STUDIA MATHEMATICA 212 (2) (2012)

# Generators of maximal left ideals in Banach algebras

by

H. G. Dales (Lancaster) and W. Żelazko (Warszawa)

**Abstract.** In 1971, Grauert and Remmert proved that a commutative, complex, Noetherian Banach algebra is necessarily finite-dimensional. More precisely, they proved that a commutative, complex Banach algebra has finite dimension over $\mathbb{C}$ whenever all the closed ideals in the algebra are (algebraically) finitely generated. In 1974, Sinclair and Tullo obtained a non-commutative version of this result. In 1978, Ferreira and Tomassini improved the result of Grauert and Remmert by showing that the statement is also true if one replaces 'closed ideals' by 'maximal ideals in the Shilov boundary of $A$'. We give a shorter proof of this latter result, together with some extensions and related examples.

We study the following conjecture. Suppose that all maximal left ideals in a unital Banach algebra $A$ are finitely generated. Then $A$ is finite-dimensional.

## 1. Introduction

**1.1. Notation.** We first recall some standard notation that we shall use in this paper.

The natural numbers and the integers are $\mathbb{N}$ and $\mathbb{Z}$, respectively. For $n \in \mathbb{N}$, we set

$$\mathbb{N}_n = \{1, \dots, n\}.$$

The unit circle and open unit disc in the complex field $\mathbb{C}$ are $\mathbb{T}$ and $\mathbb{D}$, respectively; the real line is $\mathbb{R}$, and $\mathbb{R}^+ = \{s \in \mathbb{R} : s \geq 0\}$. The algebra of all $n \times n$ matrices over $\mathbb{C}$ is denoted by $\mathbb{M}_n$; the matrix units in the matrix algebra $\mathbb{M}_n$ are denoted by $E_{i,j}$ for $i, j \in \mathbb{N}_n$; the identity matrix is denoted by $\iota_n$.

We write $c_0$ and $\ell^p$ (for $p \in [1, \infty]$) for the standard sequence spaces on $\mathbb{N}$; we write $c = c_0 \oplus \mathbb{C}1$ for the Banach space of all convergent sequences, where $1 = (1, 1, 1, \dots)$.

Let $A$ be an (associative) algebra, always taken to be over the complex field. In the case where $A$ does not have an identity, the algebra formed by

2010 *Mathematics Subject Classification*: Primary 46H10; Secondary 46J10.
*Key words and phrases*: maximal left ideal, finitely-generated, Banach algebra, Shilov boundary, Choquet boundary, strong boundary point, Jacobson radical, quasi-analytic algebra.

DOI: 10.4064/sm212-2-5

adjoining an identity to $A$ is denoted by $A^\sharp$; we take $A^\sharp$ to be $A$ in the case where $A$ already has an identity.

A linear subspace $I$ of $A$ is a *left ideal* if $AI \subset I$, a *right ideal* if $IA \subset I$, and an *ideal* if $AI + IA \subset I$. The *Jacobson radical* of an algebra $A$ is denoted by $J(A)$; it is defined to be the intersection of the maximal modular left (or right) ideals of $A$, and it is proved that it is an ideal in $A$ (see [5, §1.5]).

Let $S$ be a subset of $A$. Then the *left ideal generated by* $S$ is the intersection of the left ideals of $A$ that contain $S$; this left ideal is denoted by $\langle S\rangle$. Clearly,

$$\langle S\rangle = \Big\{\sum_{i=1}^{n} a_i s_i : a_1,\ldots,a_n \in A^\sharp,\ s_1,\ldots,s_n \in S,\ n\in\mathbb{N}\Big\}.$$

The left ideal generated by a finite subset $\{a_1,\ldots,a_n\}$ is equal to

$$I = A^\sharp a_1 + \cdots + A^\sharp a_n, \tag{1.1}$$

and it is denoted by $\langle a_1,\ldots,a_n\rangle$.

Let $I$ be a left ideal in the algebra $A$. Then: $I$ is *countably generated* if there is a countable set $S$ with $I = \langle S\rangle$; $I$ is *finitely generated* if there are elements $a_1,\ldots,a_n$ in $A$ with $I = \langle a_1,\ldots,a_n\rangle$, and in this case, $a_1,\ldots,a_n$ are *generators* of $I$; $I$ is *singly generated* (by $a$) if $I = \langle a\rangle$ for some $a\in A$.

An algebra $A$ is *left Noetherian* if the family of left ideals in $A$ satisfies the ascending chain condition; this is the case if and only if each left ideal in $A$ is finitely generated. See [11, Chapter VIII], for example.

**1.2. The families $\mathfrak{I}_\infty(A)$ and $\mathfrak{U}_\infty(A)$.** Now suppose that $A$ is an algebra. We denote by $\mathfrak{I}_\infty(A)$ the family of all left ideals in $A$ which are not finitely generated, and by $\mathfrak{U}_\infty(A)$ the family of all left ideals in $A$ which are not countably generated, so that $\mathfrak{U}_\infty(A) \subset \mathfrak{I}_\infty(A)$.

The result of Grauert and Remmert which was stated in the abstract can be formulated as follows. See the Appendix to §5 in [9].

THEOREM 1.1. *Let $A$ be a commutative Banach algebra. Suppose that every closed ideal in $A$ is finitely generated. Then $A$ is finite-dimensional.* ∎

This theorem was generalized by Sinclair and Tullo [13] to the non-commutative case; we state their result as follows.

THEOREM 1.2. *Let $A$ be a Banach algebra. Suppose that every closed left ideal in $A$ is finitely generated. Then $A$ is finite-dimensional.* ∎

In the proof of their result, Grauert and Remmert used the following fact [9, Bemerkung 2, p. 54]. Let $I$ be an ideal in a commutative Banach algebra $A$, and suppose that the closure $\overline{I}$ of $I$ is finitely generated. Then $I$ is already closed, so that $I = \overline{I}$. A non-commutative version of this result is proved by Sinclair and Tullo [13, Lemma 1], and this result is stated in [5, Proposition 2.6.37] as follows.

THEOREM 1.3. *Let $A$ be a Banach algebra, and let $I$ be a left ideal of $A$ such that $\bar{I}$ is finitely generated. Then $I$ is closed in $A$.* ∎

In fact, the result is stated for more general algebras than Banach algebras. Unfortunately, the proof of Proposition 2.6.37 in [5] is not quite correct. Let $A$ be a Banach algebra, and take $n \in \mathbb{N}$. We define $\mathfrak{A} = \mathbb{M}_n(A)$, the algebra of $n \times n$ matrices over $A$, so that $\mathfrak{A}$ is also a Banach algebra with respect to the norm $\|\cdot\|$, where

$$\|a\| = \max\{\|a_{i1}\| + \cdots + \|a_{in}\| : i = 1, \ldots, n\} \quad (a = (a_{ij}) \in \mathfrak{A});$$

in the case where the algebra $A$ is unital, with identity $e_A$, the element $\iota = (\iota_{ij})$, where $\iota_{ij} = \delta_{ij} e_A$ $(i, j = 1, \ldots, n)$, is the identity of $\mathfrak{A}$. To prove Theorem 1.3, we may suppose that $A$ is unital. The proof in [5] refers to the 'determinant' of elements in $\mathfrak{A}$; however, the determinant of such elements is only defined in the special case where $A$ is commutative. Nevertheless, the proof in [13, Lemma 1] is correct; we sketch the details.

Let $I$ be a left ideal in $A$ with $\bar{I} = \langle a_1, \ldots, a_n\rangle$, where $a_1, \ldots, a_n \in \bar{I}$. Then the open mapping theorem shows that there are $b_1, \ldots, b_n \in I$ and $x = (x_{ij}) \in \mathfrak{A}$ with $\|x\| < 1$ such that

$$a_i = b_i + x_{i1}a_1 + \cdots + x_{in}a_n \quad (i = 1, \ldots, n),$$

and so $(b_1, \ldots, b_n) = (\iota - x)(a_1, \ldots, a_n)$ in $A^n$. Since $\iota - x$ is invertible in $\mathfrak{A}$, it follows that $(a_1, \ldots, a_n) = (\iota - x)^{-1}(b_1, \ldots, b_n) \in I^n$, giving the result.

It follows that every left ideal in a Banach algebra $A$ is finitely generated whenever this is true for each closed left ideal in $A$.

For various generalizations of versions of Theorem 1.2 to certain topological algebras, see [3, 4, 8, 15, 16].

The following generalizations of Theorems 1.2 and 1.3 were given by Boudi in [2, Proposition 1 and Theorem 3].

THEOREM 1.4. *Let $A$ be a Banach algebra, and let $I$ be a left ideal of $A$ such that $\bar{I}$ is countably generated. Then $I$ is closed in $A$.* ∎

THEOREM 1.5. *Let $A$ be a Banach algebra. Suppose that every closed left ideal in $A$ is countably generated. Then $A$ is finite-dimensional.* ∎

COROLLARY 1.6. *Each closed, countably-generated left ideal in a Banach algebra is finitely generated.*

*Proof.* Let $I$ be a closed, countably-generated left ideal in a Banach algebra $A$, say $I = \langle S\rangle$, where $S = \{a_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, set $J_n = \langle a_1, \ldots, a_n\rangle$, so that $J_n \subset J_{n+1}$ $(n \in \mathbb{N})$ and $\bigcup\{J_n : n \in \mathbb{N}\} = I$. By Baire's category theorem, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{int} \bar{J}_{n_0} \neq \emptyset$. But then $\bar{J}_{n_0} = I$, and so, by Theorem 1.4, $J_{n_0}$ is closed in $A$. Thus $I = J_{n_0}$ is finitely generated. ∎

## 2. The general case

**2.1. The families $\mathfrak{M}_\infty(A)$ and $\mathfrak{N}_\infty(A)$.** In this section, we shall consider algebras which are not necessarily commutative.

Let $A$ be an algebra. Then the families $\mathfrak{I}_\infty(A)$ and $\mathfrak{U}_\infty(A)$, which were defined above, are each a partially ordered set with respect to inclusion.

THEOREM 2.1. *Let $A$ be an algebra. Then each member of the family $\mathfrak{I}_\infty(A)$ is contained in a maximal element of the family.*

*Proof.* Let $\mathcal{C}$ be a chain in the partially ordered set $(\mathfrak{I}_\infty(A), \subset)$, each member of which contains the specified member of the family, and define

$$I = \bigcup \{J : J \in \mathcal{C}\},$$

so that $I$ is a left ideal in $A$.

Assume towards a contradiction that $I$ is finitely generated as a left ideal, say $I = \langle a_1, \ldots, a_n \rangle$, where $a_1, \ldots, a_n \in I$. Since $\mathcal{C}$ is a chain, there exists $J \in \mathcal{C}$ with $a_1, \ldots, a_n \in J$, and hence $J = \langle a_1, \ldots, a_n \rangle$ is finitely generated, a contradiction. Thus $I \in \mathfrak{I}_\infty(A)$. Clearly, $I$ is an upper bound for $\mathcal{C}$.

It follows from Zorn's lemma that $\mathfrak{I}_\infty(A)$ contains a maximal element, and that this maximal element contains the specified member of the family. ∎

A similar argument to the above, with applications, appears in [1].

The following theorem applies only to Banach algebras.

THEOREM 2.2. *Let $A$ be a Banach algebra. Then each member of the family $\mathfrak{U}_\infty(A)$ is contained in a maximal element of the family.*

*Proof.* As in the above proof, consider a chain $\mathcal{C}$ in the partially ordered set $(\mathfrak{U}_\infty(A), \subset)$, and define $I = \bigcup\{J : J \in \mathcal{C}\}$. Then $I$ is a left ideal in $A$ and $\bar{I}$ is a closed left ideal in $A$.

Assume towards a contradiction that $\bar{I}$ is countably generated. By Corollary 1.6, $\bar{I}$ is finitely generated. By Theorem 1.3, $I$ is closed, and so $I$ is finitely generated. As in Theorem 2.1, this is a contradiction, and hence $\bar{I} \in \mathfrak{U}_\infty(A)$. Clearly, $\bar{I}$ is an upper bound for $\mathcal{C}$ in $\mathfrak{U}_\infty(A)$, and so Zorn's lemma again applies. ∎

The sets of maximal elements in $\mathfrak{I}_\infty(A)$ and $\mathfrak{U}_\infty(A)$ are denoted by $\mathfrak{M}_\infty(A)$ (for an algebra $A$) and $\mathfrak{N}_\infty(A)$ (for a Banach algebra $A$), respectively.

COROLLARY 2.3. *Let $A$ be an infinite-dimensional Banach algebra. Then the families $\mathfrak{M}_\infty(A)$ and $\mathfrak{N}_\infty(A)$ are non-empty and equal, and each member of this family is closed.*

*Proof.* Clearly, both families are non-empty and each member of either of these families is closed because, by Theorem 1.4, their closures belong to the respective families.

Take $M \in \mathfrak{M}_\infty(A)$. By Corollary 1.6, we have $M \in \mathfrak{I}_\infty(A)$, and so $M \in \mathfrak{N}_\infty(A)$.

Take $N \in \mathfrak{N}_\infty(A)$. Then $N \in \mathfrak{I}_\infty(A)$, and so there exists $M \in \mathfrak{M}_\infty(A)$ with $N \subset M$. By Corollary 1.6, $M \in \mathfrak{U}_\infty(A)$, and hence $N = M \in \mathfrak{M}_\infty(A)$.

We have shown that $\mathfrak{M}_\infty(A) = \mathfrak{N}_\infty(A)$. ∎

We wish to study the following conjecture.

Conjecture. *Let $A$ be a unital Banach algebra. Suppose that all maximal left ideals are finitely* (*or even singly*) *generated in $A$. Then $A$ is finite-dimensional.*

We remark that this is a question about Banach algebras, not a purely algebraic question. For consider a large, infinite-dimensional field $F$ containing $\mathbb{C}$. Then $F$ has only one proper ideal, namely $\{0\}$, and this is finitely generated, but $F$ is not finite-dimensional over $\mathbb{C}$. However, by the Gel'fand–Mazur theorem [5, Theorem 2.2.42], such a field $F$ cannot be a Banach algebra.

The above conjecture is considered in [6] in the special case in which $A = \mathcal{B}(E)$, the Banach algebra of all bounded linear operators on a Banach space $E$, and it will be established there for 'many' Banach spaces $E$. The question is left open for the Banach algebra $\mathcal{B}(C(\mathbb{I}))$, where $C(\mathbb{I})$ is the Banach space of all continuous functions on the closed unit interval $\mathbb{I}$.

We make the following remark about a special case of the conjecture. Let $A$ be a unital $C^*$-algebra. Then it can be shown rather easily that each finitely-generated left ideal in $A$ is singly generated by a self-adjoint projection, and then that $A$ is finite-dimensional whenever each maximal left ideal is finitely generated, and so our conjecture holds for the class of $C^*$-algebras.

Let $A$ be a unital Banach algebra, and take $M \in \mathfrak{M}_\infty(A)$. One might suspect that $M$ is necessarily a maximal left ideal in $A$; if this were true, then our conjecture would be immediately positively resolved. However this is not true. Trivially, it is true in the special case where $\operatorname{codim} M = 1$; the following example shows that it need not be true in the case where $\operatorname{codim} M = 2$.

Example 2.4. We begin with the unital, three-dimensional algebra $B$ which consists of the upper-triangular matrices in $\mathbb{M}_2$. Thus we identify

$$B = \mathbb{C}p \oplus \mathbb{C}q \oplus \mathbb{C}r,$$

where

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.1}$$

Hence the product in $B$ is specified by

$$p^2 = p, \quad q^2 = q, \quad pq = qp = 0, \quad r^2 = 0, \quad pr = rq = r, \quad rp = qr = 0.$$

The identity of $B$ is $e = p + q$; the radical of $B$ is $J(B) = \mathbb{C}r$.

We define $M = \mathbb{C}p$. Then $M$ is a left ideal in $B$ of codimension 2.

We further define $I = \mathbb{C}p \oplus \mathbb{C}r$ and $J = \mathbb{C}q \oplus \mathbb{C}r$. Then both $I$ and $J$ are left ideals in $B$ of codimension 1, both are maximal left ideals, and they are the only two maximal left ideals in $B$. Clearly, $M \subset I$, but $M \not\subset J$; further, $I \cap J = \mathbb{C}r = J(B)$. Since $M \subsetneq I$, the left ideal $M$ is not a maximal left ideal.

We define

$$\|\alpha p + \beta q + \gamma r\| = \max\{|\alpha|, |\beta|, |\gamma|\} \quad (\alpha, \beta, \gamma \in \mathbb{C}).$$

Then $(B, \|\cdot\|)$ is a Banach algebra with $\|e\| = 1$.

We now take $(E, \|\cdot\|)$ to be an infinite-dimensional Banach space, and set $K = E \oplus E \oplus E \oplus E$; a generic element of $K$ is regarded as a $2 \times 2$ matrix

$$\mathbf{x} = (x_{i,j}) = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},$$

where $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \in E$, and so $K = \mathbb{M}_2(E)$ as a linear space. The norm on $K$ is given by

$$\|\mathbf{x}\| = \|x_{1,1}\| + \|x_{1,2}\| + \|x_{2,1}\| + \|x_{2,2}\| \quad (\mathbf{x} \in K),$$

so that $(K, \|\cdot\|)$ is a Banach space.

The left and right actions of $B$ on $K$ are given by 'matrix multiplication on the left and right', respectively; these actions are denoted by $\cdot$. Clearly, these are associative actions, and $(K, \cdot, \|\cdot\|)$ is a unital Banach $B$-bimodule.

We now define the linear space $A = B \oplus K$, with the norm given by

$$\|(b, \mathbf{x})\| = \|b\| + \|\mathbf{x}\| \quad (b \in B,\ \mathbf{x} \in K)$$

and the product given by

$$(b, \mathbf{x})(c, \mathbf{y}) = (bc, b \cdot \mathbf{y} + \mathbf{x} \cdot c) \quad (b, c \in B,\ \mathbf{x}, \mathbf{y} \in K).$$

Then $A$ is a unital Banach algebra, with identity $(e, 0)$. We regard $B$ and $K$ as subspaces of $A$; clearly, $K$ is an ideal in $A$.

The space $K$ satisfies $K^2 = \{0\}$, and so $K \subset J(A)$. Thus

$$J(A) = \mathbb{C}r \oplus K.$$

There are just two maximal left ideals in $A$; they are $I + K$ and $J + K$, and $(I + K) \cap (J + K) = J(A)$. The closed left ideal $M + K$ has codimension 2 in $A$; the only maximal left ideal that contains $M$ is $I + K$.

We *claim* that $I + K$ is a finitely-generated left ideal of $A$; indeed, we claim that $I + K = Ap + Ar$. Since $p, r \in I$, we have $Ap + Ar \subset I + K$.

Certainly, $p, r \in Ap + Ar$, and so $I \subset Ap + Ar$. Now take $\mathbf{x} \in K$. Then

$$\mathbf{x} = \begin{pmatrix} x_{1,1} & 0 \\ x_{2,1} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_{1,2} & 0 \\ x_{2,2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Ap + Ar,$$

and so $K \subset Ap + Ar$, giving the claim.

We also *claim* that $M + K$ is not finitely generated. Indeed, assume towards a contradiction that

$$K \subset Ap + \sum_{k=1}^{n} A\mathbf{x}^{(k)},$$

where $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in K$. Since $K^2 = \{0\}$ and $Bp \cap K = \{0\}$, in fact

$$K \subset Kp + \sum_{k=1}^{n} B\mathbf{x}^{(k)}.$$

In particular, for each $x \in E$, there exist $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ for $k = 1, \dots, n$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \in Kp + \sum_{k=1}^{n} \begin{pmatrix} \alpha_k & \gamma_k \\ 0 & \beta_k \end{pmatrix}\mathbf{x}^{(k)},$$

and so

$$x = \sum_{k=1}^{n} \beta_k x_{2,2}^{(k)}.$$

This shows that $E$ is spanned by $\{x_{2,2}^{(1)}, \dots, x_{2,2}^{(n)}\}$, a contradiction of the fact that $E$ is infinite-dimensional. This gives the claim.

Thus $M + K \in \mathfrak{M}_\infty(A)$, but $M + K$ is not a maximal left ideal of the algebra $A$.

We also note that the maximal left ideal $J + K$ of $A$ is not finitely generated, and so our example is not a counter-example to our conjecture. Indeed, assume towards a contradiction that

$$K \subset Aq + Ar + \sum_{k=1}^{n} A\mathbf{x}^{(k)},$$

where $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in K$. By considering the element

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

of $K$, we see that $E$ is spanned by $\{x_{1,1}^{(1)}, \dots, x_{1,1}^{(n)}, x_{1,2}^{(1)}, \dots, x_{1,2}^{(n)}\}$, again a contradiction of the fact that $E$ is infinite-dimensional. ∎

Let $A$ be a unital Banach algebra. For a positive resolution of the above-stated conjecture, we must prove that, whenever there exists an element

$M \in \mathfrak{M}_\infty(A)$, the algebra $A$ also contains a maximal left ideal that is not finitely generated (but this ideal need not contain $M$). There is an algebraic argument that does establish this in the special case where $M$ has finite codimension in $A$.

LEMMA 2.5. *Let $I$ be a left ideal in $\mathbb{M}_n$, where $n \in \mathbb{N}$. Then:*

(i) $\dim I = kn$ *for some* $k \in \{0, \dots, n\}$*;*
(ii) *$I$ is a maximal left ideal if and only if* $\operatorname{codim} I = n$ *in* $\mathbb{M}_n$*;*
(iii) *in the case where $I$ is proper, there exists $x \in \mathbb{C}^n$ with $x \neq 0$ such that $ax = 0$ $(a \in I)$.*

*Proof.* By [12, Exercise 3, p. 173], there is a bijective map

$$F \mapsto \{a \in \mathbb{M}_n : ax = 0 \ (x \in F)\}$$

from the family of linear subspaces of $\mathbb{C}^n$ onto the family of left ideals of $\mathbb{M}_n$. The result follows easily from this. ∎

LEMMA 2.6. *Let $A$ be a unital algebra, and let $L$ be an ideal of $A$ such that $A/L = \mathbb{M}_n$ for some $n \in \mathbb{N}$. Suppose that each maximal left ideal in $A$ that contains $L$ is finitely generated in $A$. Then $L$ is finitely generated as a left ideal in $A$.*

*Proof.* We identify $A$ with $\mathbb{M}_n \oplus L$ as a linear space, and denote the product of matrices in $\mathbb{M}_n$ by $\cdot$.

Take $j \in \mathbb{N}_n$. Define $P_j = \iota_n - E_{j,j}$, where we recall that $(E_{i,j})$ is the set of matrix units of $\mathbb{M}_n$, and set $M_j = \langle P_j \rangle \subset \mathbb{M}_n$, so that $M_j$ is the space of matrices with zeros in the $j$th column. By Lemma 2.5(ii), $M_j$ is a maximal left ideal in $\mathbb{M}_n$, and so $M_j + L$ is a maximal left ideal in $A$. By hypothesis, $M_j + L$ is finitely generated in $A$, and so there is a finite subset $S_j$ in $L$ such that $L = AP_j + \langle S_j \rangle$.

Set $S = S_1 \cup \dots \cup S_n$, a finite subset of $L$. Then

$$L \subset AP_j + \langle S \rangle \quad (j \in \mathbb{N}_n).$$

By enlarging $S$ if necessary, we may suppose that $S$ contains the difference between the products in $A$ and $\mathbb{M}_n$ of any two matrix units $E_{i,j}$ in $\mathbb{M}_n$, and so we have

$$L \subset \mathbb{M}_n \cdot P_j + LP_j + \langle S \rangle \quad (j \in \mathbb{N}_n).$$

Since $LP_j + \langle S \rangle \subset L$ (where we note that $L$ is a right ideal) and since $(\mathbb{M}_n \cdot P_j) \cap L = \{0\}$, it follows that

$$L = LP_j + \langle S \rangle \quad (j \in \mathbb{N}_n).$$

Set $T = S \cup SP_1 \cup \dots \cup SP_n$, a finite subset of $L$. Then we now have

$$\begin{aligned} L &= LP_1 + \langle S \rangle = (LP_2 + \langle S \rangle)P_1 + \langle S \rangle = LP_2P_1 + \langle S \cup SP_1 \rangle \\ &= \dots = LP_n \cdots P_1 + \langle T \rangle = \langle T \rangle \end{aligned}$$

because the product of $P_n, \ldots, P_1$ in $\mathbb{M}_n$ is zero, and so the product in $A$ is an element of $T$.

This proves the result. ∎

Lemma 2.7. *Let $A$ be a unital algebra, and take ideals $I$, $J$, and $L$ in $A$ such that $L = IJ + JI$. Suppose that $I$ and $J$ are finitely-generated left ideals in $A$. Then $L$ is a finitely-generated left ideal in $A$.*

*Proof.* Suppose that $I = Ax_1 + \cdots + Ax_m$ and $J = Ay_1 + \cdots + Ay_n$, where $x_1, \ldots, x_m \in I$ and $y_1, \ldots, y_n \in J$. Then

$$x_i y_j \in IJ \subset L \quad (i \in \mathbb{N}_m,\ j \in \mathbb{N}_n).$$

Take $x \in I$ and $y \in J$. Then $y = \sum_{j=1}^{n} a_j y_j$ for some $a_1, \ldots, a_n \in A$. For each $j \in \mathbb{N}_n$, we have $xa_j \in I$, and so

$$xa_j = \sum_{i=1}^{m} b_{i,j} x_i$$

for some $b_{1,j}, \ldots, b_{m,j} \in A$. Hence

$$xy = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{i,j} x_i y_j \in \langle S_1 \rangle,$$

where $S_1$ is the finite set $\{x_i y_j : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n\}$. It follows that $IJ \subset \langle S_1 \rangle$. Similarly, $JI \subset \langle S_2 \rangle$, where $S_2$ is the finite set $\{y_j x_i : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n\}$.

It follows that $L$ is generated as a left ideal by the finite set $S_1 \cup S_2$. ∎

Lemma 2.8. *Let $A$ be a unital algebra, and let $L$ be an ideal in $A$ of finite codimension such that $A/L$ is semisimple. Suppose that each maximal left ideal in $A$ that contains $L$ is finitely generated in $A$. Then $L$ is finitely generated as a left ideal in $A$.*

*Proof.* By Wedderburn's theorem [5, Theorem 1.5.9], there exist $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that $A/L = \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_k}$. For $j \in \mathbb{N}_k$, set

$$I_j = \bigoplus \{\mathbb{M}_{n_i} : i \in \mathbb{N}_k,\ i \neq j\} + L,$$

so that $I_j$ is an ideal in $A$ with $A/I_j = \mathbb{M}_{n_j}$. By Lemma 2.6, each $I_j$ is a finitely-generated left ideal in $A$.

Take $j \in \mathbb{N}_k$. The identity matrix in $\mathbb{M}_{n_j}$ is now denoted by $p_j$, and so the identity of $A$ has the form $p_1 + \cdots + p_k + x_0$ for some $x_0 \in L$.

Take $j_1, j_2 \in \mathbb{N}_k$ with $j_1 \neq j_2$. Clearly $I_{j_1} I_{j_2} + I_{j_2} I_{j_1} \subset I_{j_1} \cap I_{j_2}$. Now take $x \in I_{j_1} \cap I_{j_2}$. We have

$$x = (p_1 + \cdots + p_k + x_0)x \in I_{j_1} I_{j_2} + I_{j_2} I_{j_1}$$

because each $p_i$ for $i \in \mathbb{N}_n$ belongs either to $I_{j_1}$ or to $I_{j_2}$ (or to both) and $x_0 \in I_{j_1} \cap I_{j_2}$. Thus $I_{j_1} \cap I_{j_2} \subset I_{j_1} I_{j_2} + I_{j_2} I_{j_1}$. We have shown that $I_{j_1} I_{j_2} + I_{j_2} I_{j_1} = I_{j_1} \cap I_{j_2}$. By Lemma 2.7, $I_{j_1} \cap I_{j_2}$ is a finitely-generated left ideal in $A$.

By repeating this argument finitely many times, we see that the ideal $I_1 \cap \cdots \cap I_n$ is a finitely generated left ideal in $A$. However, $I_1 \cap \cdots \cap I_n = L$, and so $L$ is finitely generated as a left ideal in $A$. ∎

LEMMA 2.9. *Let $A$ be a unital algebra, and let $K$ be an ideal in $A$ of finite codimension. Suppose that each maximal left ideal in $A$ that contains $K$ is finitely generated in $A$. Then $K$ is finitely generated as a left ideal in $A$.*

*Proof.* By Wedderburn's principal theorem [5, Corollary 1.5.19], we have $A/K = B \oplus R$ for a subalgebra $B$ which is a direct sum of full matrix algebras and an ideal $R$ which is the radical of $A/K$. We identify $A$ with $B \oplus R \oplus K$ as a linear space. The ideal $R$ is nilpotent in $A/K$, say $R^n = \{0\}$ in $A/K$, and so $R^n \subset K$ in $A$.

Set $L = R + K$. Then $L$ is an ideal of $A$ such that $A/L$ is semisimple, and so, by Lemma 2.8, $L$ is finitely generated as a left ideal in $A$. Thus there exist $r_1, \dots, r_m \in R$ and a finite subset $S$ of $K$ such that

$$K \subset Ar_1 + \cdots + Ar_m + \langle S \rangle.$$

By enlarging $S$ if necessary, we may suppose $Br_1, \dots, Br_m \subset S$ and that $S$ contains any product $r_{i_1} \cdots r_{i_k}$ for any $i_1, \dots, i_k \in \mathbb{N}_m$ and $k \in \mathbb{N}_n$; the enlarged set $S$ is still finite. Thus we see that

$$K \subset \sum_{i=1}^{n} Kr_i + \langle S \rangle.$$

But now

$$K \subset \sum_{i_1, i_2 = 1}^{m} Kr_{i_1} r_{i_2} + \langle S \rangle \subset \cdots \subset \sum_{i_1, \dots, i_n = 1}^{m} Kr_{i_1} \cdots r_{i_n} + \langle S \rangle = \langle S \rangle.$$

Thus $K$ is finitely generated as a left ideal in $A$. ∎

THEOREM 2.10. *Let $A$ be a unital, infinite-dimensional algebra, and suppose that some element of $\mathfrak{M}_\infty(A)$ has finite codimension in $A$. Then one of the maximal left ideals in $A$ is not finitely generated.*

*Proof.* We suppose that $\mathfrak{M}_\infty(A)$ is such that $\operatorname{codim} M = n$, where $n \in \mathbb{N}$, and we identify $A/M$ with $\mathbb{C}^n$ as a linear space. The identity of $A$ is denoted by $e$.

For each $a \in A$, set $T_a(b + M) = ab + M$ $(b \in A)$. Then

$$\theta : a \mapsto T_a, \quad A \to \mathbb{M}_n,$$

is a unital homomorphism, and its image $\theta(A)$ is a unital subalgebra of $\mathbb{M}_n$. Define

$$K = \ker \theta = \{a \in A : aA \subset M\},$$

so that $K$ is an ideal in $A$ of codimension at most $n^2$. For each $a \in A$, we have $a = ae \in M$, and so $K \subset M$. Clearly $M/K$ is a left ideal in $A/K$.

Assume towards a contradiction that each maximal left ideal in $A$ is finitely generated. By Lemma 2.9, $K$ is finitely generated as a left ideal in $A$, and so $M$ is also finitely generated as a left ideal in $A$ because $M/K$ is a finite-dimensional space, a contradiction of the hypothesis.

Thus $A$ contains a maximal left ideal that is not finitely generated. ■

We now introduce a slightly complicated algebraic condition.

THEOREM 2.11. *Let $A$ be an algebra. Suppose that $I$ is a left ideal in $A$ with the property that there exist $a, b \in A \setminus I$ such that $ba \in I$ and $Ia \subset I$. Then $I$ does not belong to either $\mathfrak{M}_\infty(A)$ or $\mathfrak{U}_\infty(A)$.*

*Proof.* Assume towards a contradiction that $I \in \mathfrak{M}_\infty(A)$. We note that every left ideal $J$ that properly contains $I$ is finitely generated.

Take $a$ and $b$ as specified.

Consider the left ideal $A^\sharp a + I$ of $A$. Since $a \notin I$, the left ideal $A^\sharp a + I$ properly contains $I$, and so it is finitely generated, say

$$A^\sharp a + I = \langle b_1, \ldots, b_m \rangle,$$

where $b_1, \ldots, b_m \in A^\sharp a + I$. Each element $b_i$ has the form $a_i a + u_i$, where $a_i \in A^\sharp$ and $u_i \in I$, and so $A^\sharp a + I = \langle a, u_1, \ldots, u_m \rangle$.

Define $J = \langle u_1, \ldots, u_m \rangle \subset I$, so that $A^\sharp a + I = A^\sharp a + J$, and define

$$K = \{c \in A : ca \in I\}.$$

Then $K$ is a left ideal in $A$, and $I \subset K$ because $Ia \subset I$. Further, we claim that $I \subset Ka + J$. For take $x \in I$. Since $I \subset A^\sharp a + J$, there exist $c \in A^\sharp$ and $j \in J$ with $x = ca + j$. Then $ca = x - j \in I$, and so $c \in K$, giving the claim. Since $Ka + J \subset I$, it follows that $Ka + J = I$.

Since $b \in K \setminus I$, we have $I \subsetneq K$, and so $K$ is finitely generated, say $K = \langle c_1, \ldots, c_n \rangle$, where $c_1, \ldots, c_n \in A$. But now

$$I = \langle c_1 a, \ldots, c_n a, u_1, \ldots, u_m \rangle,$$

and so $I$ is finitely generated, a contradiction. Thus $I \notin \mathfrak{M}_\infty(A)$.

A trivial variation of the above argument shows that $I \notin \mathfrak{U}_\infty(A)$. ■

THEOREM 2.12. *Let $A$ be a Banach algebra. Suppose that $I$ is an ideal in $A$ and that $I \in \mathfrak{M}_\infty(A)$. Then either $I$ is a maximal modular ideal of codimension $1$ in $A$ or $I = A$.*

*Proof.* We consider the case where $I \neq A$. By Corollary 2.3, $I$ is closed.

Since $I$ is a closed ideal in $A$, the space $A/I$ is a Banach algebra. Each non-zero, left ideal in $A/I$ has the form $J/I$, where $J$ is a left ideal in $A$ with $J \supsetneq I$. Since $J$ is finitely generated in $A$ as a left ideal, the ideal $J/I$ is a finitely-generated left ideal in $A/I$, and so $\mathfrak{I}_\infty(A/I)$ is empty. By Theorem 1.2, $A/I$ is a finite-dimensional algebra.

Suppose that $A/I$ is semisimple. Then $A/I$ is a finite direct sum of matrix algebras. Assume towards a contradiction that $\dim A/I \geq 2$. Then there are idempotents $p$ and $q$ in $A/I$ with $pq = 0$, and so there are $a, b \in A \setminus I$ with $ab \in I$. Since $Ia \subset I$, it is immediate from Theorem 2.11 that $I \notin \mathfrak{M}_\infty(A)$, a contradiction. Thus $\dim A/I = 1$, and so $I$ is a maximal modular ideal of codimension 1 in $A$.

Assume towards a contradiction that $A/I$ is not semisimple. Since the radical of $A/I$ is finite-dimensional, there exists $a \in A \setminus I$ with $a^2 \in I$, again a contradiction of Theorem 2.11.

Hence $I$ is a maximal modular ideal of codimension 1 in $A$. ∎

## 3. The commutative case

**3.1. Finitely-generated maximal ideals.** We shall now consider commutative Banach algebras.

THEOREM 3.1. *Let $A$ be a commutative, unital Banach algebra. Suppose that each maximal ideal in $A$ is countably generated. Then $A$ is finite-dimensional.*

*Proof.* Since maximal ideals in $A$ are closed, each maximal ideal in $A$ is finitely generated by Corollary 1.6.

Assume towards a contradiction that $\mathfrak{M}_\infty(A) \neq \emptyset$, and take $I \in \mathfrak{M}_\infty(A)$. Since $A$ is unital, with identity $e_A$, say, $A$ itself is singly generated by $e_A$, and so $I \neq A$. By Theorem 2.12, $I$ is a maximal ideal in $A$. Since each maximal ideal is finitely generated, we have a contradiction, and so $\mathfrak{M}_\infty(A) = \emptyset$. By Corollary 2.3, $A$ is finite-dimensional. ∎

The above theorem requires that $A$ be unital, and this condition cannot be dispensed with. For let $A$ be the commutative, radical algebra $L^1([0,1])$, with convolution multiplication, as in [5, Definition 4.7.38]; thus $A$ is the *Volterra algebra*. Then $A$ has no maximal ideals, but $A$ is infinite-dimensional.

Let $A$ be a commutative Banach algebra. It can happen that the set $\mathfrak{M}_\infty(A)$ has just one element, whilst the maximal ideal space of $A$ is infinite. For example, let $A$ be the algebra $c$, so that $c$ is a commutative, unital Banach algebra. A maximal ideal of $c$ of the form

$$M_k = \{(x_i) \in c : x_k = 0\},$$

where $k \in \mathbb{N}$, is generated by the sequence $(a_i)$, where $a_k = 0$ and $a_i = 1$ $(i \neq k)$. The only other maximal ideal of $c$ is

$$c_0 = \left\{(x_i) \in c : \lim_{i \to \infty} x_i = 0\right\},$$

and this ideal is not finitely generated, and hence not countably generated.

**3.2. The Shilov boundary.** We first recall some standard notation; see [5, Chapter 4], for example.

Let $K$ be a non-empty, compact space. We write $C(K)$ for the space of all continuous functions on $K$ with the pointwise operations, and set

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C(K)),$$

so that $(C(K), |\cdot|_K)$ is a commutative, unital Banach algebra. A unital subalgebra $A$ of $C(K)$ that separates the points of $K$ and is a Banach algebra for some norm $\|\cdot\|$ is a *Banach function algebra* on $K$; $A$ is a *uniform algebra* if it is closed in $(C(K), |\cdot|_K)$.

A Banach function algebra $A$ on $K$ is *natural* if every character on $A$ has the form $\varepsilon_x : f \mapsto f(x)$ for some $x \in K$, or equivalently, each maximal ideal of $A$ has the form

$$M_x := \{f \in A : f(x) = 0\}$$

for some $x \in K$.

Let $A$ be a commutative, unital Banach algebra, and let $\Phi_A$ be the character space of $A$ as in [5], so that $\Phi_A$ is identified with the maximal ideal space of $A$; the maximal ideal corresponding to a character $\varphi \in \Phi_A$ is $M_\varphi = \ker \varphi$. It is standard that $\Phi_A$ is a compact space in the relative weak-$*$ topology $\sigma(A', A)$. For $a \in A$, define $\widehat{a}$ by

$$\widehat{a}(\psi) = \psi(a) \quad (\psi \in \Phi_A).$$

The Gel'fand transformation

$$\mathcal{G} : a \mapsto \widehat{a}, \quad A \to C(\Phi_A),$$

maps $A$ onto a Banach function algebra $\widehat{A}$ which is natural on $\Phi_A$.

Let $A$ be a natural Banach function algebra on a compact space $K$. A closed subset $F$ of $K$ is a *peak set* if there exists a function $f \in A$ such that $|f(x)| = 1$ $(x \in F)$ and $|f(y)| < 1$ $(y \in K \setminus F)$; in this case, $f$ *peaks* on $F$; a point $x \in K$ is a *peak point* if $\{x\}$ is a peak set, and a *strong boundary point* if $\{x\}$ is an intersection of peak sets; the set of strong boundary points of $A$ is denoted by $S_0(A)$. A countable intersection of peak sets is always a peak set, and so, in the case where $K$ is metrizable, $S_0(A)$ is the set of peak points of $A$. (However even a uniform algebra may have strong boundary points which are not peak points.) A subset $S$ of $K$ is a *boundary* if $S \cap F \neq \emptyset$ for each peak set $F$ of $A$. The intersection of all the closed boundaries of $A$ is a closed boundary; it is called the *Shilov boundary* of $A$ and is denoted by $\Gamma(A)$ [5, Definition 4.3.1(iv)]; for $x \in K$, we have $x \in \Gamma(A)$ if and only if, for each open neighbourhood $U$ of $x$ in $K$, there exists $f \in A$ such that $|f|_K > |f|_{K \setminus U}$ [5, Theorem 4.3.5].

Let $A$ be a commutative, unital Banach algebra. Then we define $\Gamma(A)$ to be $\Gamma(\widehat{A})$.

In the case where $A$ is a natural uniform algebra on $K$, the *Choquet boundary* of $A$ is defined in [5, Definition 4.3.3]; by [5, Theorem 4.3.5], it coincides with the set $S_0(A)$; by [5, Proposition 4.3.4], $S_0(A)$ is a boundary for $A$, and by [5, Corollary 4.3.7(i)], $S_0(A)$ is a dense subset of $\Gamma(A)$.

The following two results are essentially Corollary 1.7 of [8], where an analogous result for more general topological algebras is proved (by a considerably longer argument).

THEOREM 3.2. *Let $A$ be a commutative, unital Banach algebra. Suppose that $\varphi \in \Gamma(A)$ and the ideal $M_\varphi$ is countably generated. Then $\varphi$ is isolated in $\Phi_A$.*

*Proof.* Since $M_\varphi$ is a closed ideal in $A$, it follows from Corollary 1.6 that $M_\varphi$ is finitely generated.

Set $K = \Phi_A$. By a theorem of Gleason (e.g., see [14, Theorem 15.2]), there is an open neighbourhood $U$ of $\varphi$ in $K$ and a homeomorphism $\gamma$ from $U$ onto an analytic variety $V$ in a polydisc $\Delta$ in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ such that, for each $a \in A$, there is a holomorphic function $F$ on $\Delta$ such that $\widehat{a} = F \circ \gamma$ on $U$.

Since $\varphi \in \Gamma(A)$, there exists $a \in A$ such that $L \subset U$, where

$$L = \{\psi \in \Phi_A : |\psi(a)| = |\widehat{a}|_K\}.$$

We have $\partial U \subset K \setminus L$, where $\partial U$ denotes the frontier of $U$. Assume that $\partial U \neq \emptyset$. Then there exists $z \in V$ such that $|F(z)| > |F|_{\partial V}$ for a holomorphic function $F$ on $\Delta$, a contradiction of the maximum principle for holomorphic functions on varieties [10, III, Theorem 16]. Thus $\partial U = \emptyset$ and $U$ is compact. Hence $V$ is compact. But there are no compact, infinite varieties in $\mathbb{C}^n$, and so $V$ and $U$ are finite. Thus $\varphi$ is isolated in $\Phi_A$. ∎

THEOREM 3.3. *Let $A$ be a commutative, unital Banach algebra. Suppose that $M_\varphi$ is countably generated for each $\varphi \in \Gamma(A)$. Then $A$ is finite-dimensional.*

*Proof.* Since $\Gamma(A)$ is compact and each point of $\Gamma(A)$ is isolated in $\Phi_A$, the set $\Gamma(A)$ is finite, and hence $\Phi_A = \Gamma(A)$. By Theorem 3.1, $A$ is finite-dimensional. ∎

One might suspect that $\mathfrak{M}_\infty(A) \subset \Gamma(A)$ for a commutative Banach algebra $A$, but the following examples show that this is not necessarily the case.

EXAMPLE 3.4. Let $A$ be an infinite-dimensional, commutative, unital Banach algebra that is local, so that the unique maximal ideal in $A$ is $J(A)$. Then $J(A)$ is not finitely generated.

However an infinite-dimensional radical of a commutative, unital Banach algebra can be singly generated. For example, let $B$ be any infinite-dimensional, commutative, unital Banach algebra, with identity $e$, and let $A$ be a unital subalgebra of $B$. Set $\mathfrak{A} = A \oplus B$, with

$$\|(a,b)\| = \|a\| + \|b\| \quad (a \in A,\, b \in B)$$

and product defined by

$$(a_1,b_1)(a_2,b_2) = (a_1a_2, a_1b_2 + a_2b_1) \quad (a_1,a_2 \in A,\, b_1,b_2 \in B).$$

Then $\mathfrak{A}$ is a commutative Banach algebra with identity $(e,0)$. In the case where $A = B$ is semisimple, $J(\mathfrak{A}) = \{0\} \oplus B$, which is infinite-dimensional; here the radical $J(\mathfrak{A})$ is generated by $(0,e)$ because

$$(0,b) = (b,0)(0,e) \quad (b \in B).$$

Now suppose that $B = C(\overline{\mathbb{D}})$, the uniform algebra of all continuous functions on $\overline{\mathbb{D}}$, and that $A = A(\overline{\mathbb{D}})$, the disc algebra, consisting of the functions in $C(\overline{\mathbb{D}})$ that are analytic on $\mathbb{D}$. Then $\Phi_{\mathfrak{A}} = \overline{\mathbb{D}}$ and $\Gamma(\mathfrak{A}) = \mathbb{T}$. We recall that

$$\{f \in A : f(0) = 0\} = ZA,$$

where $Z$ is the coordinate functional. Set

$$M = \{(f,g) \in \mathfrak{A} : f(0) = 0\}.$$

Then $M$ is a maximal ideal of $\mathfrak{A}$, and it corresponds to a character on $\mathfrak{A}$ which is not in $\Gamma(\mathfrak{A})$.

We *claim* that the ideal $M$ is not finitely generated in $\mathfrak{A}$. Indeed, assume towards a contradiction that $M$ is finitely generated. Then we can suppose that the generators are $(Zf_1,g_1),\dots,(Zf_k,g_k)$, where $f_1,\dots,f_k \in A$ and $g_1,\dots,g_k \in B$. Thus, for each $g \in B$, there exist $r_1,\dots,r_k \in A$ and $s_1,\dots,s_k \in B$ such that

$$(0,g) = \sum_{i=1}^{k} (r_i,s_i)(Zf_i,g_i). \tag{3.1}$$

We define $F = \operatorname{lin}\{g_1,\dots,g_k\}$, a finite-dimensional subspace of $B$. Since $r_i - r_i(0)1 \in ZA$, it follows from (3.1) that $g \in \sum_{i=1}^{k} r_i(0)g_i + ZB$. Thus $B = F + ZB$. However it is not true that $B/ZB$ is a finite-dimensional space; for example, the set $\{|Z|^{1/n} + ZB : n \in \mathbb{N}\}$ is linearly independent in $B/ZB$. This is the required contradiction.

Hence $M \in \mathfrak{M}_\infty(\mathfrak{A})$, but $M \notin \Gamma(\mathfrak{A})$. ∎

We now present a natural uniform algebra $A$ such that $\mathfrak{M}_\infty(A) \not\subset \Gamma(A)$.

Example 3.5. Let $A$ be a unital, commutative Banach algebra, and take $\varphi \in \Phi_A$; set $M = M_\varphi$.

Suppose that $M = \langle f_1,\dots,f_n \rangle$. Then $\dim(M/M^2) \le n$. Indeed, for each $f \in M$, there exist $g_1,\dots,g_n$ with $f = \sum_{j=1}^{n} g_jf_j$; for $j = 1,\dots,n$, write

$g_j = g_j(\varphi)1 + h_j$, where $h_j \in M$. Then

$$f \in \sum_{j=1}^{n} g_j(\varphi) f_j + M^2,$$

and so $M/M^2 = \operatorname{lin}\{f_1 + M^2, \dots, f_n + M^2\}$. Thus the space of point derivations at $\varphi$ is finite-dimensional.

Now let $X = \prod_{n=1}^{\infty} \Delta_n$, where each $\Delta_n$ is the closed unit disc, and let $A$ be the tensor product of countably many copies of the disc algebra, as in [14, Theorem 14.3]. Then $A$ is semisimple, the character space of $A$ is $X$, and the Shilov boundary is $\Gamma(A) = \prod_{n=1}^{\infty} \mathbb{T}_n$, where each $\mathbb{T}_n$ is the unit circle. The character corresponding to evaluation at the point $0 = (0, 0, \dots)$ is not in $\Gamma(A)$. Let $M$ be the corresponding maximal ideal. Then each $f \in M$ is an analytic function in each of the coordinate functionals $Z_1, Z_2, \dots$, and each linear functional $d_n : f \mapsto (\partial f/\partial z_n)(0)$ is a (continuous) point derivation at 0. Since these linear functionals are linearly independent, it is not true that $\dim(M/M^2)$ is finite, and so $M$ is not finitely generated. ■

**3.3. Strong boundary points.** Let $A$ be a natural Banach function algebra, and now suppose that the closed ideal $M_\varphi$ is countably generated for each $\varphi \in S_0(A)$, rather than for each $\varphi \in \Gamma(A)$. Is it still true that $A$ must be finite-dimensional? First we claim that this is true when $A$ is a uniform algebra (and when the number of isolated points in $\Phi_A$ is countable).

We write $\delta_\varphi$ for the characteristic function of the singleton set $\{\varphi\}$ when $\varphi \in \Phi_A$. By Shilov's idempotent theorem [5, Theorem 2.4.33], $\delta_\varphi \in A$ whenever $\varphi$ is isolated in $\Phi_A$.

THEOREM 3.6. *Let $A$ be a uniform algebra on $\Phi_A$. Suppose that $S_0(A)$ is countable and that the maximal ideal $M_\varphi$ is countably generated for each $\varphi \in S_0(A)$. Then $A$ is finite-dimensional.*

*Proof.* We set $S_0(A) = \{\varphi_n : n \in \mathbb{N}\}$. By Theorem 3.2, each point $\varphi \in S_0(A)$ is an isolated point of $\Phi_A$, and so $S_0(A)$ is open in $\Phi_A$.

Assume towards a contradiction that we have $S_0(A) \neq \Phi_A$, and set $L = \Phi_A \setminus S_0(A)$, a non-empty, compact subset of $\Phi_A$. Clearly each isolated point is a peak point of $\Phi_A$, and so it is in $S_0(A)$.

Consider the function

$$f := 1 - \sum_{n=1}^{\infty} \frac{1}{n} \delta_{\varphi_n},$$

so that $f \in A$. At each $\varphi \in S_0(A)$, we have $|f(\varphi)| < 1$, and, for each $\varphi \in L$, we have $f(\varphi) = 1$. Hence $L$ is a peak set for $A$. Since $S_0(A)$ is a boundary for $A$, $S_0(A) \cap L \neq \emptyset$, a contradiction. Thus $S_0(A) = \Phi_A$.

It follows from Theorem 3.3 that $A$ is finite-dimensional. ■

We do not know if the above theorem holds in the case where $S_0(A)$ is not necessarily countable.

Let $A$ be a uniform algebra, and let $\varphi \in S_0(A)$. Then we can prove that the following are equivalent without resource to Gleason's theorem, which was used in the proof of Theorem 3.2:

(a) $M_\varphi$ is singly generated;
(b) $M_\varphi$ is countably generated;
(c) $\varphi$ is isolated in $\Phi_A$.

It is sufficient to prove that (b)$\Rightarrow$(c). Thus, suppose that $M_\varphi$ is countably generated, and assume towards a contradiction that $\varphi$ is not isolated in $\Phi_A$. Then it is easy to see that, given any countable set $S$ in $M_\varphi$, there exists $h \in C(\Phi_A)$ such that $|h|_{\Phi_A} < 1$ and $\lim_{\psi\to\varphi} h(\psi)/g(\psi) = \infty$ for each $g \in \langle S\rangle$. By [14, Theorem 20.12], there exists $f \in A$ with $f(\varphi) = 1$ and $|f(\psi)| < 1 - h(\psi)$ for each $\psi \in \Phi_A$ with $\psi \neq \varphi$. But now $1 - f \in M_\varphi$, but $1 - f \notin \langle S\rangle$.

Thus we have a direct proof of Theorem 3.6 avoiding Gleason's theorem.

Let $\omega_1$ be the first uncountable ordinal, set $A = C([0, \omega_1])$, and consider the maximal ideal $M_{\omega_1}$. Then the above remark shows that $M_{\omega_1}$ is not countably generated. This is related to [17, 18].

We show finally that the above theorem does not hold if we replace 'uniform algebra' by 'Banach function algebra'.

Example 3.7. Since our example is rather long, we divide the construction into a number of steps.

(1) *The set $K$.* Our first step is the construction of a certain compact subset $K$ of the plane. We start with $\overline{\mathbb{D}}$, the closed unit disc. For each $n \in \mathbb{N}$, we consider the circle $\Gamma_n$ of radius $1 + 1/n$. Then we place $n$ points equally spaced on $\Gamma_n$. The totality of these points is $U$, and the union of $U$ with $\overline{\mathbb{D}}$ and with $\mathbb{T}$ form the sets $K$ and $L$, respectively. Clearly $K$ and $L$ are compact, $U$ is the set of isolated points of $K$, $L = \overline{U}$, and the interior of $K$ is the open disc $\mathbb{D}$.

(2) *The algebra $B$.* We now define a natural Banach function algebra $B$ of quasi-analytic functions on $\overline{\mathbb{D}}$. For the general theory of such Banach function algebras, see [5, §4.4]; the specific example that we require is given in [7]. Thus $(B, \|\cdot\|_B)$ is an algebra of infinitely-differentiable functions on $\overline{\mathbb{D}}$ such that $\|g\|_B < \infty$, where $\|\cdot\|_B$ is specified by the formula

$$\|g\|_B = \sum_{k=0}^{\infty} \frac{1}{M_k} |g^{(k)}|_{\overline{\mathbb{D}}} \quad (g \in B)$$

for a suitable sequence $(M_k : k \in \mathbb{Z}^+)$. The norm is chosen so that $(B, \|\cdot\|_B)$ is a natural Banach function algebra on $\overline{\mathbb{D}}$ and such that the algebra $B$ is quasi-analytic, in the sense that $g = 0$ whenever $g \in B$ has the property that $g^{(j)}(z_0) = 0$ $(j \in \mathbb{Z}^+)$ for some point $z_0 \in \overline{\mathbb{D}}$. [In fact, we can take

$$M_k = k! \log 2 \cdots \log(k+2) \qquad (k \in \mathbb{N}),$$

with $M_0 = 1$, for example.]

We note that the uniform closure of the algebra $B$ is $A(\overline{\mathbb{D}})$, the standard disc algebra, a natural uniform algebra on $\overline{\mathbb{D}}$.

(3) *The algebra* $C$. Next, we consider the Banach function algebra which is $C := \operatorname{Lip} L$ of Lipschitz functions on $L$. These Lipschitz algebras are also discussed in [5, §4.4]. Thus $C$ consists of the continuous functions on $L$ such that $\|\cdot\|_C < \infty$, where $\|\cdot\|_C$ is specified by the formula

$$\|h\|_C = |h|_L + \sup\left\{\frac{|h(z)-h(w)|}{|z-w|} : z, w \in L,\ z \neq w\right\}.$$

It is shown in [5, Theorem 4.4.24] that $C$ is a natural Banach function algebra on $L$. Clearly the uniform closure of $C$ is $C(L)$.

(4) *The algebra* $A$. We now form a Banach function algebra $A$ which is the combination of $B$ and $C$. Indeed,

$$A = \{f \in C(K) : f|\overline{\mathbb{D}} \in B,\ f|L \in C\}.$$

The norm $\|\cdot\|_A$ on $A$ is specified by

$$\|f\|_A = \max\{\|f|\overline{\mathbb{D}}\|_B,\ \|f|L\|_C\} \qquad (f \in A).$$

We see that $(A, \|\cdot\|_A)$ is a Banach function algebra on $K$.

(5) *The algebra* $\mathfrak{A}$. We next consider the functions $f \in A$ such that

$$\lim_{n\to\infty} \frac{f(x_n)-f(z_0)}{x_n - z_0} = f'(z_0) \tag{3.2}$$

whenever $z_0 \in \mathbb{T}$ and $(x_n)$ is a sequence in $U$ with $\lim_{n\to\infty} x_n = z_0$. These functions $f$ clearly form a subalgebra, say $\mathfrak{A}$, of $A$.

We *claim* that $\mathfrak{A}$ is a closed subalgebra of $A$.

Indeed, take a sequence $(f_k)$ in $\mathfrak{A}$ such that $f_k \to f$ in $(A, \|\cdot\|_A)$. To see that $f \in \mathfrak{A}$, take $z_0 \in \mathbb{T}$ and a sequence $(x_n)$ in $U$ with $\lim_{n\to\infty} x_n = z_0$. Fix $\varepsilon > 0$. Since $\|f_k|\overline{\mathbb{D}} - f|\overline{\mathbb{D}}\|_B \to 0$, there exists $k_1 \in \mathbb{N}$ such that

$$|f_k'(z_0) - f'(z_0)| < \varepsilon \qquad (k \geq k_1).$$

Since $\|f_k|L - f|L\|_C \to 0$, there exists $k_2 \in \mathbb{N}$ such that

$$\frac{|(f_k - f)(x_n) - (f_k - f)(z_0)|}{|x_n - z_0|} < \varepsilon \qquad (k \geq k_2).$$

Set $k_0 = \max\{k_1, k_2\}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{f_{k_0}(x_n) - f_{k_0}(z_0)}{x_n - z_0} - f_{k_0}'(z_0)\right| < \varepsilon \qquad (n \geq n_0).$$

Hence
$$\left|\frac{f(x_n)-f(z_0)}{x_n-z_0}-f'(z_0)\right|<3\varepsilon \quad (n\geq n_0).$$
It follows that $\lim_{n\to\infty}(f(x_n)-f(z_0))/(x_n-z_0)=f'(z_0)$, and so $f\in\mathfrak{A}$, giving the claim.

Since $\mathfrak{A}$ contains the restrictions to $K$ of the polynomials, it is clear that $\mathfrak{A}$ contains the constants and separates the points of $K$, and so $\mathfrak{A}$ is a Banach function algebra on $K$. Since $\mathfrak{A}$ also contains $\delta_z$ for each $z\in U$, it is easy to see that the uniform closure of $\mathfrak{A}$ is $A(K)$, the algebra of all continuous functions on $K$ that are analytic on $\operatorname{int} K=\mathbb{D}$.

(6) *The naturality of* $\mathfrak{A}$. We next *claim* that $\mathfrak{A}$ is natural on $K$. It is a general result that it suffices to prove:

(i) for each $f\in\mathfrak{A}$ such that $\mathbf{Z}(f)=\{z\in L: f(z)=0\}$ is void, it follows that $1/f\in\mathfrak{A}$;
(ii) the uniform closure $A(K)$ of $\mathfrak{A}$ is natural on $K$.

(This follows immediately from [5, Proposition 4.1.5(i)].) Clause (ii) is a standard result of Arens [5, Theorem 4.3.14]. For (i), take $f\in\mathfrak{A}$ with $\mathbf{Z}(f)=\emptyset$, and set $g=1/f\in C(K)$. Since $B$ and $C$ are each natural, it follows that $g\mid\overline{\mathbb{D}}\in B$ and $g\mid L\in C$, and hence $g\in A$. It is clear that $g$ satisfies equation (3.2), and so $g\in\mathfrak{A}$. Thus $\mathfrak{A}$ is natural on $K$.

(7) *The peak points of* $\mathfrak{A}$. Certainly each point of $U$ is an isolated peak point for $A$.

We now *claim* that there are no other peak points. It is enough to show that the point $z=1$ is not a peak point.

Assume towards a contradiction that $f\in\mathfrak{A}$ and that $f$ peaks at 1, say with $f(1)=1$. Then $f\mid\overline{\mathbb{D}}=1+g$ for a certain function $g\in B$. The function $g$ is not zero, and so, since $B$ is a quasi-analytic algebra, there exists $k\in\mathbb{N}$ such that $g^{(k)}(1)\neq 0$, where the derivative is calculated with respect to $\overline{\mathbb{D}}$; we take $k_0\in\mathbb{N}$ to be the minimum such $k$.

First, suppose that $k_0\geq 2$. Then there exists $\alpha\in\mathbb{C}\setminus\{0\}$ such that
$$f(z)=1+\alpha(z-1)^{k_0}+o(|z-1|^{k_0})$$
as $z\to 1$ with $z\in\overline{\mathbb{D}}$. But, whatever the value of $\alpha$, we can choose a sequence $(z_n)$ in $\overline{\mathbb{D}}$ with $\lim_{n\to\infty}z_n=1$ and $\Re(\alpha(z_n-1)^{k_0})>0$ for each $n\in\mathbb{N}$. This implies that $|f(z_n)|>1$ for all sufficiently large $n\in\mathbb{N}$, a contradiction.

Second, suppose that $k_0=1$. We must now use points outside $\overline{\mathbb{D}}$, and so, at this point, we regard $g\in\mathfrak{A}$ as a function on $K$ with $f=1+g$. We know that there exists $\alpha\in\mathbb{C}\setminus\{0\}$ such that
$$f(z)=1+\alpha(z-1)+o(|z-1|)$$

as $z \to 1$ with $z \in \overline{\mathbb{D}}$. Now the set $\{z \in \mathbb{C} : \Re(\alpha(z-1)) \geq 0\}$ is a half-plane with 1 on the boundary line. In the case where $\alpha \notin \mathbb{R}^+$, we can choose a sequence $(z_n)$ in $\overline{\mathbb{D}}$ as before to obtain a contradiction. Thus we may suppose that $\alpha \in \mathbb{R}^+$. But now, by the choice of the set $U$, there is a sequence $(x_n)$ in $U$ with $\Re(\alpha(x_n - 1)) > 0$ for each $n \in \mathbb{N}$. By the construction of our algebra $\mathfrak{A}$, we know that

$$\lim_{n\to\infty} \Re\left(\frac{f(x_n) - f(1)}{x_n - 1}\right) = \alpha > 0.$$

Thus $\Re f(x_n) > 1$ for all sufficiently large $n \in \mathbb{N}$. It follows that $|f(x_n)| > 1$ for all sufficiently large $n \in \mathbb{N}$, and this is the required contradiction.

(8) *The conclusion.* We have shown that $\mathfrak{A}$ is a natural Banach function algebra on $K$ such that $S_0(\mathfrak{A}) = U$, the countable set of isolated points of $K$. Let $z \in U$, with corresponding maximal ideal $M_z$. Then $M_z$ is singly generated by the function $1 - \delta_z$, and so all maximal ideals corresponding to points of $\Gamma_0(\mathfrak{A})$ are singly generated. However, $\mathfrak{A}$ is not a finite-dimensional algebra. ∎

## References


[1] S. I. Bilavska and B. V. Zabavsky, *On the structure of maximal non-finitely generated ideals of ring and Cohen's theorem*, Bul. Acad. Ştiinţe Rep. Mold. Mat. 2011, 33–41.

[2] N. Boudi, *Banach algebras in which every left ideal is countably generated*, Irish Math. Soc. Bull. 48 (2002), 17–24.

[3] R. Choukri and A. El Kinani, *Topological algebras with ascending or descending chain condition*, Arch. Math. (Basel) 72 (1999), 438–443.

[4] R. Choukri, A. El Kinani, and M. Oudadess, *Algèbres topologiques à idéaux à gauche fermés*, Studia Math. 168 (2005), 159–164.

[5] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. 24, Clarendon Press, Oxford, 2000.

[6] H. G. Dales, T. Kania, T. Kochanek, P. Koszmider, and N. J. Laustsen, *Maximal left ideals of the Banach algebra of bounded operators on a Banach space*, submitted for publication.

[7] H. G. Dales and J. P. McClure, *Completion of normed algebras of polynomials*, J. Austral. Math. Soc. Ser. A 20 (1975), 504–510.

[8] A. V. Ferreira and G. Tomassini, *Finiteness properties of topological algebras, I*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), 471–488.

[9] H. Grauert und R. Remmert, *Analytische Stellenalgebren*, Springer, Berlin, 1971.

[10] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[11] T. W. Hungerford, *Algebra*, Grad. Texts in Math. 73, Springer, New York, 1980.

[12] N. Jacobson, *Basic Algebra, II*, W. H. Freeman, San Francisco, CA, 1980.

[13] A. M. Sinclair and A. W. Tullo, *Noetherian Banach algebras are finite dimensional*, Math. Ann. 211 (1974), 151–153.

[14] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, Tarrytown-on-Hudson, New York, 1971.

[15] W. Żelazko, *A characterization of commutative Fréchet algebras with all ideals closed*, Studia Math. 138 (2000), 293–300.
[16] W. Żelazko, *A characterization of $F$-algebras with all one-sided ideals closed*, Studia Math. 168 (2005), 135–145.
[17] W. Żelazko, *When does a closed ideal of a commutative unital Banach algebra contain a dense subideal?*, Funct. Approx. Comment. Math. 44 (2011), 285–287.
[18] W. Żelazko, *Concerning dense subideals in commutative Banach algebras*, Funct. Approx. Comment. Math., to appear.

H. G. Dales
Department of Mathematics and Statistics
Fylde College
University of Lancaster
Lancaster LA1 4YF, United Kingdom
E-mail: g.dales@lancaster.ac.uk

W. Żelazko
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
P.O. Box 21
00-956 Warszawa, Poland
E-mail: W.Zelazko@impan.pl

*Received September 11 2012*
*Revised version November 29, 2012* (7615)