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On generalized property (v) for bounded linear operators

by

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Abstract. An operator T acting on a Banach space X has property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, where $\sigma_a(T)$ is the approximate point spectrum of T, $\sigma_{SBF_+^-}(T)$ is the upper semi-B-Weyl spectrum of T and $E(T)$ is the set of all isolated eigenvalues of T. We introduce and study two new spectral properties (v) and (gv) in connection with Weyl type theorems. Among other results, we show that T satisfies (gv) if and only if T satisfies (gw) and $\sigma(T) = \sigma_a(T)$.

1. Introduction and preliminaries. Throughout this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T and by $R(T) = T(X)$ the range of T. We denote by $\alpha(T) := \dim N(T)$ the *nullity* of T and by $\beta(T) := \operatorname{codim} R(T) = \dim X/R(T)$ the *defect* of T.

Other two classical quantities in operator theory are the *ascent* $p = p(T)$, defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$ (if no such exists, we put $p(T) = \infty$), and the *descent* $q = q(T)$, defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$ (if no such q exists, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent (see [22, Prop. 50.2]).

An operator $T \in L(X)$ is said to be *Fredholm* (resp., *upper semi-Fredholm*, *lower semi-Fredholm*) if $\alpha(T)$, $\beta(T)$ are both finite (resp., $R(T)$ is closed and $\alpha(T) < \infty$, $\beta(T) < \infty$), and *semi-Fredholm* if T is either upper or lower semi-Fredholm. If T is semi-Fredholm, its *index* is defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$.

Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These are defined as follows: $T \in L(X)$

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is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if it is Fredholm (resp., upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (resp., $p(T) < \infty$, $q(T) < \infty$). An operator $T \in L(X)$ is said to be *upper semi-Weyl* (resp., *lower semi-Weyl*) if it is upper Fredholm (resp., lower semi-Fredholm) and $\operatorname{ind} T \leq 0$ (resp., $\operatorname{ind} T \geq 0$); T is *Weyl* if it is both upper and lower semi-Weyl, i.e. T is a Fredholm operator of index 0.

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$
$$\sigma_W(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl, we have $\sigma_W(T) \subseteq \sigma_b(T)$. Analogously, the *upper semi-Browder spectrum*, *upper semi-Weyl spectrum*, *lower semi-Browder spectrum*, and *lower semi-Weyl spectrum* are defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$
$$\sigma_{SF_+^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\},$$
$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Browder}\},$$
$$\sigma_{SF_-^+}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Weyl}\}.$$

From the classical Fredholm theory we have

$$\sigma_{SF_+^-}(T) = \sigma_{SF_-^+}(T^*), \quad \sigma_{SF_-^+}(T) = \sigma_{SF_+^-}(T^*),$$
$$\sigma_{ub}(T) = \sigma_{lb}(T^*), \quad \sigma_{lb}(T) = \sigma_{ub}(T^*).$$

Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in L(X)$ to the subspace $R(T^n) = T^n(X)$. According [12] and [15], T is *semi-B-Fredholm* (resp., *B-Fredholm*, *upper semi-B-Fredholm*, *lower semi-B-Fredholm*) if for some integer $n \geq 0$, the range $R(T^n)$ is closed and T_n, viewed as an operator from $R(T^n)$ into itself, is semi-Fredholm (resp., Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $T \in L(X)$ is said to be *B-Browder* (resp., *upper semi-B-Browder*, *lower semi-B-Browder*) if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is Browder (resp., upper semi-Browder, lower semi-Browder).

If T_n is a semi-Fredholm operator, it follows from [15, Proposition 2.1] that so is T_m for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of a semi-B-Fredholm operator T as the index of the semi-Fredholm operator T_n.

Further $T \in L(X)$ is said to be *B-Weyl* [13, Definition 1.1] if it is B-Fredholm of index 0; *upper semi-B-Weyl* if it is upper semi-B-Fredholm with $\operatorname{ind} T \leq 0$; and *lower semi-B-Weyl* if it is lower semi-B-Fredholm with $\operatorname{ind} T \geq 0$.

An operator $T \in L(X)$ is said to be *left* (resp. *right*) *Drazin invertible* if $p(T) < \infty$ (resp. $q(T) < \infty$) and $R(T^{p(T)+1})$ (resp. $R(T^{q(T)})$) is closed. Moreover, T is *Drazin invertible* if it has finite ascent and descent. We denote by $LD(X)$, $RD(X)$ and $D(X)$ the classes of left Drazin invertible, right Drazin invertible and Drazin invertible operators respectively. It is proved in [12, Theorem 3.6] that a B-Browder (resp., upper semi-Browder, lower semi-Browder) operator is just a Drazin invertible (resp., left Drazin invertible, right Drazin invertible) operator.

The classes of operators defined above motivate the definitions of several spectra. The *left Drazin invertible spectrum* is defined by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin LD(X)\},$$

the *right Drazin invertible spectrum* is

$$\sigma_{RD}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin RD(X)\},$$

while the *Drazin invertible spectrum* is

$$\sigma_{D}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin D(X)\}.$$

Clearly, $\sigma_D(T) = \sigma_{LD}(T) \cup \sigma_{RD}(T)$. The *B-Weyl spectrum* is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}.$$

The *upper semi-B-Weyl spectrum* and *lower semi-B-Weyl spectrum* are, respectively,

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-B-Weyl}\},$$
$$\sigma_{SBF_-^+}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-B-Weyl}\}.$$

Obviously,

$$\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_-^+}(T), \quad \sigma_{SBF_+^-}(T) = \sigma_{SBF_-^+}(T^*),$$
$$\sigma_{SBF_-^+}(T) = \sigma_{SBF_+^-}(T^*), \quad \sigma_{LD}(T) = \sigma_{RD}(T^*), \quad \sigma_{RD}(T) = \sigma_{LD}(T^*).$$

Moreover,

$$\sigma_{BW}(T) \subseteq \sigma_D(T), \quad \sigma_{SBF_+^-}(T) \subseteq \sigma_{LD}(T), \quad \sigma_{SBF_-^+}(T) \subseteq \sigma_{RD}(T).$$

Another class of operators related to semi-B-Fredholm operators is the class of quasi-Fredholm operators defined below. First, we set

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1.1. $T \in L(X)$ is said to be *quasi-Fredholm of degree d* if there exists $d \in \mathbb{N}$ such that:

(a) $\operatorname{dis}(T) = d$,
(b) $T^n(X)$ is a closed subspace of X for each $n \geq d$,
(c) $T(X) + N(T^d)$ is a closed subspace of X.

By Proposition 2.5 of [11] every semi-B-Fredholm operator is quasi-Fredholm. For further information on quasi-Fredholm operators we refer to [12] and [2].

Now, we introduce an important property in local spectral theory. The localized version of this property has been introduced by Finch [20], and in the framework of Fredholm theory, this property has been characterized in several ways (see Chapter 3 of [1]). An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}$$

is $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if it has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, every $T \in L(X)$ has SVEP at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \;\Rightarrow\; T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \;\Rightarrow\; T^* \text{ has SVEP at } \lambda. \tag{2}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_a(T)$ the classical *approximate point spectrum* defined by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_s(T)$ denotes the *surjectivity spectrum*

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\},$$

then $\sigma_a(T) = \sigma_s(T^*)$ and $\sigma_s(T) = \sigma_a(T^*)$.

It is easily seen from the definition of localized SVEP that

$$\lambda \notin \operatorname{acc}\sigma_a(T) \;\Rightarrow\; T \text{ has SVEP at } \lambda, \tag{3}$$

where $\operatorname{acc} K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and

$$\lambda \notin \operatorname{acc}\sigma_s(T) \;\Rightarrow\; T^* \text{ has SVEP at } \lambda. \tag{4}$$

REMARK 1.2. The implications (1)–(4) are actually equivalences whenever $\lambda I - T$ is a quasi-Fredholm operator, in particular when $\lambda I - T$ is semi-B-Fredholm (see [2]).

LEMMA 1.3 ([3, Lemma 2.4]). *Let $T \in L(X)$. Then*

(i) *T is upper semi-B-Fredholm and $\alpha(T) < \infty$ if and only if $T \in \Phi_+(X)$.*
(ii) *T is lower semi-B-Fredholm and $\beta(T) < \infty$ if and only if $T \in \Phi_-(X)$.*

The following lemma is a particular case of [12, Theorem 3.6].

LEMMA 1.4. *For $T \in L(X)$, the following statements are equivalent:*

(i) *$\lambda_0 I - T \in LD(X) \Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite ascent,*
(ii) *$\lambda_0 I - T \in RD(X) \Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite descent,*
(iii) *$\lambda_0 I - T \in D(X) \Leftrightarrow \lambda_0 I - T$ is quasi-Fredholm with finite ascent and descent.*

Denote by $\operatorname{iso} K$ the set of all isolated points of $K \subseteq \mathbb{C}$. If $T \in L(X)$ define

$$\begin{aligned} E^0(T) &= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E^0_a(T) &= \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ E(T) &= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}, \\ E_a(T) &= \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(\lambda I - T)\}. \end{aligned}$$

Let $\Pi^0(T) = \sigma(T)\backslash\sigma_b(T)$, i.e. $\Pi^0(T)$ is the set of all poles of the resolvent of T having finite rank. Clearly, for every $T \in L(X)$ we have

$$\Pi^0(T) \subseteq E^0(T) \subseteq E^0_a(T) \quad \text{and} \quad E(T) \subseteq E_a(T).$$

Let $T \in L(X)$. Following Coburn [19], T is said to satisfy *Weyl's theorem*, in symbols (W), if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$. Following Rakočević ([24], [23]), T is said to satisfy *a-Weyl's theorem*, in symbols (aW), if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0_a(T)$, and T is said to have *property* (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. According to Berkani and Koliha [14], T is said to satisfy *generalized Weyl's theorem*, in symbols (gW), if $\sigma(T)\backslash\sigma_{BW}(T) = E(T)$. Similarly, T is said to satisfy *generalized a-Weyl's theorem*, in symbols (gaW), if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, and T is said to have *generalized property* (w), in symbols (gw), if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$.

For $T \in L(X)$, define

$$\begin{aligned} \Pi^0_a(T) &= \sigma_a(T) \setminus \sigma_{ub}(T), \\ \Pi(T) &= \sigma(T) \setminus \sigma_D(T), \\ \Pi_a(T) &= \sigma_a(T) \setminus \sigma_{LD}(T). \end{aligned}$$

Following [16], an operator $T \in L(X)$ is said to have *property* (b) (resp. (gb)) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$). It is shown in [16, Theorem 2.3] that (gb) implies (b) but not conversely. According to [17], an operator $T \in L(X)$ has *property* (ab) (resp. (gab)) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$ (resp. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$). It is proved in [17, Theorem 2.2] that (gab) implies (ab), but not conversely. According also to [17], $T \in L(X)$ has *property* (aw) (resp. (gaw)) if $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ (resp. $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$). In [17, Theorem 3.3], it is shown that (gaw) implies (aw), but not conversely.

Following [25], we say that $T \in L(X)$ has *property* (z) (resp. (gz)) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$ (resp. $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$). Property (gz) extends (z) to the context of B-Fredholm theory. It is shown in [25, Theorem 2.2] that (gz) implies (z), but not conversely. Following [25], $T \in L(X)$ is said to have *property* (az) (resp. (gaz)) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$ (resp. $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$). In [25, Corollary 3.5], it is shown that (gaz) is equivalent to (az), and [25, Corollary 3.7] states that $T \in L(X)$ satisfies (gz) if and only if it satisfies (gaz) and $E_a(T) = \Pi_a(T)$.

According to [5], $T \in L(X)$ has *property* (R) if $\Pi_a^0(T) = E^0(T)$. It is shown in [5, Theorem 2.4] that (w) implies (R), but not conversely. Also in [5] it is shown that property (R) and Weyl's theorem are independent. According to [6], an operator $T \in L(X)$ has the *generalized property* (R), abbreviated (gR), if $\Pi_a(T) = E(T)$. In [6, Theorem 2.2], it is shown that (gR) implies (R), but not conversely.

2. Generalized property (v).

According to [21], $T \in L(X)$ has *property* (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. By [21, Theorem 2.4], if $T \in L(X)$ satisfies (Bw), then generalized Browder's theorem holds for T and $\sigma(T) = \sigma_{BW}(T) \cup \operatorname{iso}\sigma(T)$.

In this section we introduce and study two new spectral properties that are independent of (Bw).

Definition 2.1. An operator $T \in L(X)$ is said to have *property* (v) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$, and *generalized property* (v), abbreviated (gv), if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$.

Theorem 2.2. *If $T \in L(X)$ has property (gv), then it has property (v).*

Proof. Assume that T satisfies (gv) and let $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$. Since $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, we have $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. As T satisfies (gv), it follows that $\lambda \in E(T)$. Thus $\lambda \in \operatorname{iso}\sigma(T)$ and $\alpha(\lambda I - T) > 0$. Since $\lambda I - T$ is upper semi-Weyl, it is upper semi-Fredholm. Therefore, $0 < \alpha(\lambda I - T) < \infty$, thus $\lambda \in E^0(T)$. This shows $\sigma(T) \setminus \sigma_{SF_+^-}(T) \subseteq E^0(T)$.

Conversely, let $\lambda \in E^0(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T) < \infty$. Since T satisfies (gv) and $E^0(T) \subseteq E(T)$, we have $\lambda \in \sigma(T)$ and $\lambda I - T$ is upper semi-B-Weyl. Thus, $\lambda I - T$ is upper semi-B-Fredholm and $\alpha(\lambda I - T) < \infty$. By Lemma 1.3, $\lambda I - T$ is upper semi-Fredholm, and hence upper semi-Weyl. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$ and consequently we have $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. ∎

The converse of Theorem 2.2 does not hold in general, as we can see in the following example.

EXAMPLE 2.3. Let Q be defined, for $x = (\xi_i) \in \ell^1$, by

$$Q(\xi_1, \xi_2, \ldots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \ldots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. It follows from [14, Example 3.12] that

$$\overline{R(Q^n)} \neq R(Q^n), \quad n = 1, 2, \ldots.$$

Define an operator T on $X = \ell^1 \oplus \ell^1$ by $T = Q \oplus 0$. Then $N(T) = \{0\} \oplus \ell^1$, $\sigma(T) = \{0\}$, $E(T) = \{0\}$, $E^0(T) = \emptyset$. Since $R(T^n) = R(Q^n) \oplus \{0\}$, $R(T^n)$ is not closed for any $n \in \mathbb{N}$; so T is not upper semi-B-Weyl (or upper semi-Weyl) and $\sigma_{SBF_+^-}(T) = \{0\}$ (and $\sigma_{SF_+^-}(T) = \{0\}$). We thus have

$$\sigma(T) \setminus \sigma_{SBF_+^-}(T) \neq E(T), \quad \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T).$$

Hence, T satisfies (v), but not (gv).

The following two examples show that properties (Bw) and (v) (or (gv)) are independent.

EXAMPLE 2.4. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$. Then, $\sigma(R) = \mathbf{D}(0,1)$, the closed unit disc on $\mathbb{C}$, and so $\operatorname{iso}\sigma(R) = E^0(R) = \emptyset$, $E(R) = \emptyset$. Moreover, $\sigma_a(R) = \mathit{\Gamma}$, $\sigma_{SBF_+^-}(R) = \mathit{\Gamma}$, $\sigma_{SF_+^-}(R) = \mathit{\Gamma}$, where $\mathit{\Gamma}$ denotes the unit circle of $\mathbb{C}$. Then R does not satisfy (v) or (gv), since $\sigma(R) \setminus \sigma_{SF_+^-}(R) = \mathbf{D}(0,1) \setminus \mathit{\Gamma} \neq \emptyset = E^0(R)$ and $\sigma(R) \setminus \sigma_{SBF_+^-}(R) = \mathbf{D}(0,1) \setminus \mathit{\Gamma} \neq \emptyset = E(R)$. On the other hand, $\sigma_{BW}(R) = \mathbf{D}(0,1)$, and so $\sigma(R) \setminus \sigma_{BW}(R) = \emptyset = E^0(R)$. Hence, R satisfies (Bw).

EXAMPLE 2.5. Consider the operator $T = 0$ on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_a(T) = \{0\}$, $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = \emptyset$ and $E_a^0(T) = \emptyset$. Since $E^0(T) \subseteq E_a^0(T)$, we have $E^0(T) = \emptyset$. Therefore, $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. Thus, T does not satisfy (Bw). Moreover, $E(T) = \{0\}$. Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \{0\} = E(T)$, and so T satisfies (gv) and hence (v).

THEOREM 2.6. *Let $T \in L(X)$. Then T has property (v) if and only if it satisfies Weyl's theorem and $\sigma_{SF_+^-}(T) = \sigma_W(T)$.*

Proof. *Sufficiency*: Suppose that T satisfies (v), i.e. $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. If $\lambda \in \sigma(T) \setminus \sigma_W(T)$, then $\lambda I - T$ is a Weyl operator, and hence it is upper semi-Weyl. Thus, $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$, and so $\sigma(T) \setminus \sigma_W(T) \subseteq E^0(T)$.

To show the opposite inclusion, let $\lambda \in E^0(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$, and so $\lambda I - T$ is upper semi-Fredholm. Since $\lambda \in \operatorname{iso} \sigma(T)$, both T, T^* have SVEP at λ. Thus, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. This implies that $\lambda I - T$ is a Browder operator, and so a Weyl operator. Therefore, $\lambda \in \sigma(T) \setminus \sigma_W(T)$, and hence T satisfies Weyl's theorem. Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ and $\sigma(T) \setminus \sigma_W(T) = E^0(T)$. Hence, $\sigma_{SF_+^-}(T) = \sigma_W(T)$.

Necessity: Suppose that T satisfies Weyl's theorem and $\sigma_{SF_+^-}(T) = \sigma_W(T)$. Then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_W(T) = E^0(T)$, and hence T satisfies (v). ∎

Similarly to Theorem 2.6, we have the following result.

THEOREM 2.7. *Let $T \in L(X)$. Then T has property (gv) if and only if T satisfies generalized Weyl's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.*

Proof. *Sufficiency*: Suppose that T satisfies (gv), i.e. $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda I - T$ is a B-Weyl operator, and hence it is upper semi-B-Weyl. Thus, $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, and so $\sigma(T) \setminus \sigma_{BW}(T) \subseteq E(T)$.

To show the opposite inclusion, let $\lambda \in E(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, so that $\lambda I - T$ is upper semi-B-Fredholm, hence quasi-Fredholm, and both T, T^* have SVEP at λ. By Remark 1.2 and Lemma 1.4, $\lambda I - T$ is Drazin invertible, and hence B-Weyl. Therefore, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, and so $E(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$. This shows that T satisfies generalized Weyl's theorem. Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$.

Necessity: Suppose that T satisfies generalized Weyl's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. Then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, and so T satisfies (gv). ∎

The next example shows that, in general, Weyl's theorem (resp. generalized Weyl's theorem) does not imply property (v) (resp. (gv)).

EXAMPLE 2.8. Let R be the shift operator defined in Example 2.4. Then $\sigma_{BW}(R) = \mathbf{D}(0,1)$, $\sigma_W(R) = \mathbf{D}(0,1)$. Therefore, $\sigma(R) \setminus \sigma_W(R) = \emptyset = E^0(R)$ and $\sigma(R) \setminus \sigma_{BW}(R) = \emptyset = E(R)$. Thus, R satisfies both Weyl's theorem and generalized Weyl's theorem, but neither (v) nor (gv).

In the next theorem we give conditions for the equivalence between (v) and (w) (resp. (gv) and (gw)).

THEOREM 2.9. *Let $T \in L(X)$. Then:*

(i) *T has property (v) if and only if T has property (w) and $\sigma(T) = \sigma_{\mathrm{a}}(T)$.*
(ii) *T has property (gv) if and only if T has property (gw) and $\sigma(T) = \sigma_{\mathrm{a}}(T)$.*

Proof. (i) Suppose that T satisfies (v) and let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Since $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$, we have $\lambda \in \pi_{00}(T)$. Thus, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subseteq E^0(T)$. Now if $\lambda \in E^0(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T) < \infty$. Consequently, $\lambda I - T$ is not injective, and hence not bounded below. So, $\lambda \in \sigma_a(T)$. Since T satisfies (v) and $\lambda \in E^0(T)$, it follows that $\lambda I - T$ is upper semi-Weyl. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus, $E^0(T) \subseteq \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and T satisfies (w). Consequently, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. Therefore, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

Conversely, suppose that T satisfies (w) and $\sigma(T) = \sigma_a(T)$. Then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. Hence, $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ and T satisfies (v).

(ii) Suppose that T satisfies (gv) and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, we have $\lambda \in E(T)$. Therefore, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E(T)$. Now if $\lambda \in E(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T)$. Consequently, $\lambda I - T$ is not injective, and hence not bounded below. Thus, $\lambda \in \sigma_a(T)$. As T satisfies (gv) and $\lambda \in E(T)$, it follows that $\lambda I - T$ is upper semi-B-Weyl. Hence, $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, so $E(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and T satisfies (gw). Consequently, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Therefore, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and $\sigma(T) = \sigma_a(T)$.

Conversely, assume that T satisfies (gw) and $\sigma(T) = \sigma_a(T)$. Then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Thus, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and T satisfies (gv). ∎

The following example shows that property (gw) (resp. (w)) does not imply (gv) (resp. (v)).

EXAMPLE 2.10. Let R be the shift operator defined in Example 2.4. Then R does not satisfy (v) or (gv). On the other hand, $\sigma_a(R) \setminus \sigma_{SF_+^-}(R) =$

$\emptyset = E^0(R)$ and $\sigma_a(R) \setminus \sigma_{SBF_+^-}(R) = \emptyset = E(R)$. Therefore, R satisfies (w) and (gw).

Recall that $T \in L(X)$ is said to satisfy *a-Browder's theorem* (resp. *generalized a-Browder's theorem*) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$ (resp. $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$). From [10, Theorem 2.2] (see also [4, Theorem 3.2(ii)]), a-Browder's theorem and generalized a-Browder's theorem are equivalent. It is well known that a-Browder's theorem for T implies Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$. Also by [10, Theorem 2.1] Browder's theorem for T is equivalent to generalized Browder's theorem for T, i.e. $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$.

We prove in Theorem 2.2 that property (gv) implies (v). The next result gives the precise relationships between these properties.

THEOREM 2.11. *For $T \in L(X)$, the following statements are equivalent:*

(i) *T has property (gv),*
(ii) *T has property (v) and $E(T) = \Pi_a(T)$.*

Proof. (i)⇒(ii). Assume that T satisfies (gv); then it also satisfies (v). If $\lambda \in E(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$. Since T satisfies (gv), $\lambda I - T$ is upper semi-B-Fredholm. Therefore, T has SVEP at λ and $\lambda I - T$ is quasi-Fredholm. By Remark 1.2 and Lemma 1.4, $\lambda I - T$ is left Drazin invertible, and so $\lambda \in \sigma(T) \setminus \sigma_{LD}(T)$. By Theorem 2.9, $\sigma(T) = \sigma_a(T)$, and hence $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T) = \Pi_a(T)$. This shows the inclusion $E(T) \subseteq \Pi_a(T)$.

To show the opposite inclusion, let $\lambda \in \Pi_a(T)$. Since T satisfies (gv), it follows that $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T) \subseteq \sigma(T) \setminus \sigma_{LD}(T) \subseteq \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Therefore, $\Pi_a(T) \subseteq E(T)$.

(ii)⇒(i). Assume that T satisfies (v) and $E(T) = \Pi_a(T)$. By Theorem 2.9, T satisfies (w) and $\sigma(T) = \sigma_a(T)$. Property (w) implies by [8, Theorem 2.6] that T satisfies a-Browder's theorem, or equivalently, generalized a-Browder's theorem. Therefore, $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Since $E(T) = \Pi_a(T)$, we have $E(T) = \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. Thus, $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and T satisfies (gv). ■

COROLLARY 2.12. *Let $T \in L(X)$. Then:*

(i) *T has property (v) if and only if T has property (z).*
(ii) *T has property (gv) if and only if T has property (gz).*

Proof. (i) Suppose that T satisfies (v). By Theorem 2.9(i), $\sigma(T) = \sigma_a(T)$, and so $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T) = E_a^0(T)$. Therefore, T satisfies (z).

Conversely, assume that T satisfies (z). By [25, Theorem 2.4(i)], $\sigma(T) = \sigma_a(T)$, and so $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T) = E^0(T)$. Therefore, T satisfies (z).

(ii) The proof is similar to the proof of (i). Just use both Theorem 2.9(ii) and [25, Theorem 2.4(ii)]. ∎

THEOREM 2.13. *Suppose that $T \in L(X)$ has property (gv). Then:*

(i) *T satisfies generalized a-Browder's theorem and $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$.*
(ii) *T satisfies generalized Browder's theorem and $\sigma(T) = \sigma_{BW}(T) \cup \operatorname{iso}\sigma(T)$.*

Proof. (i) By [7, Theorem 2.4] it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$. We have the following two cases.

CASE 1: $\lambda \notin \sigma(T)$.
CASE 2: $\lambda \in \sigma(T)$.

In Case 1, clearly T has SVEP at λ. In Case 2, we have $\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$ and since T satisfies (gv), it follows that $\lambda \in E(T)$. Therefore, $\lambda \in \operatorname{iso}\sigma(T)$, and so T has SVEP at λ again.

To show the equality $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$, observe first that $\sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T) \subseteq \sigma(T)$ holds for every $T \in L(X)$. To show the opposite inclusion, suppose that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_{SBF_+^-}(T)$. Then $E(T)$, since T satisfies (gv). Therefore, $\lambda \in \operatorname{iso}\sigma(T)$, and so $\sigma(T) \subseteq \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$. This shows that $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$.

(ii) Follows from (i), by using the fact that generalized a-Browder's theorem implies generalized Browder's theorem, and the equality $\sigma(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$ implies the inclusion $\sigma(T) \subseteq \sigma_{BW}(T) \cup \operatorname{iso}\sigma(T)$, leading to $\sigma(T) = \sigma_{BW}(T) \cup \operatorname{iso}\sigma(T)$. ∎

For $T \in L(X)$, define $\Pi_+(T) = \sigma(T) \setminus \sigma_{LD}(T)$. The precise relationship between generalized a-Browder's theorem and property (gv) is described by the following theorem.

THEOREM 2.14. *If $T \in L(X)$, then the following statements are equivalent:*

(i) *T has property (gv),*
(ii) *T satisfies generalized a-Browder's theorem and $\Pi_+(T) = E(T)$.*

Proof. (i)⇒(ii). Assume that T satisfies (gv). Then, by Theorem 2.13, it is sufficient to prove $\Pi_+(T) = E(T)$. Indeed, $E(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) =$

$\sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T)$, since T satisfies (gv) and generalized a-Browder theorem by Theorem 2.13.

(ii)⇒(i). If T satisfies generalized a-Browder's theorem and $\Pi_+(T) = E(T)$, then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T) = E(T)$. Therefore, T satisfies (gv). ∎

Corollary 2.15. *If $T \in L(X)$ has SVEP at each $\lambda \notin \sigma_{SBF_+^-}(T)$, then T has property (gv) if and only if $E(T) = \Pi_+(T)$.*

Proof. The hypothesis that T has SVEP at each $\lambda \notin \sigma_{SBF_+^-}(T)$ implies that T satisfies generalized a-Browder's theorem. Therefore, if $E(T) = \Pi_+(T)$, then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T) = \Pi_+(T) = E(T)$. ∎

For $T \in L(X)$, define $\Pi_+^0(T) = \sigma(T) \setminus \sigma_{ub}(T)$. The proofs of the following three theorems are analogous to the proofs of Theorems 2.13–2.15.

Theorem 2.16. *Suppose that $T \in L(X)$ has property (v). Then:*

(i) *T satisfies a-Browder's theorem and $\sigma(T) = \sigma_{SF_+^-}(T) \cup \operatorname{iso} \sigma(T)$.*
(ii) *T satisfies Browder's theorem and $\sigma(T) = \sigma_W(T) \cup \operatorname{iso} \sigma(T)$.*

Theorem 2.17. *If $T \in L(X)$, then the following statements are equivalent:*

(i) *T has property (v),*
(ii) *T satisfies a-Browder's theorem and $\Pi_+^0(T) = E^0(T)$.*

Corollary 2.18. *If $T \in L(X)$ has SVEP at every point $\lambda \notin \sigma_{SF_+^-}(T)$, then T has property (v) if and only if $E^0(T) = \Pi_+^0(T)$.*

It is proved in [18, Lemma 2.1] that if T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$ (resp., T has SVEP at every $\lambda \notin \sigma_{SF_-^+}(T)$), then $\sigma_W(T) = \sigma_{SF_+^-}(T)$ and $\sigma_a(T) = \sigma(T)$ (resp., $\sigma_W(T^*) = \sigma_{SF_+^-}(T^*)$ and $\sigma_a(T^*) = \sigma(T^*)$). Also, it is proved in [18, Lemma 2.4] that if T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$ (resp., T has SVEP at every $\lambda \notin \sigma_{SBF_-^+}(T)$), then $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) = \sigma_D(T)$ and $\sigma_a(T) = \sigma(T)$ (resp., $\sigma_{BW}(T^*) = \sigma_{SBF_+^-}(T^*) = \sigma_D(T^*)$ and $\sigma_a(T^*) = \sigma(T^*)$). By the above results, we clearly see that if T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$, then properties (R), (w), (v), (z), (b), (az), (aw), (ab), Weyl's theorem and a-Weyl's theorem are equivalent for T. In the same form, we obtain equivalence for the respective "generalized" properties for T. The same equivalences hold for T^* if T has SVEP at $\lambda \notin \sigma_{SBF_-^+}(T)$.

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