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Algebra of multipliers on the space of real analytic functions of one variable

by

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Abstract. We consider the topological algebra of (Taylor) multipliers on spaces of real analytic functions of one variable, i.e., maps for which monomials are eigenvectors. We describe multiplicative functionals and algebra homomorphisms on that algebra as well as idempotents in it. We show that it is never a Q-algebra and never locally m-convex. In particular, we show that Taylor multiplier sequences cease to be so after most permutations.

1. Introduction. We consider *multipliers* on the space $\mathscr{A}(I)$ of real analytic functions on an open set $I \subseteq \mathbb{R}$, i.e., continuous linear maps $M : \mathscr{A}(I) \to \mathscr{A}(I)$ such that every monomial is an eigenvector of M. Here and throughout the paper we denote by I an arbitrary open subset of $\mathbb{R}$. The corresponding sequence $(m_n)_{n\in\mathbb{N}}$ of eigenvalues is called the *multiplier sequence* of M. Since monomials are linearly dense in $\mathscr{A}(I)$, any multiplier is uniquely determined by its multiplier sequence. On the other hand the space $\mathscr{A}(I)$ has no Schauder basis [12], in particular, monomials do not form a basis in $\mathscr{A}(I)$ and therefore multipliers are not just diagonal operators. In case $0 \in I$, I connected, $M(f)(z) = \sum_{n=0}^{\infty} m_n f_n z^n$ around zero whenever $f(z) = \sum_{n=0}^{\infty} f_n z^n$ around zero and $(m_n)_{n\in\mathbb{N}}$ is the corresponding multiplier sequence (comp. [11, Prop. 2.1]).

Clearly, the space $M(I)$ of all multipliers on $\mathscr{A}(I)$ is a subalgebra of the algebra $L(\mathscr{A}(I))$ of all continuous linear operators on $\mathscr{A}(I)$, and $M(I)$ is complete when we equip it with the topology of uniform convergence on bounded subsets of $\mathscr{A}(I)$. Composition is a separately continuous multiplication on $M(I)$. In fact, it was proved in [10] that for connected open I the space $M(I)$ is either a Fréchet space or an LF-space. On the other hand,

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$M(I)$ as an algebra can be identified with the family of multiplier sequences with pointwise multiplication.

In the present paper we study properties of $M(I)$ as a topological algebra. In case I is an open connected subset of the real line (i.e., an interval) we describe all multiplicative functionals on $M(I)$ (Corollaries 4.2 and 5.3), idempotents in $M(I)$ (Theorem 3.1), and algebra homomorphisms on $M(I)$ (Corollary 4.4, Theorems 5.6 and 5.7). In particular, we show that all such algebra homomorphisms and multiplicative functionals are automatically continuous (Corollary 4.4, Proposition 5.5). As a consequence, we show that $M(I)$ is never a Q-algebra (Corollaries 4.3 and 5.4) and never locally m-convex (Corollary 3.3). Finally, we describe closed ideals in $M(I)$ whenever $0 \in I$ (Theorem 5.2) or $I = (a, b)$, $0 < a < b < \infty$ (Theorem 4.5). The invertible elements in $M(I)$, $0 \notin I$, are described in Corollary 4.3. The more challenging problem of the case $0 \in I$ will be postponed to the forthcoming paper [11]. The natural candidates for homomorphisms of the algebra $M(I)$ are permutations of the corresponding multiplier sequences. Our description of the homomorphisms shows that very few such permutations indeed act on $M(I)$ (Theorem 5.7), which seems to be the most interesting conclusion of the paper.

The main tools we use are the representation theorems proved in [10] and repeated for the sake of convenience without proof in Section 2. In Section 3 we give results true for arbitrary open sets $I \subset \mathbb{R}$. However, in most cases the methods for $0 \in I$ and $0 \notin I$ are different. We consider the latter case in Section 4, and the former in Section 5.

In [10] we studied multipliers and proved some fundamental representation theorems. In the forthcoming paper [11] we will study invertibility of multipliers on $\mathscr{A}(I)$ in the challenging case of $0 \in I$. This is a part of a broader project of studying operators on the space of real analytic functions: see, for instance, [8], [9].

Let us introduce some notation. We call any (possibly unbounded) open connected subset of $\mathbb{R}$ an interval. We denote by $\hat{\mathbb{C}}$ the Riemann sphere and by $H(S)$ the space of holomorphic functions on some open neighbourhood of $S \subseteq \hat{\mathbb{C}}$, while $H_0(S)$ denotes the subspace consisting of holomorphic functions vanishing at ∞. Clearly,

$$H_0(\hat{\mathbb{C}} \setminus S) = \bigcup_{K \subset S,\, K \text{ compact}} H_0(\hat{\mathbb{C}} \setminus K)$$

and $H_0(\hat{\mathbb{C}} \setminus K)$ are Fréchet spaces. Hence, if S has a fundamental sequence of compact subsets then $H_0(\hat{\mathbb{C}} \setminus S)$ as well as $H(\hat{\mathbb{C}} \setminus S)$ are LF-spaces.

For any $a \in \mathbb{R}$ we denote by M_a the dilation map

$$M_a(f)(x) := f(ax).$$

By η_k we denote the monomial function

$$\eta_k(x) := x^k, \qquad k \in \mathbb{N}.$$

Analogously, for $z \in \mathbb{C}$ and $x \in \mathbb{R} \setminus \{0\}$, we define

$$\eta_z^+(x) := |x|^z \quad \text{and} \quad \eta_z^-(x) := \operatorname{sgn}(x)|x|^z.$$

It is well-known via the so-called Köthe–Grothendieck duality (see [1, Th. 1.3.5]) that every continuous linear functional T on $H(K)$, $K \Subset \mathbb{C}$ compact, corresponds to a holomorphic function $f \in H_0(\hat{\mathbb{C}} \setminus K)$ (called the Cauchy transform of T) where

$$T(g) = \frac{1}{2\pi i} \int_\gamma g(z) f(z)\, dz, \quad f(z) := \left\langle \frac{1}{z - \cdot}, T \right\rangle$$

and γ is a curve surrounding K once and separating K from the singularities of f. An analogous representation holds for continuous linear functionals on $H(U)$, $U \subset \mathbb{C}$ open.

We use without reference many facts on the spaces $\mathscr{A}(I)$—for them we refer to [7]. For unexplained notions from functional analysis see the book [16].

2. Representations. First, we recall the representation theorems proved in [10]. They will be very useful later on. We define the *dilation set* $V(I)$ for an open set $I \subset \mathbb{R}$ by

$$V(I) := \bigcap_{y \in I} \{x : xy \in I\} = \bigcap_{\varepsilon \in I,\, \varepsilon \neq 0} (1/\varepsilon) I.$$

The name comes from the fact that the dilation $M_a : \mathscr{A}(I) \to \mathscr{A}(I)$ is well-defined if and only if $a \in V(I)$. Observe that $1 \in V(I)$, and that $0 \in I$ if and only if $0 \in V(I)$.

The dilation set $V(I)$ is crucial for the representation theorems presented in this section. It has been calculated in [10, Section 3] for many open sets I. We recall here the results for intervals to indicate what $V(I)$ may be like.

Remark 2.1. (a) $V(\mathbb{R}) = \mathbb{R}$ and $V(]-\infty, 0[) = V(]0, \infty[) =]0, \infty[$.

(b) Let $-\infty < a < 0 < b < \infty$. Then $V(]a, \infty[) = V(]-\infty, b[) = [0, 1]$, $V(]a, 0[) = V(]0, b[) =]0, 1]$ and $V(]a, b[) = [-\min(|a/b|, |b/a|), 1]$.

(c) Let $0 < a < b < \infty$. Then $V(]a, b[) = \{1\}$.

We need the following notion.

Definition 2.2. An open set $I \subset \mathbb{R}$ is *nice* if for every open neighbourhood U of $V(I)$ there are finitely many nonzero $\varepsilon_1, \ldots, \varepsilon_p \in I$ such that

$$\bigcap_{j=1}^{p} (1/\varepsilon_j) I \subset U.$$

Notice the following easy observation:

PROPOSITION 2.3 ([10, Prop. 2.4]). *Every open interval* (*bounded or unbounded*) *is nice.*

For more on dilation sets and nice sets see [10, Section 3]. The following is the fundamental representation theorem for multipliers via analytic functionals.

THE FIRST REPRESENTATION THEOREM 2.4 ([10, Th. 2.6]). *Let $I \subset \mathbb{R}$ be an open set. The map*

$$\mathscr{B} : \mathscr{A}(V(I))'_b \to M(I), \qquad \mathscr{B}(T)(g)(y) := \langle g(y\,\cdot), T\rangle,$$

is bijective and continuous, and the multiplier sequence of $\mathscr{B}(T)$ is the sequence of moments of the analytic functional T, i.e. $(\langle z^n, T\rangle)_{n\in\mathbb{N}}$. If I is nice then $\mathscr{B}$ is even a homeomorphism.

Using the Köthe–Grothendieck duality we get the following representations.

THE SECOND REPRESENTATION THEOREM 2.5 ([10, Th. 2.8]). *For any open set $I \subseteq \mathbb{R}$ the algebra of multipliers $M(I)$ is isomorphic as an algebra* (*even topologically isomorphic whenever I is a nice set*) *to the LF-algebra of holomorphic functions $H(\hat{\mathbb{C}} \setminus (1/V(I)))$ with Hadamard multiplication of Taylor series, i.e.,*

$$f * g(z) = \sum_{n=0}^{\infty} m_n l_n z^n$$

around zero where

$$f(z) = \sum_{n=0}^{\infty} m_n z^n, \quad g(z) = \sum_{n=0}^{\infty} l_n z^n$$

around zero. Here $1/V(I) := \{1/z : z \in V(I)\}$. The multiplier sequence of the given multiplier is equal to the Taylor coefficients (m_n) of the corresponding function f.

REMARK 2.6. (a) In fact, the identification $H(\hat{\mathbb{C}} \setminus (1/V(I))) \to M(I)$ is continuous for an arbitrary open set $I \subset \mathbb{R}$.

(b) If $V(I)$ is a compact set, with I nice, then $M(I)$ is a Fréchet space. In general, if I is nice then $M(I)$ is an LF-space.

(c) Moreover, the identification in the Second Representation Theorem for compact $V(I)$ represents the multiplier algebra as $H(G)$ for some domain $G \subset \hat{\mathbb{C}}$ where $0 \in G$ and G is *admissible*, i.e., $H(G)$ is an algebra with Hadamard multiplication. In particular, $G \subseteq \mathbb{C}$ is admissible if and only if the complement of G is a semigroup with multiplication (see the introduction to [23]). Such algebras are studied, for instance, in [4], [23], [24], [22].

(d) Since $M(I)$ is algebra isomorphic to $H_0(\hat{\mathbb{C}} \setminus V(I))$, on $M(I)$ there are two different algebra structures: one given by composition of multipliers, the other given by pointwise multiplication on $H_0(\hat{\mathbb{C}} \setminus V(I))$. In case I is nice these multiplications give two different topological algebra structures for the same topology.

Let us denote the *support function* of a convex compact set K by

$$H_K(y) := \sup_{z\in K}(\operatorname{Re} zy).$$

Then for any convex compact set K and any convex set Ω we define

$$\operatorname{Exp}(K) := \{f \in H(\mathbb{C}) : \forall\varepsilon > 0 : \|f\|_{K,\varepsilon} < \infty\},\ \operatorname{Exp}(\Omega) := \bigcup_{K\Subset\Omega} \operatorname{Exp}(K),$$

where

$$\|f\|_{K,\varepsilon} := \sup_{z\in\mathbb{C}} |f(z)| \exp(-H_K(z) - \varepsilon|z|).$$

For a set $A \subset \mathbb{R}_+$ we put $\log A := \{\log x : x \in A\}$.

For open sets I with $0 \notin I$ there is an additional representation of multipliers which is especially simple and precise if I is connected.

The Third Representation Theorem 2.7 ([10, Th. 5.3]). *For any open set $I \subset \mathbb{R}$ with $0 \notin I$ and $V(I)$ connected, the map*

$$\mathscr{M} : M(I) \to \operatorname{Exp}(\log V(I)), \quad \mathscr{M}(\mathscr{B}(T))(z) := \langle \eta_z^+, T\rangle, \quad T \in \mathscr{A}(V(I))',$$

is an algebra isomorphism where $\mathscr{B}(T)$ is defined as in Theorem 2.4 and $\operatorname{Exp}(\log V(I))$ *is equipped with pointwise multiplication. If I is nice then $\mathscr{M}$ is a homeomorphism.*

The multiplier sequence corresponding to M is $(\mathscr{M}(M)(n))_{n\in\mathbb{N}}$.

Corollary 2.8 ([10, Cor. 5.5]). *For any open set $I \subset \mathbb{R} \setminus \{0\}$ the map*

$$\mathscr{M}^+ \times \mathscr{M}^- : M(I) \to \operatorname{Exp}(\mathbb{R}) \times \operatorname{Exp}(\mathbb{R}), \quad \mathscr{M}^\pm(M)(z) = \langle \eta_z^\pm, T\rangle,$$

is an injective algebra homomorphism, where $T \in \mathscr{A}(V(I))'$, $\mathscr{B}(T) = M$ and $\operatorname{Exp}(\mathbb{R})$ *is equipped with pointwise multiplication. The multiplier sequence corresponding to M is $(\mathscr{M}^{(-1)^n}(M)(n))_{n\in\mathbb{N}}$.*

3. Algebra of multipliers. Summarizing the consequences of the section above: the multipliers form a commutative subalgebra $M(I)$ of $L(\mathscr{A}(I))$ with composition as a separately continuous multiplication. It is either an LF-algebra or a Fréchet algebra (at least for nice open sets I) according to the First and Second Representation Theorems. Now, we explore the algebra structure of $M(I)$.

Especially interesting are those multipliers which are projections, i.e., idempotents. Surprisingly, there exist very few such multipliers.

Theorem 3.1. *Let $I \subset \mathbb{R}$ be an arbitrary open set. Every idempotent in $M(I)$ has a multiplier sequence which is the characteristic function of a set $A \subset \mathbb{N}$ belonging to an algebra $\mathscr{E}(I)$ of subsets of $\mathbb{N}$. Moreover,*

(a) *if $0 \notin I$ and I is not symmetric, then $\mathscr{E}(I) = \{\emptyset, \mathbb{N}\}$;*
(b) *if $0 \notin I$ and I is symmetric, then $\mathscr{E}(I) = \{\emptyset, 2\mathbb{N}, \mathbb{N}, \mathbb{N} \setminus 2\mathbb{N}\}$;*
(c) *if $0 \in I$ and I is not symmetric, then $\mathscr{E}(I)$ is generated by all finite sets;*
(d) *if $0 \in I$ and I is symmetric, then $\mathscr{E}(I)$ is generated by all finite sets and the set $2\mathbb{N}$.*

We call the algebra $\mathscr{E}(I)$ of sets the *idempotent algebra* for $\mathscr{A}(I)$.

Proof. We use the Second Representation Theorem (i.e., Theorem 2.5). Idempotents correspond to the functions $f(z) = \sum_{n \in S} z^n$ for some set S with only real singularities. Moreover, the family of possible sets S is easily seen to be an algebra. Clearly, either f is a polynomial or the radius of convergence is equal 1. By Szegö's Theorem (see [3, Satz 6.1] or [20, Ch. 11.4, p. 260]), f is of the form $P(z)/(1 - z^m)$ for some polynomial and $m \in \mathbb{N}$. In order that $f \in \bigcup_{K \subset 1/V(I)} H(\hat{\mathbb{C}} \setminus K)$ we must have $m = 1$ or $m = 2$. This shows that $\mathscr{E}(I)$ is contained in the algebra of sets generated by all finite sets and the set of even numbers.

It is easy to observe that in the representation via $H(\hat{\mathbb{C}} \setminus (1/V(I)))$ the unit multiplier is given by $1/(1 - z)$, and the idempotent with multiplier sequence being the characteristic function of the set of even numbers is given by $1/(1 - z^2)$. Idempotents corresponding to finite sets are polynomials. By the Second Representation Theorem, singularities of the representation of idempotents in $H(\hat{\mathbb{C}} \setminus (1/V(I)))$ must be in $1/V(I)$. For open sets I the set of even numbers belongs to the idempotent algebra $\mathscr{E}(I)$ if and only if $-1 \in V(I)$, i.e., I is symmetric with respect to 0. Analogously, any finite set belongs to $\mathscr{E}(I)$ for an open set $I \subset \mathbb{R}$ if and only if $0 \in V(I)$, i.e., if $0 \in I$. ∎

The algebra $M(I)$ has plenty of multiplicative functionals.

Let us define $\tau_j : M(I) \to \mathbb{C}$, $\tau_j(M) = m_j$, where $(m_n)_{n \in \mathbb{N}}$ is the multiplier sequence of M. Clearly, τ_j is a multiplicative functional on $M(I)$ for any open subset $I \subseteq \mathbb{R}$.

Proposition 3.2. *For every $j \in \mathbb{N}$ and every open set $I \subseteq \mathbb{R}$ the functional τ_j is a non-zero continuous linear multiplicative functional on $M(I)$.*

Proof. For any $\eta_k \in \mathscr{A}(I)$ there is a continuous functional $f_k \in \mathscr{A}(I)'$ such that $f_k(\eta_k) = 1$. Then $\tau_k(M) = f_k(M(\eta_k))$, hence τ_k is continuous. It is clear that it is multiplicative. ∎

Corollary 3.3. *For an arbitrary open set $I \subset \mathbb{R}$ the algebra $M(I)$ is never locally m-convex.*

Proof. By [10, Section 4], the operator $\theta : \mathscr{A}(I) \to \mathscr{A}(I)$, $(\theta f)(z) := zf'(z)$, is always a multiplier. Assume that $M(I)$ is locally m-convex. Then it is easily seen that every entire function φ acts on $M(I)$ (comp. [26, Section 16]), i.e., if $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, then for any $M \in M(I)$ the series $\sum_{n=0}^{\infty} a_n M^n$ is convergent. Let us apply that to $M = \theta$. By Proposition 3.2,

$$\tau_j\Big(\sum_{n=0}^{\infty} a_n \theta^n\Big) = \sum_{n=0}^{\infty} a_n \tau_j(\theta)^n = \sum_{n=0}^{\infty} a_n j^n = \varphi(j).$$

We have proved that for any $\varphi \in H(\mathbb{C})$ the sequence $(\varphi(j))_{j\in\mathbb{N}}$ is a multiplier sequence for some $M \in M(I)$. Since an arbitrary sequence is of that form, this contradicts Theorem 2.5. ∎

4. Case of sets not containing zero. For I open not containing zero, the algebra $M(I)$ has many more multiplicative functionals than in the general case.

PROPOSITION 4.1 ([10, Prop. 5.1]). *If $I \subset \mathbb{R}$ is an arbitrary open set not containing zero then every function $\eta_z^{\pm}$ is an eigenvector of every multiplier on $\mathscr{A}(I)$. Hence, for any $z \in \mathbb{C}$ the map*

$$\tau_z^{\pm}(M) := \frac{M(\eta_z^{\pm})}{\eta_z^{\pm}}$$

is a continuous multiplicative functional on $M(I)$.

If I is an interval, we even have a precise description. From the Third Representation Theorem (Theorem 2.7) we will deduce

COROLLARY 4.2. *If $I \subset \mathbb{R}$ is an arbitrary open set not containing zero such that $V(I)$ is connected then the only non-zero multiplicative functionals on $M(I)$ are defined as follows:*

$$\tau_z^{+}(M) := \mathscr{M}(M)(z) \quad \textit{for any } z \in \mathbb{C}.$$

In particular, all multiplicative functionals are automatically continuous.

Proof. It is enough to find multiplicative functionals on the isomorphic algebra $\operatorname{Exp}(\log V(I))$. Clearly, point evaluations at every $z \in \mathbb{C}$ are non-zero multiplicative functionals. Since they coincide with the functionals defined in Proposition 4.1, they are continuous on $M(I)$.

Let δ be a non-zero multiplicative functional on $\operatorname{Exp}(\log V(I))$. Obviously, $\eta_j(z) := z^j$, $j \in \mathbb{N}_0$, belongs to the algebra and

$$\delta(\eta_1) =: w, \quad \delta(\eta_0) = 1.$$

It is easily seen that the function $\frac{f - f(w)\cdot\eta_0}{\eta_1 - w\cdot\eta_0}$ belongs to $\operatorname{Exp}(\log V(I))$ for any

f in the same class. Hence

$$\delta(f-f(w)\eta_0)=\delta\left(\frac{f-f(w)\cdot\eta_0}{\eta_1-w\cdot\eta_0}\right)\delta(\eta_1-w\eta_0)=0.$$

On the other hand,

$$\delta(f-f(w)\eta_0)=\delta(f)-f(w).$$

We have proved that δ is the point evaluation at w. ∎

Corollary 4.3. *Let $I\subset\mathbb{R}$ be an arbitrary open set not containing zero. If I is not symmetric with respect to zero then the invertible elements in $M(I)$ are exactly of the form*

$$M=B\cdot M_a,\qquad M_a(f)(x)=f(ax),$$

where $B\in\mathbb{C}\setminus\{0\}$, $a,a^{-1}\in V(I)$. If I is symmetric with respect to zero then the invertible elements in $M(I)$ are either as above or of the form

$$M=B_+\left(\frac{M_{a_+}+M_{-a_+}}{2}\right)+B_-\left(\frac{M_{a_-}-M_{-a_-}}{2}\right),$$

where $B_+,B_-\in\mathbb{C}\setminus\{0\}$, $\pm a_+,\pm a_-,\pm a_+^{-1},\pm a_-^{-1}\in V(I)$.

In particular, the set of non-invertible elements is dense (i.e., $M(I)$ is not a Q-algebra) and if $V(I)=\{1\}$ then the only invertible multipliers are non-zero constants.

In the non-symmetric case the closure of the set $\mathrm{Inv}(M(I))$ *of invertible elements of $M(I)$ is equal to* $\mathrm{Inv}(M(I))\cup\{0\}$. *In the symmetric case* $\overline{\mathrm{Inv}(M(I))}$ *consists of all multipliers such that the even and odd parts of their multiplier sequences are either zero, or they are the even and odd parts of the multiplier sequence of an invertible multiplier of the form $B\cdot M_a$, i.e., $B\in\mathbb{C}\setminus\{0\}$, $a,a^{-1}\in V(I)$.*

Proof. By Corollary 2.8, if M is invertible then $\mathscr{M}^+(M),\mathscr{M}^-(M)\in\mathrm{Exp}(\mathbb{R})$ cannot have zeroes. By the classical Hadamard representation theorem for entire functions of finite order, for some $A_+,A_-,B_+,B_-\in\mathbb{C}$ we have

$$\mathscr{M}^+(M)(z)=B_+\exp(A_+z),\quad \mathscr{M}^-(M)(z)=B_-\exp(A_-z).$$

By Corollary 2.8, the corresponding multiplier sequences are of the form $(m_n)_{n\in\mathbb{N}}$, where

$$m_n=\begin{cases}B_+\exp(A_+n) & \text{for even } n,\\ B_-\exp(A_-n) & \text{for odd } n.\end{cases}$$

Invertibility of M implies that $B_+,B_-\neq 0$. By the Second Representation Theorem 2.5, M corresponds to $f\in H(\hat{\mathbb{C}}\setminus(1/V(I)))$,

$$f(z)=\sum_{n=0}^{\infty}m_nz^n,$$

so for $a_+ := \exp(A_+)$, $a_- := \exp(A_-)$,

$$f(z) = \begin{cases} \dfrac{B_+}{1-a_+z} & \text{for } B_+ = B_- \text{ and } A_+ = A_-, \\ \dfrac{B_+}{1+a_+z} & \text{for } B_+ = -B_- \text{ and } A_+ = A_-, \\ \dfrac{B_+}{2}\left(\dfrac{1}{1-a_+z} + \dfrac{1}{1+a_+z}\right) + \dfrac{B_-}{2}\left(\dfrac{1}{1-a_-z} - \dfrac{1}{1+a_-z}\right) & \text{otherwise.} \end{cases}$$

Thus if $B_+ = B_-$, $A_+ = A_-$ then M is just $B_+ \cdot M_{a_+}$. Since the singularities of f must be contained in $1/V(I)$ we get $a_+ \in V(I)$. If $B_+ = -B_-$, $A_+ = A_-$ then $M = B_+ \cdot M_{-a_+}$ with negative $-a_+ \in V(I)$. In both cases invertibility means that the singularities a_+^{-1} or $-a_+^{-1}$ of the function corresponding to the inverse belong to $1/V(I)$ as well.

In the other case, $\pm a_+, \pm a_- \in V(I)$ and

$$M = B_+\left(\frac{M_{a_+} + M_{-a_+}}{2}\right) + B_-\left(\frac{M_{a_-} - M_{-a_-}}{2}\right).$$

Hence M is invertible iff $\pm a_+^{-1}, \pm a_-^{-1} \in V(I)$. Since $V(I)$ is a semigroup (see [10, Prop. 2.1]), $-1 = (-a_+) \cdot a_+^{-1} \in V(I)$ and I is symmetric with respect to zero.

Let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of invertible multipliers convergent to M. Without loss of generality we may assume that either

$$M^{(n)} = C_n M_{a_n}, \qquad (a_n) \subset V(I),\ C_n \neq 0 \text{ for any } n \in \mathbb{N},$$

or

$$M^{(n)} = B_+^{(n)}\left(\frac{M_{a_{n,+}} + M_{-a_{n,+}}}{2}\right) + B_-^{(n)}\left(\frac{M_{a_{n,-}} - M_{-a_{n,-}}}{2}\right)$$

with $(a_{n,+}), (a_{n,-}) \subset V(I)$, $B_+^{(n)}, B_-^{(n)} \neq 0$ for any $n \in \mathbb{N}$.

Let us consider the first case. Let $C_n \in \mathbb{C}$, $(a_n) \subset V(I)$ and $C_n M_{a_n} \to M \in M(I)$. Then $C_n M_{a_n}(1) = C_n \to M(1)$ and $C_n M_{a_n}(\eta_k)(1) = C_n a_n^k \to M(\eta_k)(1)$. If $M(1) \neq 0$ then $a_n \to a$ and $C_n M_{a_n} \to C M_a$. If $M(1) = 0$ then $C_n a_n^k$ is convergent for every $k \in \mathbb{N}$ so $C_n a_n^k \to 0$ and $C_n M_{a_n} \to 0$.

Let us consider the second case. Here

$$M^{(n)}(\eta_0^+)(1) = B_+^{(n)} \to M(\eta_0^+)(1), \quad M^{(n)}(\eta_0^-)(1) = B_-^{(n)} \to M(\eta_0^-)(1)$$

and

$$M^{(n)}(\eta_k^+)(1) = B_+^{(n)} a_{n,+}^k \to M(\eta_k^+)(1).$$

If $M(\eta_0^+)(1) \neq 0$ then $a_{n,+} \to a_+$ for some $a_+ \in \mathbb{R}$. If $M(\eta_0^+)(1) = 0$ then $B_+^{(n)} a_{n,+}^k$ is convergent for every $k \in \mathbb{N}$, hence $B_+^{(n)} a_{n,+}^k \to 0$. Therefore the even part of the multiplier sequence $(m_n)_{n\in\mathbb{N}}$ of M is either zero or equal to the even part of the multiplier sequence of the invertible multiplier $M(\eta_0^+)(1) \cdot M_{a_+}$. Analogously,

$$M^{(n)}(\eta_k^-)(1) = B_-^{(n)} a_{n,-}^k \to M(\eta_k^-)(1).$$

If $M(\eta_0^-)(1) \neq 0$ then $a_{n,-} \to a_-$ for some $a_- \in \mathbb{R}$. As in the previous case, the odd part of the multiplier sequence is either zero or the odd part of the multiplier sequence of the invertible multiplier $M(\eta_0^-)(1) \cdot M_{a_-}$.

It is easily seen that $M_a + \varepsilon E$ for $a \in V(I)$ and a non-zero multiplier E tends to M_a as $\varepsilon \to 0$ but it can be chosen non-invertible according to the above description. ∎

Corollary 4.4. *Let $I \subset \mathbb{R} \setminus \{0\}$ be an open set with $V(I)$ connected. Then every algebra homomorphism $H : M(I) \to M(I)$ can be represented as*

$$C_\varphi : \operatorname{Exp}(\log V(I)) \to \operatorname{Exp}(\log V(I)), \quad C_\varphi(f) = f \circ \varphi, \ \ \varphi = H'|_{\mathbb{C}},$$

where $z \in \mathbb{C}$ is identified with a multiplicative functional τ_z, and φ belongs to $\operatorname{Exp}(\log V(I))$. *If I is nice then H is automatically continuous.*

Proof. Continuity follows from Corollary 4.2 and the closed graph theorem (by the First Representation Theorem, $M(I)$ is an LF-space). Moreover, $\varphi = \eta_1 \circ \varphi \in \operatorname{Exp}(\log V(I))$. ∎

The ideal structure of $\operatorname{Exp}(\{0\})$ is described in [19]. In particular, the following result holds:

Theorem 4.5 ([5, Ths. 4. and 5, Cor. 2]). *Let I be a nice open set with $V(I) = \{1\}$, $0 \notin I$. Every non-trivial closed ideal in $M(I)$ is of the form*

$$I(\alpha) := \{M \in M(I) : |x|^{\alpha_n} \in \ker M\}$$

for some $\alpha = (\alpha_n)$ either finite or with $|\alpha_n|/n \to \infty$ as $n \to \infty$. Moreover, every closed ideal in $M(I)$ has either one or two generators (*both cases can happen*).

Proof. Let us note that by the Second Representation Theorem 2.5, every $M \in M(I)$ corresponds via the isomorphism to $f \in H_0^*$ as defined in [5]. Then by the Third Representation Theorem 2.7, $\mathscr{M}(M) = \hat{f}$ as defined in [5]. Vanishing of $\mathscr{M}(M)(\alpha_n)$ means exactly that $|\cdot|^{\alpha_n}$ belongs to the kernel of M. Via [5, Th. 4] this completes the proof of the description of closed ideals. The rest follows from [5, Th. 5]. ∎

Problem 4.6. *Describe the closed ideals in $M(I)$ for $0 \notin I$, $V(I) \neq \{1\}$.*

Problem 4.7. *Describe all multiplicative functionals on $M(I)$ for arbitrary $I \subset \mathbb{R}$, $0 \notin I$.*

5. Case of sets containing zero. By the representation of $M(I)$ as $H(\hat{\mathbb{C}} \setminus (1/V(I)))$ (see Theorem 2.5), we observe that there is a finite non-zero multiplier sequence on $\mathscr{A}(I)$ if and only if $0 \in I$. In that case all finite non-zero sequences are multiplier sequences. So a special role is played by multipliers e_n, $n \in \mathbb{N}$, with multiplier sequence of zeros except 1 at the nth place. It is easily seen that $e_n \cdot e_n = e_n$ and $M \cdot e_n \in \operatorname{lin} e_n$ for all $n \in \mathbb{N}$ and any $M \in M(I)$. Moreover, if $0 \in I$ then $\operatorname{lin}\{e_n : n \in \mathbb{N}\}$ is dense in $M(I)$ whenever I is nice and $V(I)$ is connected. By [10, Prop. 3.1], this holds if I is an interval. Summarizing, the algebra $M(I)$, when I is an open interval containing zero, satisfies the assumptions of [23, Section 3]. Hence we obtain:

LEMMA 5.1 ([23, Lemma 3.2]). *Let I be an open interval containing zero. If J is an ideal in $M(I)$ then for each n, either $J \subseteq \ker \tau_n$ or $e_n \in J$.*

Recall that $\tau_n(M) = m_n$ whenever $(m_n)_{n\in\mathbb{N}}$ is the multiplier sequence of M.

THEOREM 5.2 ([23, Theorem 3.3, Corollary 3.4, Theorem 3.5]). *Let I be an open interval containing zero. Let J be an ideal in $M(I)$.*

(1) *The following assertions are equivalent:*
   - *J is a prime ideal contained in a closed ideal;*
   - *J is a closed prime ideal;*
   - *J is a closed maximal ideal;*
   - *$J = \ker \tau_n$ for some $n \in \mathbb{N}$.*

(2) *J is not dense if and only if $J \subset \ker \tau_n$ for some $n \in \mathbb{N}$.*
(3) *J is closed if and only if $J = J_B = \bigcap_{n\in B} \ker \tau_n$, where $B := \{n \in \mathbb{N} : \tau_n(M) = 0$ for all $M \in J\}$.*

In the case $0 \in I$, I an open interval, it turns out that all multiplicative functionals are described in Proposition 3.2. Since Theorem 2.5 gives a representation of $M(I)$ as the algebra $H(\hat{\mathbb{C}} \setminus (1/V(I)))$ of holomorphic functions with Hadamard multiplication, we can use the description of multiplicative functionals due to Render and Sauer [23, Th. 3.10]. In fact, if $0 \in I$, I an open interval, then $V(I)$ is $[0, 1]$, $[-s, 1]$ for some $s \in (0, 1]$, or $\mathbb{R}$ (see [10, Prop. 3.2]). So the first two cases are directly covered by the result of Render and Sauer, while the case $V(I) = \mathbb{R}$ is obtained with verbatim the same proof:

COROLLARY 5.3. *Let I be an open interval containing zero. Every multiplicative functional on $M(I)$ is of the form τ_n, $n \in \mathbb{N}$, so it is automatically continuous.*

This means that $M(I)$ can be represented as a function algebra with pointwise multiplication only taking functions on the set of natural numbers.

Corollary 5.4. *Let I be an open interval containing zero. The algebra $M(I)$ is never a Q-algebra.*

Proof. Since elements of the linear span of e_n, $n \in \mathbb{N}$, are not invertible, the set of non-invertible elements is dense (e_n corresponds to monomials in $H(\hat{\mathbb{C}} \setminus (1/V(I)))$). ∎

The *weak-algebra topology* on an algebra is the weak*-topology with respect to all non-zero multiplicative functionals. It is easily seen that every algebra homomorphism is weak-algebra continuous. As a consequence we have:

Proposition 5.5. *For every open interval I containing zero, every algebra homomorphism on $M(I)$ is continuous.*

Proof. Just observe that, since multiplicative functionals are all continuous, the weak-algebra topology is weaker than the original topology of $M(I)$. Then apply the closed graph theorem on the LF-space $M(I)$. ∎

The following result is a generalization of [24, Th. 1.1], [25, Prop. 3.1].

Theorem 5.6. *Let I be an open interval containing zero. For every algebra homomorphism $\Phi : M(I) \to M(I)$, there exists a set-valued map $\kappa_\Phi = \kappa : \mathbb{N} \to 2^{\mathbb{N}}$ with pairwise disjoint values where the characteristic functions of values are multiplier sequences of idempotents in $M(I)$ such that*

$$\Phi\Big(\sum_{n=0}^{\infty} m_n e_n\Big) = \sum_{n=0}^{\infty} m_n \sum_{j\in\kappa(n)} e_j,$$

the series being weak-algebra convergent.

Proof. Since e_n are idempotents, their Φ-images have to be idempotents as well. Thus there is a set-valued map $\kappa : \mathbb{N} \to 2^{\mathbb{N}}$ such that

$$\Phi(e_n) = \sum_{j\in\kappa(n)} e_j.$$

Since $e_n \cdot e_m = 0$ for $n \neq m$ we get

$$\Phi(e_n) \cdot \Phi(e_m) = \Phi(0) = 0,$$

hence $\kappa(n) \cap \kappa(m) = \emptyset$ for $n \neq m$. Since Φ is automatically weak-algebra continuous, the conclusion follows. ∎

For every algebra homomorphism $\Phi : M(I) \to M(I)$, I an open interval containing zero, we define $F_\Phi := \{n : \kappa_\Phi(n) \neq \emptyset \text{ finite}\}$.

Every algebra homomorphism Φ maps idempotents into idempotents. Let us note that by Theorem 3.1 the set $\kappa(n)$ has to be of a very special form. So clearly, only one or at most two (only in the symmetric case) sets $\kappa(n)$ can be infinite. Moreover, if Φ is infinite-dimensional then F_Φ must be infinite and no $\kappa(n)$ can be infinite (in case I is non-symmetric) or at most one $\kappa(n)$ can be infinite (in case I is symmetric).

The bijective case of the following theorem for compact $V(I)$ is due to Render and Sauer (see [24, Th. 1.3, Th. 4.3], [25, Th. 3.4]). The non-compact case (i.e., $I = \mathbb{R}$) requires a new idea. The generalization to non-surjective Φ is interesting since it shows that multiplier sequences of $\mathscr{A}(I)$ are permutation sensitive even if the permutation is not onto.

THEOREM 5.7. *Let I be an open interval containing zero. Let $\Phi : M(I) \to M(I)$ be an algebra homomorphism.*

(a) *If Φ is infinite-dimensional then the set F_Φ is infinite, and there are two polynomials h_+, h_- and $n_0 \in \mathbb{N}$ such that for $n > n_0$,*

$$\kappa(n) = (h_+^{-1}(n) \cap 2\mathbb{N}) \cup (h_-^{-1}(n) \cap (2\mathbb{N}+1)),$$

where $h_+ = h_-$ in case I is non-symmetric. Moreover,

$$\lim_{n\to\infty,\, n\in F_\Phi} \frac{\min \kappa(n)}{\log n} = \infty, \qquad \sup_{n\in F_\Phi} \frac{\max \kappa(n)}{n} < \infty.$$

(b) *If Φ is injective then $F_\Phi = \mathbb{N}$ and there are $p, r \in \mathbb{Z}$ with $p + r$ even and $n_0 \in \mathbb{N}$ such that for every $n > n_0$,*

$$\kappa(n) = \begin{cases} \{n+p\} & \text{for } n \text{ even,} \\ \{n+r\} & \text{for } n \text{ odd.} \end{cases}$$

If I is non-symmetric then $p = r$.

(c) *If Φ is bijective then there are $p \in \mathbb{Z}$ and $n_0 \in \mathbb{N}$ such that for every $n > n_0$,*

$$\kappa(n) = \begin{cases} \{n+p\} & \text{for } n \text{ even,} \\ \{n-p\} & \text{for } n \text{ odd.} \end{cases}$$

If I is non-symmetric then $p = 0$.

REMARK. (a) The above result shows that the set of multiplier sequences for $M(I)$ is very permutation sensitive for open intervals I containing zero, much more than one can suspect. Assume that the sequence $(m'_n)_{n\in\mathbb{N}}$ is obtained from a multiplier sequence $(m_n)_{n\in\mathbb{N}}$ via rearrangement and adding some zeros, i.e, there is an injective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $m_n = m'_{\pi(n)}$ and $m_j = 0$ for $j \notin \pi(\mathbb{N})$. Then all $(m'_n)_{n\in\mathbb{N}}$ are multiplier sequences if and only if π is as in part (b) of the theorem above, i.e., for $n > n_0$ we have $\pi(n) = n + p$ for n even and $\pi(n) = n + r$ for n odd, with $p + r$ even (for non-symmetric I we necessarily have $p = r$).

(b) It follows that for every injective algebra homomorphism Φ its image $\Phi(M(I))$ has a finite codimension.

(c) One can easily construct an injective Φ satisfying the condition in (b) above for any given n_0, p and r as in (b). Similarly, one can easily construct a bijective Φ for any n_0 and p as in (c).

Proof of Theorem 5.7. Let us identify $M(I)$ with $H(\hat{\mathbb{C}}\setminus(1/V(I)))$; here e_n identifies with the monomial z^n, and the unit identifies with $\gamma(z)=1/(1-z)$ according to Theorem 2.5. Then Φ acts as an algebra homomorphism on that space with Hadamard multiplication.

(a) The fact that F_Φ is infinite has been proved just before this theorem.

We will use the functions $\gamma_k := \theta^k(\gamma)$, $\theta(f)(z) = z\frac{d}{dz}f(z)$, identified with $\sum_{n=0}^\infty n^k e_n$. If $I \neq \mathbb{R}$ then the unit disc is contained in $\hat{\mathbb{C}}\setminus(1/V(I))$, thus the Taylor series of $\Phi(\gamma_1)$ at zero, which is equal to $\sum_{n=0}^\infty n\sum_{j\in\kappa(n)} z^j$, has radius of convergence ≥ 1, i.e.,

$$\lim_{n\to\infty,\, n\in F_\Phi} \frac{\min\kappa(n)}{\log n} = \infty.$$

Since $\Phi : M(\mathbb{R}) \to M(\mathbb{R})$ acts on an LF-space, by the Grothendieck factorization theorem, it maps every step into some step space, i.e., there is $a>0$ such that if f belongs to $H\big(\hat{\mathbb{C}}\setminus((-\infty,-1]\cup[1,\infty))\big)$ then $\Phi(f)$ is in $H\big(\hat{\mathbb{C}}\setminus((-\infty,-a]\cup[a,\infty))\big)$. Let us observe that all the functions γ_k identified with $\sum_{n=0}^\infty n^k e_n$ belong to $H\big(\hat{\mathbb{C}}\setminus((-\infty,-1]\cup[1,\infty))\big)$ so the Taylor series at zero of every function $\Phi(\gamma_k)$ has to have convergence radius at least a for some fixed a not depending on k. The Taylor series of $\Phi(\gamma_k)$ at zero is equal to

$$\sum_{n\in\mathbb{N}} n^k \sum_{j\in\kappa(n)} z^j.$$

If infinitely many of $n\in F_\Phi$ satisfy $\min\kappa(n) < A\log n$, $A>0$ (or equivalently, the radius of convergence of the above series for $k=1$ is strictly less than 1), then for infinitely many $n\in F_\Phi$ we have

$$\sqrt[\min\kappa(n)]{n} > \exp(1/A),$$

so the convergence radius of the Taylor series of $\Phi(\gamma_k)$ at zero is smaller than $\exp(-k/A)$; a contradiction.

We have proved that for any I with $0\in I$,

$$\lim_{n\to\infty,\, n\in F_\Phi} \frac{\min\kappa(n)}{\log n} = \infty$$

and moreover that $\Phi(\gamma_1)\in H\big(\hat{\mathbb{C}}\setminus((-\infty,-1]\cup[1,\infty))\big)$. Further, the Taylor series of $\Phi(\gamma_1)$ at zero has only natural coefficients. By the Pólya–Carlson Theorem [20, Ch. 11.4, p. 265], every such function is of the form

$$\Phi(\gamma_1) = p_1(z) + \frac{q(z)}{(1-z^l)^m}$$

with $\deg q(z) < lm$. In the symmetric case $\hat{\mathbb{C}}\setminus(1/V(I))$ does not contain roots of unity of order larger than 2 and in the non-symmetric case larger than 1, so $l=2$ or $l=1$, respectively. In case $l=2$ the second summand is of the form

$$\frac{q_1(z)}{(1-z)^m} + \frac{q_2(z)}{(1+z)^m}.$$

Moreover, by [3, Satz 1.3.X],

$$\frac{q_1(z)}{(1-z)^m} = \sum_{n=0}^{\infty} w_1(n)z^n, \quad \frac{q_2(z)}{(1+z)^m} = \sum_{n=0}^{\infty} w_2(n)(-z)^n$$

for some polynomials w_1, w_2. Summarizing,

$$\Phi(\gamma_1) = p_1(z) + \sum_{n=0}^{\infty} h_+(2n)z^{2n} + \sum_{n=0}^{\infty} h_-(2n+1)z^{2n+1}$$

for some polynomials h_+, h_- which are equal in the non-symmetric case. On the other hand, as we have seen before,

$$\Phi(\gamma_1) = \sum_{n\in\mathbb{N}} n \sum_{j\in\kappa(n)} z^j.$$

This shows that

$$\kappa(n) = (h_+^{-1}(n) \cap 2\mathbb{N}) \cup (h_-^{-1}(n) \cap (2\mathbb{N}+1))$$

for n large enough, which implies immediately that

$$\sup_{n\in F_\Phi} \frac{\max \kappa(n)}{n} < \infty.$$

(b) If Φ is injective then no $\kappa(n)$ can be empty and Φ has to be infinite-dimensional. If some $\kappa(n)$ is infinite then from the statements above it follows that I has to be symmetric and $\kappa(n)$ has to contain all but finitely many even numbers or odd numbers. Assume for simplicity that n is even. Then $S = \bigcup_{j\in\mathbb{N}} \kappa(2j)$ is an element of the idempotent algebra containing $\kappa(n)$ such that $S \setminus \kappa(n)$ is infinite. Therefore $\mathbb{N} \setminus S$ is finite; a contradiction, since infinitely many sets $\kappa(2j+1)$ have to be empty. We have proved that no $\kappa(n)$ is infinite.

Therefore, $h_+(2\mathbb{N}) \cup h_-(2\mathbb{N}+1)$ contains all sufficiently large positive integers. In the case of I non-symmetric, $h_+ = h_- = h$ and the latter must be a polynomial of order one and leading coefficient 1, i.e., $h(n) = n + p$ for some $p \in \mathbb{Z}$. In the case of I symmetric, $\bigcup_{n\in\mathbb{N}} \kappa(2n)$ and $\bigcup_{n\in\mathbb{N}} \kappa(2n+1)$ are disjoint infinite idempotent sets, thus $h_+(2\mathbb{N})$ contains all sufficiently large even integers or all sufficiently large odd integers. Let us consider the first case. Then $h_+(n) = n + p$ for some $p \in \mathbb{Z}$ even but $h_-(2\mathbb{N}+1)$ must contain all sufficiently large odd integers so $h_-(n) = n + r$ with r even. The other case is similar but then p and r are odd.

(c) In that case $\kappa(n)$ is always a singleton and κ can be treated as a bijective permutation of $\mathbb{N}$. In (b) we may assume without loss of generality that $n_0 = 2k_0 - 1$ thus $\bigcup_{j\geq k_0} \kappa(2j) \cup \kappa(2j+1) = \{2k_0 + p, 2k_0 + 2 + p, \dots\} \cup$

$\{2k_0+1+r, 2k_0+3+r, \dots\} =: S$. Since there must be a bijective map from $\{0,1,2,\dots,2k_0-1\}$ onto $\mathbb{N}\setminus S$ it follows that $r=-p$. ∎

PROBLEM 5.8. *Describe all non-injective infinite-dimensional algebra homomorphisms on $M(I)$ for open intervals I containing zero.*

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