1}^k \int_0^1 |c_\ell(s)|\, |(X_\ell(x\varphi(s))\Psi_j)(x\varphi(s))|\, ds\, dx \\ &\le \sum_{\ell=1}^k \int_0^1 |c_\ell(s)|\, ds\, \|X_\ell \Psi_j\|_{L^1}\end{aligned}$$

by the Fubini theorem and a change of variables. Using (2.2) we get

$$\forall j \in \mathbb{N}, \quad \|\tau_w \Psi_j - \Psi_j\|_{L^1} \lesssim 2^j \sum_{\ell=1}^k \int_0^1 |c_\ell(s)|\, ds,$$

so by definition of $|w|$ and the Cauchy–Schwarz inequality we find

$$\forall j \in \mathbb{N}, \qquad \|\tau_w \Psi_j - \Psi_j\|_{L^1} \lesssim 2^j |w|.$$

This implies that there is a sequence (c_j) in the unit ball of ℓ^q such that

$$\forall j \in \mathbb{N}, \quad \|\tau_w \Delta_j f - \Delta_j f\|_{L^p} \lesssim c_j |w| 2^{j(1-s)} \|f\|_{B^s_{p,q}}. \tag{3.2}$$

On the other hand one has of course

$$\|\tau_w \Delta_j f - \Delta_j f\|_{L^p} \lesssim c_j 2^{-js} \|f\|_{B^s_{p,q}}. \tag{3.3}$$

Now let $j_w \in \mathbb{Z}$ be such that $1/|w| \le 2^{j_w} \le 2/|w|$. Using (3.2) for low frequencies and (3.3) for high frequencies yields

$$\|\tau_w f - f\|_{L^p} \lesssim \|f\|_{B^s_{p,q}} \Big(\sum_{j \le j_w} c_j 2^{j(1-s)} |w| + \sum_{j > j_w} c_j 2^{-js} \Big).$$

Let us first consider the case $q = \infty$. Then one finds directly that

$$\|\tau_w f - f\|_{L^p} \lesssim |w|^s \|f\|_{B^s_{p,\infty}},$$

which proves one side of the equivalence.

The case $q < \infty$ is slightly more technical but is very close to the euclidean case. We include it here for the sake of completeness. We have

$$\bigg\| \frac{\|\tau_w f - f\|_{L^p}}{|w|^s} \bigg\|^q_{L^q(\mathbb{G}, \mathbf{1}_{|w|\le 1}/V(|w|))} \lesssim 2^q \|f\|^q_{\dot{B}^s_{p,q}} (I_1 + I_2)$$

where

$$I_1 = \int_{\mathbb{G}} \mathbf{1}_{|w|\le 1} \Big(\sum_{j \le j_w} c_j 2^{j(1-s)} \Big)^q \frac{|w|^{q(1-s)}\, dw}{V(|w|)},$$

$$I_2 = \int_{\mathbb{G}} \mathbf{1}_{|w|\le 1} \Big(\sum_{j > j_w} c_j 2^{-js} \Big)^q \frac{|w|^{-qs}\, dw}{V(|w|)}.$$

Hölder's inequality with weight $2^{j(1-s)}$ and the definition of j_w imply

$$\Big(\sum_{j \le j_w} c_j 2^{j(1-s)} \Big)^q \lesssim |w|^{-(1-s)(q-1)} \sum_{j \le j_w} c_j^q 2^{j(1-s)}.$$

By Fubini's theorem, we deduce that

$$I_1 \lesssim \sum_{j \in \mathbb{N}} \int_{B(0, 2^{-j+1})} |w|^{1-s} \frac{dw}{V(|w|)}\, 2^{j(1-s)} c_j^q \lesssim 1,$$

since $\|(c_j)\|_{\ell^q} \le 1$. The estimate on I_2 is very similar. Note that it is crucial here that $s \in (0,1)$.

The converse inequality is easy to prove and only depends on the fact that the mean value of Ψ_j is zero. We write indeed

$$\Delta_j f(w) = \int \tau_v f(w) \Psi_j(v)\, dv = \int (\tau_v f(w) - f(w)) \Psi_j(v)\, dv$$

so that

$$2^{js}\|\Delta_j f\|_{L^p} \le \sup_{v\in\mathbb{G}} \frac{\|\tau_v f - f\|_{L^p}}{|v|^s} \int 2^{js}|v|^s|\Psi_j(v)|\,dv.$$

Then (2.3) implies that

$$2^{js}\|\Delta_j f\|_{L^p} \le \sup_{v\in\mathbb{G}} \frac{\|\tau_v f - f\|_{L^p}}{|v|^s}.$$

Since

$$\sup_{|v|\ge 1} \frac{\|\tau_v f - f\|_{L^p}}{|v|^s} \le 2\|f\|_{L^p},$$

we get finally

$$\sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p} \le \sup_{|v|\le 1} \frac{\|\tau_v f - f\|_{L^p}}{|v|^s} + 2\|f\|_{L^p},$$

and the result follows in the case $q=\infty$. The case $q<\infty$ is similar though a little more technical, as above.

The homogeneous case is dealt with in a similar fashion. We leave the details to the reader. This proves Proposition 3.1. ∎

Using Remark 3.2, the same proof provides the following result, which will be useful in the next section.

PROPOSITION 3.3. *Let $\mathbb{G}$ be a Lie group of polynomial growth. For every $0<s<1$ and $1\le p,q\le\infty$ one has, writing $1/p = 1/a_i + 1/b_i$,*

$$\|fg\|_{B^s_{p,q}(\mathbb{G})} \le \|f\|_{L^{a_1}(\mathbb{G})}\|g\|_{B^s_{b_1,q}(\mathbb{G})} + \|g\|_{L^{a_2}(\mathbb{G})}\|f\|_{B^s_{b_2,q}(\mathbb{G})}$$

and

$$\|fg\|_{\dot B^s_{p,q}(\mathbb{G})} \le \|f\|_{L^{a_1}(\mathbb{G})}\|g\|_{\dot B^s_{b_1,q}(\mathbb{G})} + \|g\|_{L^{a_2}(\mathbb{G})}\|f\|_{\dot B^s_{b_2,q}(\mathbb{G})}.$$

3.2. The case $s\ge 1$ (inhomogeneous spaces). We shall first deal with the case when s is not an integer. We use the well-known fact that the "local Riesz transforms" $X_i(\mathrm{Id}-\Delta_{\mathbb{G}})^{-1/2}$ are bounded on $L^p(\mathbb{G})$ for $1<p<\infty$ (see for instance [Du]). This implies easily (see the next section where the same result is proved in the more difficult homogeneous case) that

$$f\in B^{s+1}_{p,q} \;\Leftrightarrow\; f\in B^s_{p,q} \text{ and } X_i f\in B^s_{p,q}\ \forall i=1,\dots,k.$$

We can then follow the lines of [CRT], by writing $\|fg\|_{B^{s+1}_{p,q}} \sim \|fg\|_{B^s_{p,q}} + \sum_{i=1}^k \|X_i(fg)\|_{B^s_{p,q}}$ and by arguing by induction.

Let us detail the case $s=1+s'$ with $0<s'<1$. On the one hand we know that for all $1\le p,q\le\infty$, if $1/a_i+1/b_i=1/p$,

$$\|fg\|_{B^{s'}_{p,q}} \lesssim \|f\|_{L^{a_1}}\|g\|_{B^{s'}_{b_1,q}} + \|g\|_{L^{a_2}}\|f\|_{B^{s'}_{b_2,q}}.$$

Then we write, by the Leibniz rule,

$$\|X_i(fg)\|_{B^{s'}_{p,q}} \le \|fX_ig\|_{B^{s'}_{p,q}} + \|gX_if\|_{B^{s'}_{p,q}},$$

and we have, by Proposition 3.3,

$$\|fX_ig\|_{B^{s'}_{p,q}} \lesssim \|f\|_{L^{a_1}}\|X_ig\|_{B^{s'}_{b_1,q}} + \|f\|_{B^{s'}_{a_2,q}}\|X_ig\|_{L^{b_2}}. \tag{3.4}$$

The estimate on gX_if in $B^{s'}_{p,q}$ is similar so we shall not write out the details for that term.

The first term on the right-hand side of (3.4) is very easy to estimate since

$$\|X_ig\|_{B^{s'}_{b_1,q}} \lesssim \|g\|_{B^{s}_{b_1,q}}.$$

So let us turn to the second term. Let us first estimate f in $B^{s'}_{a_2,q}$. We have clearly, since $s' \le s$,

$$\|f\|_{B^{s'}_{a_2,q}} \lesssim \|f\|_{B^{s}_{a_2,q}}.$$

Now we estimate X_ig in L^{b_2}, choosing $1 < b_2 < \infty$. We use the fact that

$$\begin{aligned}\|X_ig\|_{L^{b_2}} &\lesssim \|X_i(\mathrm{Id}-\Delta)^{-1/2}(\mathrm{Id}-\Delta)^{1/2}g\|_{L^{b_2}}\\ &\lesssim \|(\mathrm{Id}-\Delta)^{1/2}g\|_{L^{b_2}}\end{aligned}$$

by the continuity of the local Riesz transforms. Since $[\Delta_j, \Delta_{\mathbb{G}}] = 0$, Bernstein's lemma (see Proposition 4.3 of [FMV]) implies

$$\|\Delta_j(\mathrm{Id}-\Delta)^{1/2}g\|_{L^{b_2}} \lesssim 2^j\|\Delta_jg\|_{L^{b_2}}.$$

This implies that

$$\begin{aligned}\|(\mathrm{Id}-\Delta)^{1/2}g\|_{L^{b_2}} &\lesssim \|S_0(\mathrm{Id}-\Delta)^{1/2}g\|_{L^{b_2}} + \sum_{j\ge 0}2^j\|\Delta_jg\|_{L^{b_2}}\\ &\le \|g\|_{L^{b_2}} + \sum_{j\ge 0}2^{js}\|\Delta_jg\|_{L^{b_2}}2^{j(1-s)} \lesssim \|g\|_{B^{s}_{b_2,q}}\end{aligned}$$

since $s > 1$. This gives the required estimate for the second term in (3.4) and allows us to conclude the proof of Theorem 1.1 when $s \in \mathbb{R}^+ \setminus \mathbb{N}$.

The general case $s > 0$ is then obtained by interpolation: we recall indeed that the following complex interpolation is true (see [BL, Theorem 6.4.5], whose proof only relies on the dyadic decomposition and may be easily adapted to our situation):

$$[B^{1-\varepsilon}_{p,q}; B^{1+\varepsilon}_{p,q}]_{1/2} = B^{1}_{p,q}. \tag{3.5}$$

The multilinear interpolation result of [BL, Theorem 4.4.1] provides the case $s = 1$ and the other integer cases are obtained similarly. ∎

Note that the above proof actually gives the following result.

Proposition 3.4. *Let $\mathbb{G}$ be a Lie group of polynomial growth. For every $s \geq 1$, $1 \leq q \leq \infty$ and $1 < p < \infty$ one has, writing $1/p = 1/a_i + 1/b_i$ and choosing $1 < a_i, b_i < \infty$,*

$$\|fg\|_{B^s_{p,q}(\mathbb{G})} \leq \|f\|_{L^{a_1}(\mathbb{G})}\|g\|_{B^s_{b_1,q}(\mathbb{G})} + \|g\|_{L^{a_2}(\mathbb{G})}\|f\|_{B^s_{b_2,q}(\mathbb{G})}.$$

4. Proof of Theorem 1.3. As in the previous case, the idea is to argue by induction for noninteger values of s, and then by interpolation. To do so, we need the following result, which is new to our knowledge, even for the Heisenberg group.

Proposition 4.1. *Let $\mathbb{G}$ be a nilpotent Lie group and let $s > 0$ and $p \in (1, \infty)$ be given. Then $f \in \dot{B}^{s+1}_{p,q}(\mathbb{G})$ if and only if $X_i f \in \dot{B}^s_{p,q}(\mathbb{G})$ for all $i = 1, \dots, k$.*

Proof. On the one hand we need to prove that for all $i = 1, \dots, k$ and $j \in \mathbb{N}$,

$$\|\Delta_j X_k f\|_{L^p} \lesssim 2^j \|\Delta_j f\|_{L^p}.$$

Using Bernstein's inequalities and by density of polynomials in the space of continuous functions it is then actually enough to prove that for all integers m,

$$\|(-\Delta_{\mathbb{G}})^{m/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p} \lesssim \|(-\Delta_{\mathbb{G}})^{m/2} f\|_{L^p}. \tag{4.1}$$

Indeed, if (4.1) holds, then one also has, multiplying both sides by 2^{jm},

$$\|(-2^{2j}\Delta_{\mathbb{G}})^{m/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p} \lesssim \|(-2^{2j}\Delta_{\mathbb{G}})^{m/2} f\|_{L^p},$$

so for smooth compactly supported function φ, we get by functional calculus

$$\|\varphi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p} \lesssim \|\varphi(-2^{2j}\Delta_{\mathbb{G}}) f\|_{L^p}. \tag{4.2}$$

But recalling that $\Delta_j = \psi(-2^{2j}\Delta_{\mathbb{G}})$ we have

$$\begin{aligned}\|\Delta_j X_k f\|_{L^p} &= \|\psi(-2^{2j}\Delta_{\mathbb{G}})X_k f\|_{L^p} \\ &= \|\psi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{1/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p}.\end{aligned}$$

Then we can write

$$\begin{aligned}&\|\psi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{1/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p} \\ &\qquad \lesssim 2^j\|\psi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{-1/2}X_k f\|_{L^p} \lesssim 2^j \|\Delta_j f\|_{L^p}\end{aligned}$$

due to Bernstein's inequality

$$\|\Delta_j(-\Delta_{\mathbb{G}})^{1/2} f\|_{L^p} \lesssim 2^j \|\Delta_j f\|_{L^p}$$

along with (4.2).

So let us prove (4.1). Actually according to [LV] the operator $\mathcal{L}^k_m = (-\Delta_{\mathbb{G}})^{(m-1)/2}X_k(-\Delta_{\mathbb{G}})^{-m/2}$ is bounded on $L^p(\mathbb{G})$ for $1 < p < \infty$. That is false if the group is not nilpotent (see for instance [A]) so it is here that the assumption that $\mathbb{G}$ is nilpotent is used. Now writing

$$\begin{aligned}(-\Delta_{\mathbb{G}})^{m/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k f&\\ &= (-\Delta_{\mathbb{G}})^{m/2}(-\Delta_{\mathbb{G}})^{-1/2}X_k(-\Delta_{\mathbb{G}})^{-m/2}(-\Delta_{\mathbb{G}})^{m/2}f\\ &= \mathcal{L}_m^k(-\Delta_{\mathbb{G}})^{m/2}f,\end{aligned}$$

the result follows.

On the other hand, using again the fact that polynomials are dense in the space of continuous functions, we also need to check that for all f,

$$\|(-\Delta_{\mathbb{G}})^{(m+1)/2}f\|_{L^p} \lesssim \sup_k \|(-\Delta_{\mathbb{G}})^{m/2}X_k f\|_{L^p}.$$

To prove that, we simply use again the fact that $\mathcal{L}_m^k$ is bounded on $L^p(\mathbb{G})$ for every $1 < p < \infty$. Indeed, we can write

$$\begin{aligned}\|(-\Delta_{\mathbb{G}})^{(m+1)/2}f\|_{L^p} &\le \sum_k \|(-\Delta_{\mathbb{G}})^{(m-1)/2}X_k^2 f\|_{L^p}\\ &= \sum_k \|(-\Delta_{\mathbb{G}})^{(m-1)/2}X_k(-\Delta_{\mathbb{G}})^{-m/2}(-\Delta_{\mathbb{G}})^{m/2}X_k f\|_{L^p}\\ &= \sum_k \|\mathcal{L}_m^k(-\Delta_{\mathbb{G}})^{m/2}X_k f\|_{L^p},\end{aligned}$$

whence the result. Proposition 4.1 is proved. ∎

Proposition 4.1 allows us to obtain Theorem 1.3 rather easily when $s \in \mathbb{R}^+ \setminus \mathbb{N}$, using also Proposition 3.3. Let us give the details.

The fact that $\dot{B}^s_{d/s,1}(\mathbb{G})$ is embedded in $L^\infty(\mathbb{G})$ follows from easy calculations:

$$\|f\|_{L^\infty} \le \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^\infty} \lesssim \sum_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^{d/s}}$$

by the Bernstein inequalities (Proposition 4.2 of [FMV]).

Now let us prove that $\dot{B}^s_{d/s,1}(\mathbb{G})$ is an algebra, and then that for every $s > 1$ and $1 < p < \infty$, if $f, g \in \dot{B}^s_{p,(s-1)/s} \cap L^\infty(\mathbb{G})$ then $fg \in \dot{B}^s_{p,1} \cap L^\infty(\mathbb{G})$. We follow the lines of the inhomogeneous case treated above, but we need to be careful because the norms are now homogeneous. Let us define $s = 1 + s'$ with $s' \in (0,1)$. As in the inhomogeneous case, by the Leibniz rule, we have

$$\|X_i(fg)\|_{\dot{B}^{s'}_{p,q}} \le \|fX_i g\|_{\dot{B}^{s'}_{p,q}} + \|gX_i f\|_{\dot{B}^{s'}_{p,q}},$$

and we study in more detail the first term on the right-hand side, which satisfies due to Proposition 3.3, for $1/a_i + 1/b_i = 1/p$ (and choosing from now on $1 < a_i, b_i < \infty$),

$$\|fX_i g\|_{\dot{B}^{s'}_{p,q}} \lesssim \|f\|_{L^{a_1}}\|X_i g\|_{\dot{B}^{s'}_{b_1,q}} + \|f\|_{\dot{B}^{s'}_{a_2,q}}\|X_i g\|_{L^{b_2}}. \tag{4.3}$$

On the one hand
$$\|X_i g\|_{\dot B^{s'}_{b_1,q}} \lesssim \|g\|_{\dot B^{s}_{b_1,q}},$$
so it suffices to estimate $\|f\|_{\dot B^{s'}_{a_2,q}} \|X_i g\|_{L^{b_2}}$.

In the case when $q=1$ and $p=d/s$ we choose $a_2 = d/(s-1) = d/s'$ and $b_2 = d$ and use Bernstein's inequality which states that
$$\|f\|_{\dot B^{s'}_{a_2,1}} \lesssim \|f\|_{\dot B^{s}_{d/s,1}}.$$
Then according to [CRT] we have
$$\|X_i g\|_{L^d} \lesssim \|(-\Delta_{\mathbb G})^{s/2} g\|_{L^{d/s}}^{1/s} \|g\|_{L^\infty}^{1-1/s}$$
and moreover
$$\begin{aligned}\|(-\Delta_{\mathbb G})^{s/2} g\|_{L^{d/s}} &\le \sum_{j\in\mathbb Z} \|\Delta_j (-\Delta_{\mathbb G})^{s/2} g\|_{L^{d/s}} \\ &\lesssim \sum_{j\in\mathbb Z} 2^{js} \|\Delta_j g\|_{L^{d/s}} \lesssim \|g\|_{\dot B^{s}_{d/s,1}}\end{aligned}$$
by Bernstein's inequalities, so we infer that
$$\|X_i g\|_{L^d} \lesssim \|g\|_{\dot B^{s}_{d/s,1}}^{1/s} \|g\|_{L^\infty}^{1-1/s}$$
and the result follows in the case $1<s<2$. The other noninteger cases are obtained by induction.

To prove the result in the integer case we use a nonlinear interpolation argument as in the inhomogeneous case above. Let us detail the case $s=1$ for instance. We have indeed (as pointed out in [BL], the interpolation results hold in the homogeneous case)
$$[\dot B^{1-\varepsilon}_{d/(1-\varepsilon),1}; \dot B^{1+\varepsilon}_{d/(1+\varepsilon),1}]_{1/2} = \dot B^{1}_{d,1},$$
so the result follows. The other cases are obtained similarly.

In the case when $f, g \in \dot B^{s}_{p,(s-1)/s}$, we use as above the fact that
$$\|f\|_{L^{a_1}} \|X_i g\|_{\dot B^{s'}_{b_1,q}} \lesssim \|f\|_{L^{a_1}} \|g\|_{\dot B^{s}_{b_1,q}},$$
and in particular we can take $a_1 = \infty$ and $b_1 = p$, and we choose $a_2 = ps/(s-1)$ and $b_2 = ps$. Then Hölder's inequality gives
$$\begin{aligned}2^{js'} \|\Delta_j f\|_{L^{ps/(s-1)}} &\lesssim 2^{js'} \|\Delta_j f\|_{L^p}^{(s-1)/s} \|\Delta_j f\|_{L^\infty}^{1/s} \\ &\lesssim (2^{js} \|\Delta_j f\|_{L^p})^{(s-1)/s} \|f\|_{L^\infty}^{1/s}.\end{aligned}$$
Since as above
$$\|X_i g\|_{L^{ps}} \lesssim \|(-\Delta_{\mathbb G})^{s/2} g\|_{L^p}^{1/s} \|g\|_{L^\infty}^{1-1/s} \lesssim \|g\|_{\dot B^{s}_{p,1}}^{1/s} \|g\|_{L^\infty}^{1-1/s}$$
the result follows when $s>1$ is noninteger.

Let us detail for instance how to deduce the case $s = 2$. We write for instance

$$[\dot{B}^{9/4}_{p,5/9}; \dot{B}^{9/5}_{p,4/9}]_{5/9} = \dot{B}^2_{p,1/2},$$

so the result follows by bilinear interpolation. The other cases are obtained similarly.

Finally, let us turn to the last statement of the theorem, namely that if $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, $1 \le q \le \infty$, and $f \in \dot{B}^s_{p_1,q} \cap L^{p_1}(\mathbb{G})$ and $g \in \dot{B}^s_{p_2,q} \cap L^{p_2}(\mathbb{G})$ then $fg \in \dot{B}^s_{p,q} \cap L^p(\mathbb{G})$. We recall the real interpolation result [BL, Theorem 6.4.5] (which holds also in the homogeneous case as indicated in [BL])

$$[\dot{B}^{s_1}_{p,q_1}; \dot{B}^{s_2}_{p,q_2}]_{\theta,r} = \dot{B}^s_{p,r}, \quad s = \theta s_1 + (1-\theta)s_2,\ s_1 \neq s_2 \tag{4.4}$$

so in particular

$$[\dot{B}^0_{a_2,\infty}; \dot{B}^s_{a_2,q}]_{(s-1)/s,1} = \dot{B}^{s'}_{a_2,1} \hookrightarrow \dot{B}^{s'}_{a_2,q}.$$

We infer that f belongs to $\dot{B}^{s'}_{a_2,q}$ as soon as f belongs to $L^{a_2} \cap \dot{B}^s_{a_2,q} \hookrightarrow \dot{B}^0_{a_2,\infty} \cap \dot{B}^s_{a_2,q}$. Similarly

$$[\dot{B}^0_{b_2,\infty}; \dot{B}^s_{b_2,q}]_{1/s,1} = \dot{B}^1_{b_2,1},$$

so using the fact that

$$\|X_i g\|_{L^{b_2}} \lesssim \|X_i g\|_{\dot{B}^0_{b_2,1}} \lesssim \|g\|_{\dot{B}^1_{b_2,1}}$$

we infer that $X_i g$ belongs to L^{b_2} as soon as $g \in L^{b_2} \cap \dot{B}^s_{b_2,q} \subset \dot{B}^0_{b_2,\infty} \cap \dot{B}^s_{b_2,q}$. The result follows for $1 < s < 2$, and the theorem is proved by an easy induction and interpolation, as in the inhomogeneous case. ∎

5. Paradifferential calculus on H-type groups. In this section, we consider several topics related to harmonic analysis on H-type groups, which we recall are particular cases of Carnot groups, where it turns out that an explicit Fourier transform is available.

5.1. Fourier transforms. In order to construct paradifferential and pseudodifferential calculus on H-type groups, one needs to introduce a suitable Fourier transform. This is classically done through infinite-dimensional unitary irreducible representations on a suitable Hilbert space since H-type groups are noncommutative. Two representations are available: the Bargmann representation (see [KR] for instance) and the Schrödinger representation (see [CG] for instance).

5.1.1. *General definitions.* Let us define generally what a Fourier transform is on noncommutative groups. The irreducible unitary representations of G (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are parametrized by $\lambda \in \mathbb{R}^n \setminus \{0\}$ where n is the dimension of the center of the group. Each such representation π_λ acts on a

Hilbert space $\mathcal{H}_\lambda(\mathbb{K}^\ell)$ of functions defined on $\mathbb{K}^\ell$. We then have the following definition.

Definition 5.1. Let $f \in L^1(\mathbb{G})$. Then the *Fourier transform* of f is the operator on $\mathcal{H}_\lambda(\mathbb{K}^\ell)$ parametrized by $\lambda \in \mathbb{R}^n \setminus \{0\}$ defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{G}} f(w)\pi_\lambda(w)\,dw.$$

Note that $\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda)$. Let $F_{\alpha,\lambda}$, $\alpha \in \Lambda$, be a Hilbert basis of $\mathcal{H}_\lambda(\mathbb{K}^\ell)$. We recall that an operator $A(\lambda)$ on $\mathcal{H}_\lambda$ such that

$$\sum_{\alpha\in\Lambda} |(A(\lambda)F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda}| < \infty$$

is said to be of *trace class*. One then sets

$$\operatorname{tr}(A(\lambda)) = \sum_{\alpha\in\Lambda} (A(\lambda)F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda},$$

and the following inversion theorem holds.

Theorem 5.2. *If a function f on $\mathbb{G}$ satisfies*

$$\sum_{\alpha\in\Lambda} \int_{\mathbb{R}^n} \|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} |\lambda|^\ell \,d\lambda < \infty$$

then for almost every $w \in \mathbb{G}$,

$$f(w) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R}^n} \operatorname{tr}\big(\pi_\lambda(w^{-1})\mathcal{F}(f)(\lambda)\big)|\lambda|^\ell \,d\lambda.$$

By [Y], the representation π_λ of $\mathbb{G}$ determines a representation π^*_λ of its Lie algebra $\mathcal{G}$ on the space of C^∞ vectors. The representation π^*_λ is defined by

$$\pi^*_\lambda(X)f = \left(\frac{d}{dt}\pi_\lambda(\exp(tX))f\right)\Big|_{t=0}$$

for every X in $\mathcal{G}$. We can extend π^*_λ to the universal enveloping algebra of left-invariant differential operators on $\mathbb{G}$. Let $\mathcal{K}$ be a left-invariant operator on $\mathbb{G}$. Then

$$\mathcal{K}(\pi_\lambda f, g) = (\pi_\lambda \pi^*_\lambda(\mathcal{K})f, g)$$

where $(\cdot,\cdot)$ stands for the $\mathcal{H}_\lambda$ inner product.

5.1.2. *Bargmann representations on H-type groups.* For $\lambda \in \mathbb{R}^n \setminus \{0\}$, consider the Hilbert space (called the *Fock space*) $\mathcal{H}_\lambda(\mathbb{C}^\ell)$ of all entire holomorphic functions F on $\mathbb{C}^\ell$ such that

$$\|F\|^2_{\mathcal{H}_\lambda} = \left(\frac{2|\lambda|}{\pi}\right)^\ell \int_{\mathbb{C}^\ell} |F(\xi)|^2 e^{-|\lambda|\,|\xi|^2}\,d\xi.$$

The corresponding irreducible unitary representation π_λ of the group $\mathbb{G}$ is realized on $\mathcal{H}_\lambda(\mathbb{C}^\ell)$ by (recall that $t \in \mathbb{R}^n$ and $z, \xi \in \mathbb{C}^\ell$; see [Do])

$$(\pi_\lambda(z,t)F)(\xi) = F(\xi - z)e^{i\langle\lambda,t\rangle - |\lambda|(|z|^2 + \langle z,\xi\rangle)}.$$

It is well known that the Fock space admits an orthonormal basis given by the monomials

$$F_{\alpha,\lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\,\xi)^\alpha}{\sqrt{\alpha\,!}}, \qquad \alpha \in \mathbb{N}^\ell.$$

The following classical diagonalization result (see [KR], or [BFG] and the references therein) is important for us:

Proposition 5.3. *Let $\mathcal{F}_B$ be the Fourier transform associated to the Bargmann representation π. The following diagonalization property holds: for every $f \in \mathcal{S}(\mathbb{G})$,*

$$\mathcal{F}_B(\Delta_{\mathbb{G}} f)(\lambda)F_{\alpha,\lambda} = -4|\lambda|(2|\alpha| + \ell)\mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda}.$$

This allows us to define, for every $\rho \in \mathbb{R}$,

$$\mathcal{F}_B((-\Delta_{\mathbb{G}})^\rho f)(\lambda)F_{\alpha,\lambda} = (4|\lambda|(2|\alpha| + \ell))^\rho \mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda}.$$

5.1.3. *The L^2 representation on H-type groups.* Another useful representation is the so-called *Schrödinger*, or *L^2 representation*. In this case, the unitary irreducible representations are given on $L^2(\mathbb{R}^\ell)$ by setting, for $\lambda \in \mathbb{R}^n$ (and writing $z = (x, y)$),

$$(\tilde{\pi}_\lambda(z,t)F)(\xi) = e^{i\langle\lambda,t\rangle + |\lambda| i(\sum_{j=1}^\ell x_j\xi_j + \frac{1}{2}x_j y_j)}F(\xi + y).$$

The intertwining operator between the Bargmann and the L^2 representations is the Hermite–Weber transform $K_\lambda : \mathcal{H}_\lambda \to L^2(\mathbb{R}^\ell)$ given by

$$(K_\lambda\phi)(\xi) = C_\ell|\lambda|^{\ell/4}e^{|\lambda|\,|\xi|^2/2}\phi\biggl(-\frac{1}{2|\lambda|}\frac{\partial}{\partial\xi}\biggr)e^{-|\lambda|\,|\xi|^2},$$

which is unitary and satisfies $K_\lambda\pi_\lambda(z,t) = \tilde{\pi}_\lambda(z,t)K_\lambda$. Following [Y] and the previous description, we have $\tilde{\pi}^*_\lambda(X_j) = i|\lambda|\xi_j$ and $\tilde{\pi}^*_\lambda(Y_j) = \partial/\partial\xi_j$ for $j = 1, \dots, \ell$, and similarly $\tilde{\pi}^*_\lambda(\partial_{t_k}) = i\lambda_k$ for $k = 1, \dots, n$. Therefore,

$$\tilde{\pi}^*_\lambda(-\Delta_{\mathbb{G}}) = -\sum_{j=1}^n \frac{\partial^2}{\partial\xi_j^2} + |\lambda|^2|\xi|^2.$$

Notice that this is a Hermite operator and the eigenfunctions of $\pi^*_\lambda(-\Delta_{\mathbb{G}})$ are $\Phi^\lambda_\alpha(\xi) = |\lambda|^{n/4}\Phi_\alpha(\sqrt{|\lambda|}\,\xi)$, $\alpha = (\alpha_1, \dots, \alpha_\ell)$, where $\Phi_\alpha(\xi)$ is the product $\psi_{\alpha_1}(\xi_1)\dots\psi_{\alpha_\ell}(\xi_\ell)$ and $\psi_{\alpha_j}(\xi_j)$ is the eigenfunction of $-\partial^2/\partial\xi_j^2 + \xi_j^2$ with eigenvalue $2\alpha_j + 1$. This leads to the following formula, where $|\alpha| = \alpha_1 + \dots + \alpha_\ell$:

$$\tilde{\pi}^*_\lambda(-\Delta_{\mathbb{G}})\Phi^\lambda_\alpha = (2|\alpha| + \ell)|\lambda|\Phi^\lambda_\alpha.$$

As a consequence, one has the following lemma.

Lemma 5.4. *Let $\mathcal{F}_S$ be the Fourier transform associated to the Schrödinger representation $\tilde{\pi}$. The following diagonalization property holds: for every f in $\mathcal{S}(\mathbb{G})$,*

$$\mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda = (2|\alpha| + \ell)|\lambda|\mathcal{F}_S(f)\Phi_\alpha^\lambda.$$

Proof. We have by definition

$$\mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda = \int_{\mathbb{G}} (-\Delta_{\mathbb{G}})f(z,t)\tilde{\pi}_\lambda(z,t)\Phi_\alpha^\lambda = \int_{\mathbb{G}} f(z,t)(-\Delta_{\mathbb{G}})\tilde{\pi}_\lambda(z,t)\Phi_\alpha^\lambda.$$

Using the definition of the dual representation, we have

$$\mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda = \int_{\mathbb{G}} f(z,t)\pi_\lambda(z,t)\pi_\lambda^*(-\Delta_{\mathbb{G}})\Phi_\alpha^\lambda,$$

and by the properties of the Hermite operator, this gives the result. ∎

5.2. A localization lemma. As in [BG], one can prove a localization lemma (also called the Bernstein lemma), which we state here in the context of the Bargmann representation. The proof is omitted as it is identical to the Heisenberg situation treated in [BG]. Note that using Proposition 4.1, the last statement of the lemma could be extended to iterated vector fields X^I. We denote by $\mathcal{C}_0$ the annulus $\{\tau \in \mathbb{R} : 1/2 \le |\tau| \le 4\}$ and by $\mathcal{B}_0$ the ball $\{\tau \in \mathbb{R} : |\tau| \le 2\}$.

Lemma 5.5. *Let $p, q \in [1,\infty]$ with $p \le q$, and let $u \in \mathcal{S}(\mathbb{G})$ satisfy*

$$\mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)^{-1}2^{2j}\mathcal{B}_0}(\lambda)\mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda}$$

for all $\alpha \in \mathbb{N}^\ell$. Then

$$\forall k \in \mathbb{N}, \quad \sup_{|\beta|=k} \|\mathcal{X}^\beta u\|_{L^q(\mathbb{G})} \le C_k 2^{Nj(1/p-1/q)+kj}\|u\|_{L^p(\mathbb{G})}.$$

On the other hand, if

$$\mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)^{-1}2^{2j}\mathcal{C}_0}(\lambda),$$

then for all $\rho \in \mathbb{R}$,

$$C_\rho^{-1}2^{-j\rho}\|(-\Delta_{\mathbb{G}})^{\rho/2}u\|_{L^p(\mathbb{G})} \le \|u\|_{L^p(\mathbb{G})} \le C_\rho 2^{-j\rho}\|(-\Delta_{\mathbb{G}})^{\rho/2}u\|_{L^p(\mathbb{G})}.$$

5.3. Paraproduct on H-type groups. In order to develop a paraproduct on H-type groups, one has to prove that the product of two functions is localized in frequencies whenever the functions are localized. This is the object of the next lemma, whose proof is the same as that of Proposition 4.2 of [BG].

Lemma 5.6. *There is a constant $M_1 \in \mathbb{N}$ such that the following holds. Let $f, g \in \mathcal{S}(\mathbb{G})$ be such that*

$$\mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)2^{2m}\mathcal{C}_0}(\lambda)\mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda},$$
$$\mathcal{F}_B(g)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)2^{2m'}\mathcal{C}_0}(\lambda)\mathcal{F}_B(g)(\lambda)F_{\alpha,\lambda},$$

for some integers m and m'. If $m' - m > M_1$, then there exists an annulus $\widetilde{\mathcal{C}}$ such that

$$\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)2^{2m'}\widetilde{\mathcal{C}}}(\lambda)\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda},$$

while if $|m' - m| \leq M_1$, then there exists a ball $\tilde{\mathcal{B}}$ such that

$$\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+\ell)2^{2m'}\widetilde{\mathcal{B}}}(\lambda)\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda}.$$

Definition 5.7. We define the *paraproduct* of v by u, denoted by $T_u v$, to be the bilinear operator

$$T_u v = \sum_j S_{j-1} u \Delta_j v.$$

We define the *remainder* of u and v, denoted by $R(u,v)$, to be

$$R(u,v) = \sum_{|j-j'|\leq 1} \Delta_j u \Delta_{j'} v.$$

Remark 5.8. It is clear that formally

$$uv = T_u v + T_v u + R(u,v).$$

One of the classical consequences of Lemma 5.6 is the following result, which is obtained using the previous decomposition as well as localization properties of the paraproduct and remainder terms.

Corollary 5.9. *Let $\rho > 0$ and $(p,r) \in [1,\infty]^2$. Then*

$$\|fg\|_{B^{\rho}_{p,r}(\mathbb{G})} \leq C(\|f\|_{L^\infty(\mathbb{G})}\|g\|_{B^{\rho}_{p,r}(\mathbb{G})} + \|g\|_{L^\infty(\mathbb{G})}\|f\|_{B^{\rho}_{p,r}(\mathbb{G})}).$$

If $\rho_1+\rho_2 > 0$ and if p_1 is such that $\rho_1 < Q/p_1$, then for all $(p_2,r_2) \in [1,\infty]^2$, writing $\rho = \rho_1 + \rho_2 - Q/p_1$, we have

$$\|fg\|_{B^{\rho}_{p_2,r_2}(\mathbb{G})} \leq C(\|f\|_{B^{\rho_1}_{p_1,\infty}(\mathbb{G})}\|g\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})} + \|g\|_{B^{\rho_1}_{p_1,\infty}(\mathbb{G})}\|f\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})}).$$

Moreover, if $\rho_1 + \rho_2 \geq 0$, $\rho_1 < Q/p_1$ and $1/r_1 + 1/r_2 = 1$, then

$$\|fg\|_{B^{\rho}_{p,\infty}(\mathbb{G})} \leq C(\|f\|_{B^{\rho_1}_{p_1,r_1}(\mathbb{G})}\|g\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})} + \|g\|_{B^{\rho_1}_{p_1,r_1}(\mathbb{G})}\|f\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})}).$$

Finally if $\rho_1 + \rho_2 > 0$, $\rho_j < Q/p_j$ and $p \geq \max(p_1,p_2)$, then for all (r_1,r_2),

$$\|fg\|_{B^{\rho_{12}}_{p,r}(\mathbb{G})} \leq C\|f\|_{B^{\rho_1}_{p_1,r_1}(\mathbb{G})}\|g\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})},$$

with

$$\rho_{12} = \rho_1 + \rho_2 - Q\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \quad \textit{and} \quad r = \max(r_1,r_2),$$

and if $\rho_1 + \rho_2 \geq 0$, $\rho_j < Q/p_j$ and $1/r_1 + 1/r_2 = 1$, then for all $p \geq \max(p_1,p_2)$,

$$\|fg\|_{B^{\rho_{12}}_{p,\infty}(\mathbb{G})} \leq C\|f\|_{B^{\rho_1}_{p_1,r_1}(\mathbb{G})}\|g\|_{B^{\rho_2}_{p_2,r_2}(\mathbb{G})}.$$

The same results hold in the case of homogeneous Besov spaces. Once the paraproduct algorithm is in place, one can obtain (refined) Sobolev and Hardy inequalities (see [CX] and [BG] for Sobolev embeddings in the euclidean case and for the Heisenberg group, and [BCG] for the Hardy inequalities; see also [C1] for recent extensions). One can also construct, in the context of H-type groups, an algebra of pseudo-differential operators exactly as on the Heisenberg group. We refer to [BFG] for details.

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