Operator ideal properties of vector measures with finite variation

by

Susumu Okada (Kambah), Werner J. Ricker (Eichstätt) and Luis Rodríguez-Piazza (Sevilla)

Abstract. Given a vector measure $m$ with values in a Banach space $X$, a desirable property (when available) of the associated Banach function space $L^1(m)$ of all $m$-integrable functions is that $L^1(m) = L^1(|m|)$, where $|m|$ is the $[0, \infty]$-valued variation measure of $m$. Closely connected to $m$ is its $X$-valued integration map $I_m : f \mapsto \int f \, dm$ for $f \in L^1(m)$. Many traditional operators from analysis arise as integration maps in this way. A detailed study is made of the connection between the property $L^1(m) = L^1(|m|)$ and the membership of $I_m$ in various classical operator ideals (e.g., the compact, $p$-summing, completely continuous operators). Depending on which operator ideal is under consideration, the geometric nature of the Banach space $X$ may also play a crucial role. Of particular importance in this regard is whether or not $X$ contains an isomorphic copy of the classical sequence space $\ell^1$. The compact range property of $X$ is also relevant.

1. Introduction. Let $X$ be a Banach space (with closed unit ball $B[X]$ and dual space $X^*$) and $m : \Sigma \to X$ be a vector measure, i.e., $m$ is $\sigma$-additive on the $\sigma$-algebra $\Sigma$ (of subsets of some non-empty set $\Omega$). The variation measure $|m| : \Sigma \to [0, \infty]$ of $m$ is defined analogously to that for scalar measures [14, Ch. I, Definition 1.4]. Then the classical space $L^1(|m|)$ delivers a certain collection of integrable functions associated with $m$. There are also others. Namely, a $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is called $m$-integrable if

(I1) $\int_{\Omega} |f| \, d\langle m, x^* \rangle < \infty$ for all $x^* \in X^*$, and

(I2) for each $A \in \Sigma$ there is $\int_A f \, dm \in X$ satisfying

$$\left\langle \int_A f \, dm, x^* \right\rangle = \int_A f \, d\langle m, x^* \rangle, \quad \forall x^* \in X^*$$

[25, 26]. Here, for each $x^* \in X^*$, the scalar measure $A \mapsto \langle m(A), x^* \rangle$, for $A \in \Sigma$, is denoted by $\langle m, x^* \rangle$. The space of all $m$-integrable functions is

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denoted by $L^1(m)$; it is identified with its quotient space modulo $m$-null functions, where an $m$-integrable function $f$ is $m$-null if $\int_A f \, dm = 0$ for all $A \in \Sigma$. These two spaces of integrable functions associated with $m$ are related via

\begin{equation}
L^1(|m|) \subseteq L^1(m) \tag{1.1}
\end{equation}

[38, Lemma 3.14]. If a set $A \in \Sigma$ is called $m$-null whenever $\chi_A$ is an $m$-null function, then one can also form the Banach space $L^\infty(m)$ of all (equivalence classes of) bounded $\Sigma$-measurable functions, equipped with the essential sup-norm $\| \cdot \|_{L^\infty(m)}$ in the usual way, in which case $L^\infty(m) \subseteq L^1(m)$ [26, p. 161]. Of course, the family $\text{sim} \Sigma$ of all $\Sigma$-simple functions is contained in $L^\infty(m)$. The space $L^1(m)$ is known to be complete with respect to the lattice norm

\begin{equation}
\|f\|_{L^1(m)} := \sup_{x^* \in B[X^*]} \int |f| \, d\langle m, x^* \rangle, \quad \forall f \in L^1(m), \tag{1.2}
\end{equation}

i.e., $L^1(m)$ is a (complex) Banach lattice, the space $\text{sim} \Sigma$ is dense in $L^1(m)$, and the integration map $I_m : L^1(m) \to X$ defined by

\begin{equation}
I_m : f \mapsto \int \Omega f \, dm, \quad \forall f \in L^1(m), \tag{1.3}
\end{equation}

is linear, continuous and has operator norm $\|I_m\|_{\text{op}} = 1$ [38, p. 152]. Its restriction $I_{|m|} : L^1(|m|) \to X$ is also continuous because $|\langle m, x^* \rangle|(A) \leq |m|(A)\|x^*\|$ for all $A \in \Sigma$ and $x^* \in X^*$ implies that

\begin{equation}
\|I_{|m|}(f)\|_X = \left\| \int \Omega f \, dm \right\|_X \leq \int \Omega |f| \, d|m|, \quad \forall f \in L^1(|m|). \tag{1.4}
\end{equation}

The inclusion [1.1] may be proper, even if $m$ has finite variation, i.e., $|m|(\Omega) < \infty$. Since $L^1(|m|)$ is a classical $L^1$-space, it is surely more tractable than $L^1(m)$ in general. So, there is some interest in the situation when [1.1] is actually an equality. For an arbitrary vector measure $m$, G. P. Curbera showed that $L^1(m) = L^1(|m|)$ iff the integration map $I_m$ is positive 1-summing (also called cone absolutely summing) [8, Proposition 3.1]. It was recently shown that $I_m$ is positive $p$-summing for some $1 \leq p < \infty$ iff $L^1(m)$ is isomorphic to an AL-space [5, Theorem 2.7]. Combining this with [38, Lemma 3.14(iii)] it follows that $L^1(m) = L^1(|m|)$ iff $I_m$ is positive $p$-summing for some $1 \leq p < \infty$. However, given a specific vector measure $m$ of finite variation, it is not always easy to identify the space $L^1(m)$ explicitly, although this is needed to test directly whether $L^1(m) = L^1(|m|)$ or whether $I_m$ is positive $p$-summing. Moreover, even if equality in [1.1] is established, the positive $p$-summing nature of $I_m$ alone reveals little about its possible finer structure, i.e., whether it is maybe weakly compact, or compact, or completely continuous, etc. Our aim is to provide new results and
techniques which can be used in practise to decide when equality holds in (1.1) and which also allow for a finer analysis of $I_m$. We begin by formulating a general “Operator Ideal Principle” (cf. Proposition 1.1 below), which reduces the question of equality in (1.1) to determining solely whether or not $m$ has finite variation. First some notation and terminology.

Given Banach spaces $X$ and $Y$, let $L(X, Y)$ be the Banach space of all continuous linear operators from $X$ to $Y$. An operator ideal $A$ is a method of assigning to every couple $(X, Y)$ of Banach spaces a linear subspace $A(X, Y) \subseteq L(X, Y)$ which contains the finite rank operators and such that $R \circ S \circ T \in A(W, Z)$ for every pair $(W, Z)$ of Banach spaces and all choices of operators $T \in L(W, X)$, $R \in L(Y, Z)$ and $S \in A(X, Y)$ [13, p. 131]. Some examples relevant for this paper are when $A$ is all compact, or all $p$-summing, or all completely continuous operators. We point out that the collection of positive $p$-summing operators, which require a Banach lattice as domain space, does not form an operator ideal.

Let $A$ be an operator ideal. A Banach space $X$ is called $A$-variation admissible if $|m|$ is a finite measure for every $X$-valued vector measure $m$ whose integration map satisfies $I_m \in A$. For instance, if $A_c$ is the operator ideal of all compact operators, then every Banach space is $A_c$-variation admissible [36, Theorem 4]. Or, if $A_p$ is the operator ideal of all $p$-summing operators for some $1 \leq p < \infty$, then every Banach space is $A_p$-variation admissible. This result was already presented in [39]; its complete proof will be given in Section 2.

The following result is a useful tool for establishing equality in (1.1).

**Proposition 1.1.** Let $A$ be an operator ideal and $X$ be any $A$-variation admissible Banach space. Then $L^1(|m|) = L^1(m)$ for every $X$-valued vector measure $m$ whose integration map satisfies $I_m \in A$.

The Banach sequence space $\ell^1$ turns out to play a central role. Recall that a continuous linear map between Banach spaces is completely continuous if it maps weakly convergent sequences to norm convergent sequences. Such operators are also called Dunford–Pettis operators. Let $A_{cc}$ denote the operator ideal consisting of all completely continuous operators [13, p. 49].

**Theorem 1.2.** Every Banach space $X$ with an unconditional basis and not containing an isomorphic copy of $\ell^1$ (briefly, $\ell^1 \not\hookrightarrow X$) is $A_{cc}$-variation admissible. In particular, $L^1(m) = L^1(|m|)$ whenever $m$ is an $X$-valued vector measure such that $I_m \in A_{cc}$.

According to a classical result of H. P. Rosenthal [2, p. 247], all non-reflexive, weakly sequentially complete Banach spaces $X$ have the property that $\ell^1 \hookrightarrow X$. If $m$ is any vector measure of infinite variation with values in such a space $X$ (such measures always exist as $X$ is infinite-dimensional),
then necessarily \( L^1(|m|) \subsetneq L^1(m) \). Restrict \( X \) further to come from the subclass of all infinite-dimensional Banach spaces with the Schur property (i.e., weakly convergent sequences are norm convergent or, equivalently, relatively weakly compact sets are relatively norm compact). Then, in addition to \( L^1(|m|) \subsetneq L^1(m) \), we also automatically have \( I_m \in \mathcal{A}_{cc} \). Examples of such vector measures \( m \) also exist with finite variation; see Section 4. These comments show that the requirement \( \ell^1 ightleftharpoons X \) cannot be omitted from Theorem 1.2.

The class of Banach spaces covered by Theorem 1.2 includes all reflexive spaces with an unconditional basis, the sequence space \( c_0 \), and many more.

Recall that a Banach space \( X \) has the weak Radon–Nikodým property (briefly, WRNP) if, whenever \((\Omega, \Sigma, \mu)\) is any complete probability space and \( m : \Sigma \to X \) is any \( \mu \)-continuous vector measure of finite variation, then \( m \) has a Pettis \( \mu \)-integrable, \( X \)-valued density. Of relevance to this paper is that a Banach space \( X \) satisfies \( \ell^1 \rightleftharpoons X \) iff \( X^* \) has the WRNP [15, Theorem 6.8], [11, Corollary 7.3.8]. A Banach space \( X \) has the compact range property (briefly, CRP) if every \( X \)-valued vector measure of finite variation has relatively compact range [30, Definition 2]. If \( X \) has the WRNP, then \( X \) has the CRP [30, Proposition 4]. Also, in any weakly compactly generated Banach space (hence, in all separable spaces and in all reflexive spaces), the WRNP and the Radon–Nikodým property (briefly, RNP) are equivalent [29, Corollary 3]. The same is true in arbitrary Banach lattices [17, Theorem 5]. In particular, every reflexive Banach space has the CRP [14, p. 218]. Such spaces, even if separable, need not have an unconditional basis [27, p. 27].

The following “converse type” result should be compared with Theorem 1.2.

**Theorem 1.3.** For a Banach space \( X \) the following assertions are equivalent:

(i) \( X \) has the CRP.

(ii) Every \( X \)-valued vector measure \( m \) with \( L^1(m) = L^1(|m|) \) satisfies \( I_m \in \mathcal{A}_{cc} \).

It is worth noting that the condition \( L^1(m) = L^1(|m|) \) in part (ii) of Theorem 1.3 cannot be relaxed to the requirement that \( m \) has finite variation. Indeed, in Example 3.69 of [38] there is a vector measure \( m \) taking its values in the reflexive space \( \ell^p \), \( 1 < p < \infty \) (hence, has the CRP), such that \( m \) has finite variation and \( L^1(|m|) \subsetneq L^1(m) \). Since its integration map \( I_m : L^1(m) \to \ell^p \) is an isomorphism onto \( \ell^p \) (and \( \ell^p \) does not have the Schur property), \( I_m \) is surely not completely continuous. For a non-atomic example (in the reflexive space \( L^r([0,1]) \), \( 1 < r < \infty \) which exhibits the same features we refer to the \( L^r([0,1]) \)-valued vector measure \( m_r \) induced by the classical Volterra kernel operator, namely \( m_r(A) \) is the function in \( L^r([0,1]) \) given
by $t \mapsto \int_0^t \chi_A(s) \, ds$ for $t \in [0,1]$ and every Borel set $A \in \mathcal{B}([0,1])$ [38]. Example 3.26 & Proposition 3.52. In this case $I_{m_r}$ is not an isomorphism but still fails to be completely continuous.

Suppose that $I_m$ is $p$-summing for some $1 \leq p < \infty$ [13, Ch. 2], in which case $L^1(m) = L^1(|m|)$. Then $I_m$ is also weakly compact and completely continuous [13, Theorem 2.17]. In general, neither the weak compactness of $I_m$ alone implies that $L^1(m) = L^1(|m|)$ (cf. the Volterra measures $m_r : \mathcal{B}([0,1]) \to L^r([0,1])$ for $1 < r < \infty$, mentioned above), nor does complete continuity of $I_m$ alone imply that $L^1(m) = L^1(|m|)$; consider $\ell^1$-valued measures as discussed after Theorem 1.2. We construct several non-trivial examples of vector measures $m$ (mostly arising in classical analysis) which show that $I_{m_f}$ may belong to $A_{cA_1}$, or to $A_1 \setminus A_c$, or even to $(\bigcap_{p<r<\infty} A_r) \setminus (A_p \cup A_c)$ for every $2 < p < \infty$.

2. Operator Ideal Principle and $p$-summing integration maps. Let $m : \Sigma \to X$ be a Banach-space-valued vector measure and let $L^0(\Sigma)$ denote the space of all $\mathbb{C}$-valued, $\Sigma$-measurable functions on $\Omega$. Given $f \in L^1(m)$, its indefinite integral is the vector measure $m_f : \Sigma \to X$ given by

$$m_f : A \mapsto \int_A f \, dm, \quad \forall A \in \Sigma;$$

the $\sigma$-additivity of $m_f$ follows from (I1), (I2) and the Orlicz–Pettis Theorem [13, Ch. I, Corollary 4.4].

LEMM 2.1. Let $m : \Sigma \to X$ be a Banach-space-valued vector measure.

(i) A function $f \in L^1(m)$ belongs to $L^1(|m|)$ iff its indefinite integral $m_f$ has finite variation, in which case $\|f\|_{L^1(m)} \leq \|f\|_{L^1(|m|)}$.

(ii) Let $f \in L^1(m)$. As an equality of vector spaces we have

$$L^1(m_f) = \{g \in L^0(\Sigma) : gf \in L^1(m)\}.$$

Moreover, for each $g \in L^1(m_f)$,

$$\int_A g \, dm_f = \int_A gf \, dm, \quad \forall A \in \Sigma,$$

and also

$$\|g\|_{L^1(m_f)} = \|gf\|_{L^1(m)}.$$

The multiplication operator $M_f : L^1(m_f) \to L^1(m)$ defined by $g \mapsto gf$ for $g \in L^1(m_f)$ is a linear isometry onto its range in $L^1(m)$ and

$$I_{m_f} = I_m \circ M_f.$$

Proof. (i) See Lemma 3.14(i) in [38].

(ii) To establish (2.2), let $g \in L^1(m_f)$. By Theorem 3.5 of [38] applied to $m_f$ there exists a sequence $\{s_n\}_{n=1}^\infty \subseteq \text{sim} \Sigma$ such that $s_n \to g$ pointwise
and \( \lim_{n \to \infty} \int_A s_n \, dm_f = \int_A g \, dm_f \) for \( A \in \Sigma \). Since also \( s_n f \to g f \) pointwise and \( \int_A s_n \, dm_f = \int_A s_n f \, dm \) for \( n \in \mathbb{N} \) and \( A \in \Sigma \), again by Theorem 3.5 of \cite{38}, now applied to \( m \), we can conclude that \( g f \in L^1(m) \) and (2.3) holds. So, \( g \) belongs to the right side of (2.2).

Conversely, let \( g \) belong to the right side of (2.2). Choose a sequence \( \{s_n\}_{n=1}^{\infty} \subseteq \Sigma \) with \( s_n \to g \) pointwise and \( |s_n| \leq |g| \) pointwise for each \( n \in \mathbb{N} \). Then \( |s_n f| \leq |gf| \) pointwise for \( n \in \mathbb{N} \), with \( gf \in L^1(m) \). The Dominated Convergence Theorem for vector measures \cite{38} Theorem 3.7(i)]\) then yields \( s_n f \to g f \) in \( L^1(m) \). In particular, \( \int_A s_n \, dm_f = \int_A g f \, dm \) for \( A \in \Sigma \) \cite{38} p. 112. Since \( \int_A s_n f \, dm = \int_A s_n \, dm_f \) for \( A \in \Sigma \) and \( n \in \mathbb{N} \), we can conclude from Theorem 3.5 of \cite{38} that \( g \in L^1(m_f) \) and that (2.3) holds. So, (2.2) is valid. At the same time we have established (2.3).

The formula (2.4) follows from the identities
\[
|\langle m_f, x^* \rangle(A)| = \int_A |f| \, d|\langle m_f, x^* \rangle|, \quad \forall x^* \in X^*, \; A \in \Sigma
\]
(cf. (I2) and (2.1)) together with the definition
\[
\|g\|_{L^1(m_f)} := \sup_{x^* \in B[X^*]} \int_{\Omega} |g| \, d|\langle m_f, x^* \rangle|.
\]
Then (2.2) and (2.4) ensure that \( M_f \) is a linear isometry from \( L^1(m_f) \) onto its range in \( L^1(m) \). Finally, (2.5) follows routinely from the definitions involved.

We can now establish the Operator Ideal Principle.

**Proof of Proposition 1.1.** Let \( A, X \) and \( m : \Sigma \to X \) be as in the statement of the result. Fix any \( f \in L^1(m) \). Then the composition \( I_{m_f} = I_m \circ M_f : L^1(m_f) \to X \) (cf. (2.5)) belongs to \( A \) because \( I_m \in A \) (by assumption). Since \( X \) is \( A \)-variation admissible, we can conclude that \( m_f \) has finite variation, and hence, by Lemma 2.1(i), that \( f \in L^1(|m|) \). This establishes that \( L^1(m) \subseteq L^1(|m|) \) and, via (1.1), it follows that \( L^1(m) = L^1(|m|) \).

The following result, presented in \cite{39}, will now be established. The techniques in the proof are of independent interest.

**Theorem 2.2.** Let \( 1 \leq p < \infty \) and \( A_p \) be the operator ideal consisting of all \( p \)-summing operators. Then every Banach space is \( A_p \)-variation admissible.

**Proof.** Let \( X \) be any Banach space. Fix an \( X \)-valued vector measure \( m \) defined on a measurable space \( (\Omega, \Sigma) \). Select any positive finite measure \( \mu : \Sigma \to [0, \infty) \) which has the same null sets as \( m \), written briefly as \( m \simeq \mu \) \cite[Ch. I, Corollary 2.6]{14}. So, \( L^\infty(\mu) = L^\infty(m) \) is continuously embedded, via the natural embedding, say \( \alpha \), into \( L^1(m) \). The proof is in several steps.
STEP 1. If $I_m \circ \alpha : L^\infty(\mu) \to X$ is $q$-summing for some $1 \leq q < \infty$, then there is $g_q \in L^1(\mu)$ with $g_q(w) > 0$ for every $w \in \Omega$ such that, with continuous inclusions,

$$L^\infty(\mu) \subseteq L^q(g_q \, d\mu) \subseteq L^1(m).$$

To verify this, observe that $L^\infty(\mu)$ is a (complex) AM-Banach lattice with unit $\chi_\Omega$. By the Kakutani Representation Theorem \[40, Theorem 7.4 & p. 138\], there is an isometric Banach lattice isomorphism $\beta$ of $L^\infty(\mu)$ onto $C(K)$ with $\beta(\chi_\Omega) = \chi_K$, for some (extremely disconnected) compact Hausdorff space $K$. Since $\beta(|f|) = |\beta(f)|$, it follows that $\beta(|f|^q) = |\beta(f)|^q$ for each $f \in L^\infty(\mu)$. By assumption, the composition $I_m \circ \alpha \circ \beta^{-1} : C(K) \to X$ is $q$-summing so a factorization theorem of Pietsch \[13, Corollary 2.15\] implies that there exists a finite regular Borel measure $\lambda : \mathcal{B}(K) \to [0, \infty)$ satisfying

$$\|(I_m \circ \alpha \circ \beta^{-1}(\psi))\|_X \leq \left( \int_K |\psi|^q \, d\lambda \right)^{1/q}, \quad \forall \psi \in C(K).$$

Let $\xi \in L^\infty(\mu)^*$ denote the positive linear functional corresponding to $\lambda \in M(K) = C(K)^*$ via $\beta^*$, i.e., $\xi := \beta^*(\lambda)$ where $\beta^* \in \mathcal{L}(C(K)^*, L^\infty(\mu)^*)$ is the dual operator of $\beta \in \mathcal{L}(L^\infty(\mu), C(K))$. Then (2.7) implies that

$$\|(I_m \circ \alpha)(f)\|_X \leq \left( \|f|^q, \xi \right)^{1/q}, \quad \forall f \in L^\infty(\mu).$$

Define a finitely additive set function $\eta_\xi : A \mapsto \langle \chi_A, \xi \rangle = \int_K \beta(\chi_A) \, d\lambda \in [0, \infty)$ for $A \in \Sigma$. Whenever $A \in \Sigma$ is $\mu$-null we have $\chi_A = 0$ in $L^\infty(\mu)$ and so $\beta(\chi_A) = 0$ in $C(K)$, i.e., $\eta_\xi(A) = 0$. So, there exist a positive finite measure $\eta_1 : \Sigma \to [0, \infty)$ and a purely finitely additive set function $\eta_2 : \Sigma \to [0, \infty)$ such that $\eta_\xi = \eta_1 + \eta_2$ on $\Sigma$ \[46, Theorem 1.23\]; for the definition of a positive purely finitely additive set function see \[46, Definition 1.13\]. Apply \[46, Theorem 1.22\] to find a decreasing sequence $\{B(n)\}_{n=1}^\infty$ in $\Sigma$ such that $\lim_{n \to \infty} \mu(B(n)) = 0 = \eta_2(B(n))$. Consequently, with $A(n) := \Omega \setminus B(n)$ for $n \in \mathbb{N}$, the increasing sequence $\{A(n)\}_{n=1}^\infty$ in $\Sigma$ has the property that $\Omega \setminus \bigcup_{n=1}^\infty A(n)$ is $\mu$-null and $\eta_2(A(n)) = 0$ for each $n \in \mathbb{N}$. For each $A \in \Sigma$ which is $\mu$-null it follows from 0 $\leq \eta_1(A) + \eta_2(A) = \eta_\xi(A) = 0$ that $\eta_1(A) = 0 = \eta_2(A)$. In particular, $\eta_1$ is absolutely continuous with respect to $\mu$, and hence there is $0 \leq h \in L^1(\mu)$ such that $\eta_1 = h \, d\mu$, i.e., $\eta_1(A) = \int_A h \, d\mu$ for $A \in \Sigma$. Via Theorem (20.33) of \[20\], the set functions $\eta_1, \eta_2$ correspond to positive linear functionals $\xi_1, \xi_2 \in L^\infty(\mu)^*$ such that $\eta_\xi(A) = \langle \chi_A, \xi_j \rangle$ for $A \in \Sigma$ and $j \in \{1, 2\}$. Of course, $\xi = \xi_1 + \xi_2$, as $\eta_\xi = \eta_1 + \eta_2$ and sim $\Sigma$ is dense in $L^\infty(\mu)$.

Fix $f \in L^\infty(\mu)$. The claim is that

$$\|(I_m \circ \alpha)(f)\|_X \leq \left( \int_\Omega |f|^q h \, d\mu \right)^{1/q}.$$
In fact, given \( n \in \mathbb{N} \), we have \( \langle |f|^q \chi_{A(n)}, \xi_2 \rangle = 0 \) because \( \xi_2 \geq 0 \) implies that
\[
0 \leq \langle |f|^q \chi_{A(n)}, \xi_2 \rangle \leq \langle |f|^q \|L^\infty(\mu)\chi_{A(n)}, \xi_2 \rangle
= \| |f|^q \|_{L^\infty(\mu)} \langle \chi_{A(n)}, \xi_2 \rangle = \| f \|_{L^\infty(\mu)}^q \eta_2(A(n)) = 0.
\]
This and (2.8) with \( f \chi_{A(n)} \) in place of \( f \) give, for each \( n \in \mathbb{N} \),
\[
\|(I_m \circ \alpha)(f \chi_{A(n)})\|_X^q \leq \langle |f|^q \chi_{A(n)}, \xi \rangle = \langle |f|^q \chi_{A(n)}, \xi_1 \rangle = \int_A |f|^q h \, d\mu.
\]
Since \( f \in L^\infty(m) \subseteq L^1(m) \), we know that \( m_f \) (cf. (2.1)) is a vector measure and it is clearly absolutely continuous with respect to \( \mu \) as \( m \simeq \mu \). Hence,
\[
\lim_{n \to \infty} \|(I_m \circ \alpha)(f - f \chi_{A(n)})\|_X = \lim_{n \to \infty} \|m_f(\Omega \setminus A(n))\|_X
= \lim_{n \to \infty} \|m_f(B(n))\|_X = 0.
\]
This gives (2.9) because \( \chi_{A(n)} \uparrow \chi_\Omega \) \( \mu \)-a.e. and
\[
\|(I_m \circ \alpha)(f)\|_X^q = \lim_{n \to \infty} \|(I_m \circ \alpha)(f \chi_{A(n)})\|_X^q
\leq \lim_{n \to \infty} \int_A |f|^q h \, d\mu = \int_\Omega |f|^q h \, d\mu.
\]
Define \( g_q := h + \chi_\Omega \in L^1(\mu)^+ \). It follows from (2.9) that \( \|(I_m \circ \alpha)(f)\|_X \leq \left( \int_\Omega |f|^q g_q d\mu \right)^{1/q} \) for each \( f \in L^\infty(\mu) \). In other words, \( I_m \circ \alpha \) admits a continuous linear extension \( T : L^q(\mu) \rightarrow X \) because \( g_q \geq 1 \) pointwise everywhere implies that \( L^\infty(\mu) = L^\infty(g_q d\mu) \subseteq L^q(\mu) \) continuously. Moreover, the continuous inclusion \( L^q(g_q d\mu) \subseteq L^1(m) \) holds by [38, Theorem 4.14] because \( T(\chi_A) = m(A) \) for \( A \in \Sigma^\prime \) and because \( m \simeq g_q d\mu \).

**Step 2.** If \( I_m \) is \( p \)-summing and \( L^\infty(\mu) \subseteq L^q(g_q d\mu) \subseteq L^1(m) \) continuously for some \( 1 \leq q < \infty \) and some everywhere strictly positive function \( g_q \in L^1(\mu) \), then \( I_m \circ \alpha : L^\infty(\mu) \rightarrow X \) is max \( \{(1, \frac{pq}{p+q})\} \)-summing.

To prove this, let \( \alpha_1 : L^\infty(\mu) \rightarrow L^q(g_q d\mu) \) and \( \alpha_2 : L^q(g_q d\mu) \rightarrow L^1(m) \) denote the corresponding natural embeddings. Then \( \alpha_1 \) is \( q \)-summing [13, Example 2.9(d)]. So, [13, Theorem 2.22] gives that \( I_m \circ \alpha_1 \) is max \( \{(1, \frac{pq}{p+q})\} \)-summing, and hence so is \( I_m \circ \alpha = (I_m \circ \alpha_1) \circ \alpha_2 \).

**Step 3.** If \( I_m : L^1(m) \rightarrow X \) is \( p \)-summing, then there is \( g_1 \in L^1(\mu) \) with \( g_1 > 0 \) pointwise everywhere such that \( L^1(g_1 d\mu) \subseteq L^1(m) \) continuously.

Indeed, since \( I_m \circ \alpha \) is \( p \)-summing, Step 1 with \( q:= p \) gives \( L^\infty(\mu) \subseteq L^p(g_p d\mu) \subseteq L^1(m) \) continuously for some \( g_p \in L^1(\mu) \) with \( g_p > 0 \) pointwise everywhere. By Step 2 with \( q:= p \) we know that \( I_m \circ \alpha \) is max \( \{1, \frac{p}{2}\} \)-summing. Again Steps 1 and 2, now with \( q:= \max\{1, \frac{p}{2}\} \), give that \( I_m \circ \alpha \) is max \( \{1, \frac{p}{3}\} \)-summing. We can continue this process to conclude that \( I_m \circ \alpha \)
is max\{1, p/n\}-summing for all \( n \in \mathbb{N} \). Selecting \( n \in \mathbb{N} \) with \( n \geq p \) shows that \( I_m \circ \alpha \) is 1-summing. Now apply Step 1 with \( q := 1 \) to obtain Step 3.

**Step 4.** If \( I_m \) is \( p \)-summing, then \( m \) has finite variation.

By Step 3, we have the continuous inclusion \( L^1(g_1 \, d\mu) \subseteq L^1(m) \) for some \( g_1 \in L^1(\mu) \) which is positive everywhere. With \( C \) denoting the operator norm of this continuous inclusion, we have

\[
\|m(A)\|_X = \|I_m(\chi_A)\|_X \leq \|\chi_A\|_{L^1(m)} \\
\leq C \int_{\Omega} \chi_A g_1 \, d\mu = C \int_{A} g_1 \, d\mu, \quad \forall A \in \Sigma,
\]

which implies that \( |m|(\Omega) < \infty \). This establishes Step 4.

Since \( m \) is an arbitrary \( X \)-valued vector measure, Step 4 implies that \( X \) is \( A_p \)-variation admissible. This completes the proof of Theorem 2.2. \( \blacksquare \)

**Corollary 2.3.** Let \( m \) be a Banach-space-valued vector measure. Then the integration map satisfies \( I_m \in A_1 \) iff \( I_m \in A_2 \).

**Proof.** Since \( A_1 \subseteq A_2 \) [13, Inclusion Theorem 2.8], we only need to consider the case when \( I_m \) is 2-summing. Suppose that \( m \) is \( X \)-valued. According to [13, Corollary 2.16], there is a probability measure \( \mu \) and continuous linear maps \( \alpha : L^1(m) \to L^2(\mu) \) and \( \beta : L^2(\mu) \to X \) such that \( I_m = \beta \circ \alpha \). But \( L^1(m) = L^1(|m|) \) by Theorem 2.2 and so Grothendieck’s Theorem [13, Theorem 3.4], [44, p. 202] implies that \( \alpha : L^1(|m|) \to L^2(\mu) \) is 1-summing. Hence, also \( I_m = \beta \circ \alpha \in A_1 \). \( \blacksquare \)

Proposition 1.1 and Theorem 2.2 show, for any vector measure \( m \) with \( I_m \in A_p \) for some \( 1 \leq p < \infty \), that \( L^1(m) = L^1(|m|) \). In certain situations a converse is possible.

**Proposition 2.4.** Let \( 2 < p < \infty \) and \( m \) be an \( \ell^p \)-valued vector measure satisfying \( L^1(m) = L^1(|m|) \). Then \( I_m \in A_r \) for all \( p < r < \infty \).

**Proof.** By a result of P. Saphar, every operator from \( \mathcal{L}(\ell^1, \ell^p) \) is \( r \)-summing for all \( p < r < \infty \) (see [12, p. 321, Corollary], for example). By the local technique lemma for operator ideals [12, p. 301], the same statement holds if we replace \( \ell^1 \) with any \( \mathcal{L}_1 \)-space, in particular, by \( L^1(|m|) \) (see also [38, Example 2.61(ii)]). Since \( I_m \in \mathcal{L}(L^1(m), \ell^p) = \mathcal{L}(L^1(|m|), \ell^p) \), it follows that \( I_m \in A_r \) for all \( p < r < \infty \). \( \blacksquare \)

From the proof, it is clear that the condition \( L^1(m) = L^1(|m|) \) in the statement of Proposition 2.4 can be replaced with the requirement that \( L^1(m) \) is an \( \mathcal{L}_1 \)-space.

**Remark 2.5.** The method of proof of Theorem 2.2 relies on three classical results, namely the Kakutani Representation Theorem, the Yosida–Hewitt Decomposition Theorem and a factorization theorem of Pietsch. We
point out that this argument can be adapted to provide a completely different proof of the result mentioned in Section 1, namely that \( L^1(m) = L^1(|m|) \) iff the integration map \( I_m \) is positive \( p \)-summing for some \( 1 \leq p < \infty \) \[5\]. The proof given in \[5\] is based on \( p \)-concavity arguments.

Concerning a proof via the methods of this paper recall, for \( 1 \leq p < \infty \) and \( X \) a Banach space, that an \( X \)-valued, continuous linear operator \( T \) defined on a (complex) Banach lattice \( W \) is called positive \( p \)-summing if there exists \( C > 0 \) such that
\[
\left( \sum_{j=1}^{n} \|Tw_j\|_X^p \right)^{1/p} \leq C \sup_{w^* \in B[W^*]} \left( \sum_{j=1}^{n} |\langle w_j, w^* \rangle|^p \right)^{1/p}
\]
whenever \( \{w_j\}_{j=1}^{n} \) is a finite set of positive elements in \( W \) and \( n \in \mathbb{N} \) \[4\] Definition 1]. For such an operator \( T : W \to X \) and for any \( W \)-valued positive operator \( S \) defined on a (complex) Banach lattice, the composition \( T \circ S \) is again positive \( p \)-summing \[3\] Proposition 1(d)]. Fix an \( X \)-valued vector measure \( m \) defined on a measurable space \( (\Omega, \Sigma) \) and let \( \mu : \Sigma \to [0, \infty) \) be a finite measure such that \( m \simeq \mu \). The natural embedding of \( L^\infty(\mu) = L^\infty(m) \) into \( L^1(m) \) is denoted by \( \alpha \). When referring to Steps 1 to 4 we mean those in the proof of Theorem 2.2.

**Step 1'.** If \( I_m \circ \alpha : L^\infty(\mu) \to X \) is positive \( q \)-summing for some \( 1 \leq q < \infty \), then there is \( g_q \in L^1(\mu) \) with \( g_q(w) > 0 \) for every \( w \in \Omega \) such that (2.6) holds with continuous inclusions.

To establish Step 1' let \( \beta : L^\infty(\mu) \to C(K) \) be the isometric Banach lattice isomorphism as in the proof of Step 1. Then \( \beta^{-1} : C(K) \to L^\infty(\mu) \) is a positive operator and hence, as noted above, \((I_m \circ \alpha) \circ \beta^{-1} : C(K) \to X \) is then positive \( q \)-summing. According to \[4\] Proposition 3] we see that \( I_m \circ \alpha \circ \beta^{-1} \) is actually \( q \)-summing, and hence so is \( I_m \circ \alpha = (I_m \circ \alpha \circ \beta^{-1}) \circ \beta \) as \( \beta \) is a positive operator. Thus, Step 1 can be applied to obtain (2.6) with continuous inclusions.

**Step 2'.** If \( I_m \) is positive \( p \)-summing and (2.6) holds continuously for some \( 1 \leq q < \infty \) and some everywhere strictly positive function \( g_q \in L^1(\mu) \), then \( I_m \circ \alpha : L^\infty(\mu) \to X \) is positive max \( \{1, \frac{pq}{p+q}\} \)-summing.

To see this, let \( \alpha_1 : L^\infty(\mu) \to L^q(g_q d\mu) \) and \( \alpha_2 : L^q(g_q d\mu) \to L^1(m) \) denote the respective embeddings determined by (2.6). As noted in the proof of Step 2, \( \alpha_1 \) is \( q \)-summing, and hence so is \( \alpha := \alpha_2 \circ \alpha_1 \). An examination of the proof of the Composition Theorem 2.22 (and Lemma 2.23) in \[13\] shows that it can be adapted to show that the composition \( I_m \circ \alpha \) is positive max \( \{1, \frac{pq}{p+q}\} \)-summing.

**Step 3'.** If \( I_m \) is positive \( p \)-summing, then there is \( g_1 \in L^1(\mu) \) with \( g_1 > 0 \) pointwise everywhere such that \( L^1(g_1 d\mu) \subseteq L^1(m) \) continuously.
The proof of Step 3’ is similar to that of Step 3, by applying Steps 1’ and 2’ repeatedly.

**STEP 4’**. If \( I_m \) is positive \( p \)-summing, then \( m \) has finite variation.

The continuous inclusion \( L^1(g_1 \, dm) \subseteq L^1(m) \), guaranteed by Step 3’, establishes Step 4’ as in the proof of Step 4.

It remains to show that \( L^1(m) = L^1(|m|) \) whenever \( I_m \) is positive \( p \)-summing for some \( 1 \leq p < \infty \). Since the positive \( p \)-summing operators do not form an operator ideal, we cannot appeal to Proposition 1.1. Now, by Lemma 2.1(i) we have \( L^1(|m|) \subseteq L^1(m) \). To prove the reverse inclusion, let \( f \in L^1(m) \). Select non-negative functions \( f^{(j)} \in L^1(m) \) for \( j = 1, \ldots, 4 \) such that \( f = f^{(1)} - f^{(2)} + i(f^{(3)} - f^{(4)}) \). For \( j \) fixed the continuous multiplication operator \( M_{f^{(j)}} : L^1(m_{f^{(j)}}) \to L^1(m) \), as given in Lemma 2.1(ii), is positive, and hence the composition \( I_{m_{f^{(j)}}} = I_m \circ M_{f^{(j)}} \) is positive \( p \)-summing. Via Step 4’, with \( m_{f^{(j)}} \) in place of \( m \), it follows that \( m_{f^{(j)}} \) has finite variation. Since \( \|m_f(A)\|_X \leq \sum_{j=1}^4 \|m_{f^{(j)}}(A)\|_X \) for \( A \in \Sigma \), it follows that \( m_f \) also has finite variation. So, \( f \in L^1(|m|) \) (cf. Lemma 2.1(i)). Thus, \( L^1(m) \subseteq L^1(|m|) \).

This establishes that \( L^1(m) = L^1(|m|) \).

As already noted in Section 1, the operator ideal \( \mathcal{A}_c \) has the property that every Banach space is \( \mathcal{A}_c \)-admissible [36, Theorem 4]. According to Theorem 2.2, the same is true for the operator ideal \( \mathcal{A}_p \), \( 1 \leq p < \infty \). We now show that there exist vector measures \( m \) for which \( I_m \in \mathcal{A}_c \setminus \mathcal{A}_p \) and others for which \( I_m \in \mathcal{A}_p \setminus \mathcal{A}_c \).

**Example 2.6.** Let \( G \) be any infinite compact abelian group with normalized Haar measure \( \mu \). For each \( 1 \leq p < \infty \) and each regular, complex Borel measure \( \lambda \in M(G) \), the linear operator \( C^{(p)}_\lambda \) of convolution with \( \lambda \) belongs to \( \mathcal{L}(L^p(G)) \) where, for \( f \in L^p(G) := L^p(\mu) \), we have

\[
C^{(p)}_\lambda(f) : x \mapsto \int_G f(x-y) \, d\lambda(y) \quad \mu\text{-a.e. } x \in G.
\]

Indeed, \( \|C^{(p)}_\lambda\|_{op} \leq |\lambda|(G) \). The operator \( C^{(p)}_\lambda \) induces the vector measure \( m^{(p)}_\lambda : \mathcal{B}(G) \to L^p(G) \) defined by

\[
m^{(p)}_\lambda : A \mapsto C^{(p)}_\lambda(\chi_A) = \chi_A * \lambda, \quad \forall A \in \mathcal{B}(G).
\]

It is known that \( m^{(p)}_\lambda \simeq \mu \) (provided \( \lambda \neq 0 \)) and, with continuous (natural) inclusions, that \( L^p(G) \subseteq L^1(m^{(p)}_\lambda) \subseteq L^1(G) \). Moreover, the integration map \( I_{m^{(p)}_\lambda} : L^1(m^{(p)}_\lambda) \to L^p(G) \) is also given by convolution, i.e.,

\[
I_{m^{(p)}_\lambda}(f) = f * \lambda, \quad \forall f \in L^1(m^{(p)}_\lambda).
\]
For \( \lambda \ll \mu \), i.e., there exists \( g \in L^1(G) \) such that \( \lambda(A) = \int_A g \, d\mu \) for \( A \in \mathcal{B}(G) \), we write \( g \, d\mu \) for \( \lambda \) and \( C_{g}^{(p)} \) (resp. \( m_{g}^{(p)} \)) for \( C_{\lambda}^{(p)} \) (resp. \( m_{\lambda}^{(p)} \)). All of the above claims can be found in [38, Ch. 7, §7.4], for example. The following characterization occurs in Theorem 7.67 of [38].

**FACT.** Let \( 1 < p < \infty \) and \( \lambda \in M(G) \setminus \{0\} \). The following assertions are equivalent:

(i) There exists \( g \in L^p(G) \) such that \( \lambda = g \, d\mu \).

(ii) The integration map \( I_{m_{\lambda}^{(p)}} : L^1(m_{\lambda}^{(p)}) \to L^p(G) \) is compact.

(iii) \( L^1(m_{\lambda}^{(p)}) = L^1(|m_{\lambda}^{(p)}|) = L^1(G) \).

Concerning 1-summing operators we require the following result.

**Lemma 2.7.** Let \( 1 \leq p < 2 \) and \( g \in L^p(G) \). Then \( L^1(m_{g}^{(p)}) = L^1(G) \), and the integration map \( I_{m_{g}^{(p)}} : L^1(m_{g}^{(p)}) \to L^p(G) \) is 1-summing iff \( g \in L^2(G) \).

**Proof.** In view of (2.10) and the above Fact we see that \( L^1(m_{g}^{(p)}) = L^1(G) \) and that \( I_{m_{g}^{(p)}} \) is precisely the bounded operator \( C_g^{(1,p)} : L^1(G) \to L^p(G) \) of convolution with \( g \).

Suppose that \( g \in L^2(G) \). Let \( J^{(2,p)} : L^2(G) \to L^p(G) \) denote the natural inclusion and \( C_{g}^{(1,2)} : L^1(G) \to L^2(G) \) be the bounded operator of convolution with \( g \). Since \( C_{g}^{(1,2)} \) is necessarily 1-summing [13, Theorem 3.4], so is the composition \( I_{m_{g}^{(p)}} = C_g^{(1,p)} = J^{(2,p)} \circ C_{g}^{(1,2)} \).

Conversely, suppose that \( C_g^{(1,p)} \) is 1-summing. Since \( g \in L^p(G) \), the set \( S(\hat{g}) := \{ \gamma \in \Gamma : \hat{g}(\gamma) \neq 0 \} \) is countable, where \( \Gamma \) is the dual group of \( G \) and \( \hat{g} \) is the Fourier transform of \( g \), i.e., \( \hat{g}(\gamma) := \int_G (x, \gamma)g(x) \, d\mu(x) \) for \( \gamma \in \Gamma \), with \((x, \gamma)\) denoting the value of the character \( \gamma \) at \( x \in G \). The trigonometric monomial \( x \mapsto (x, \gamma) \) on \( G \) is denoted by \((\cdot, \gamma)\). Since \( \sum_{\gamma \in S(\hat{g})} |(\cdot, \gamma)\varphi|^2 = \sum_{\gamma \in S(\hat{g})} |(\varphi)(\gamma)|^2 < \infty \) for each \( \varphi \in L^\infty(G) = L^1(G)^* \subseteq L^2(G) \), the sequence \( \{(\cdot, \gamma) : \gamma \in S(\hat{g})\} \subseteq L^1(G) \) is weakly 2-summable in \( L^1(G) \) (cf. [13, p. 32] for the definition). But \( C_{g}^{(1,p)} \) is also 2-summable [13, Inclusion Theorem 2.8], and hence \( C_{g}^{(1,p)} \) maps \( \{(\cdot, \gamma) : \gamma \in S(\hat{g})\} \) to a norm 2-summable sequence in \( L^p(G) \) [13, Proposition 2.1], i.e.,

\[
\sum_{\gamma \in S(\hat{g})} |\hat{g}(\gamma)|^2 = \sum_{\gamma \in S(\hat{g})} \|\hat{g}(\cdot, \gamma)\|_{L^p(G)}^2 = \sum_{\gamma \in S(\hat{g})} \|C_{g}^{(1,p)}((\cdot, \gamma))\|_{L^p(G)}^2 < \infty.
\]

Hence, \( \hat{g} \in \ell^2(\Gamma) \), i.e., \( g \in L^2(G) \). \( \blacksquare \)

It follows from the above Fact and Lemma 2.7 that, for every \( 1 < p < 2 \) and \( g \in L^p(G) \setminus L^2(G) \) (such functions \( g \) exist as \( \mu \) is non-atomic [38].
Lemma 7.97], the integration map \( I_{m_g^{(p)}} \) is compact, but fails to be 1-summing.

Consider now \( p := \infty \), i.e., \( X = L^\infty(G) \), and the finitely additive \( L^\infty(G) \)-valued set function \( m^{(\infty)}_\lambda : A \mapsto C^{(\infty)}_\lambda(\chi_A) \) for \( A \in \mathcal{B}(G) \), where \( C^{(\infty)}_\lambda \in \mathcal{L}(L^\infty(G)) \) is again the operator of convolution with \( \lambda \in M(G) \). Then \( m^{(\infty)}_\lambda \) is norm \( \sigma \)-additive (i.e., a vector measure) iff \( \lambda = g \, d\mu \) for some \( g \in L^1(G) \) [34, Theorem 1], in which case the integration map \( I_{m_g^{(\infty)}} \) is compact iff \( g \in C(G) \) [34, Corollary 3]. On the other hand, \( I_{m_g^{(\infty)}} \) is 1-summing iff \( \hat{g} \in \ell^1(\Gamma) \) [34, Proposition 4], i.e., iff \( g \in A(G) \) in the notation of Example 3.5(ii) below. So, \( I_{m_g^{(\infty)}} \) is compact but fails to be 1-summing whenever \( g \in C(G) \setminus A(G) \). That the inclusion \( A(G) \subsetneq C(G) \) is always proper is known [19, Theorem (37.4)].

In the proof of Lemma 2.7, it was observed that the integration map \( I_{m_g^{(p)}} : L^1(G) \to L^p(G) \) is the continuous convolution operator \( C_g^{(1,p)} : f \mapsto f * g \) from \( L^1(G) \) into \( L^p(G) \). A detailed study of those convolution operators belonging to \( \mathcal{L}(L^p(G)) \) and to \( \mathcal{L}(L^1(G), L^p(G)) \), for \( 1 \leq p \leq \infty \), and which are 1-summing appears in [34]. For the latter case, i.e., for \( \mathcal{L}(L^1(G), L^p(G)) \), such convolution operators always arise as the integration map of an \( L^p(G) \)-valued vector measure.

We now present some examples of \( m \) with \( I_m \in A_1 \setminus A_c \).

**Example 2.8.** (i) Let \((\Omega, \Sigma, \mu)\) be a finite positive measure space for which there exists an infinite partition \( \{A(n)\}_{n=1}^\infty \subseteq \Sigma \) of \( \Omega \) with \( \mu(A(n)) > 0 \) for \( n \in \mathbb{N} \). Let \( X \) be any infinite-dimensional Hilbert space. According to Lemma 2.5 in [36] (see also its proof), there exists a sequence \( \{x_n\}_{n=1}^\infty \subseteq X \) of unit vectors which is not a relatively compact subset of \( X \) and a finite set \( \mathcal{F} \subseteq X^* \) such that

\[
1 = \|x_n\|_X \leq \sum_{x^* \in \mathcal{F}} |\langle x_n, x^* \rangle|, \quad \forall n \in \mathbb{N}.
\]

Define \( H : \Omega \to X \) by \( H(w) := \sum_{n=1}^\infty \chi_{A(n)}(w)x_n \) for \( w \in \Omega \), in which case \( H \) is strongly measurable (its range is separable) and bounded, as \( \|H(w)\| = \sum_{n=1}^\infty \chi_{A(n)}(w) = 1 \) for \( w \in \Omega \). Hence, \( H \) is Bochner \( \mu \)-integrable and so the vector measure \( m : \Sigma \to X \) defined by \( m(A) := \int_A H \, d\mu \) for \( A \in \Sigma \) has finite variation. Indeed, \( |m| = \mu \) since \( |m|(A) = \int_A \|H(w)\| \, d\mu(w) \) for \( A \in \Sigma \) [14, Ch. II, Theorem 1.4]. For each \( x^* \in \mathcal{F} \) and \( A \in \Sigma \) we have

\[
|\langle m, x^* \rangle|(A) = \int_A |\langle H(w), x^* \rangle| \, d\mu(w) = \sum_{n=1}^\infty |\langle x_n, x^* \rangle| \mu(A \cap A(n)),
\]

and hence
\[
\sum_{x^* \in F} |\langle m, x^* \rangle|(A) = \sum_{n=1}^{\infty} \mu(A \cap A(n)) \sum_{x^* \in F} |\langle x_n, x^* \rangle| = \sum_{n=1}^{\infty} \alpha_n \mu(A \cap A(n)),
\]
where \(\alpha_n := \sum_{x^* \in F} |\langle x_n, x^* \rangle| \geq 1\) for all \(n \in \mathbb{N}\) (cf. (2.11)). Accordingly,

\[
|m|(A) = \mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A(n)) \leq \sum_{n=1}^{\infty} \alpha_n \mu(A \cap A(n)) = \sum_{x^* \in F} |\langle m, x^* \rangle|(A)
\]
for each \(A \in \Sigma\), and so \(L^1(m) = L^1(|m|) = L^1(\mu)\) \cite{36} Lemma 2.6(i).

Observe that \(\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty} \subseteq \mathbb{B}[L^1(\mu)] = \mathbb{B}[L^1(|m|)]\) with \(\mathbb{B}[L^1(|m|)] \subseteq \mathbb{B}[L^1(m)]\) (see Lemma 2.1(i)). Since

\[
I_m \left(\frac{1}{\mu(A(n))}\chi_{A(n)}\right) = \frac{1}{\mu(A(n))} \int_{A(n)} H \, d\mu = x_n, \quad \forall n \in \mathbb{N},
\]
and \(\{x_n\}_{n=1}^{\infty}\) is not relatively compact, it follows that the integration map \(I_m\) fails to be compact. However, since \(L^1(m) = L^1(\mu)\) with equivalence of norms, \(I_m : L^1(m) \to X\) is surely 1-summing by Grothendieck’s Theorem \cite{44} p. 202. So, \(I_m \in A_1 \setminus A_c\).

(ii) Let the measure space \((\Omega, \Sigma, \mu)\) and the partition \(\{A(n)\}_{n=1}^{\infty}\) of \(\Omega\) be as in (i) above. Define a vector measure \(m : \Sigma \to \ell^1\) by

\[
(2.12) \quad m(A) := \mu(A) f_1 + \sum_{n=1}^{\infty} \mu(A \cap A(n)) f_{n+1}, \quad \forall A \in \Sigma,
\]
where \(\{f_n\}_{n=1}^{\infty}\) is the canonical basis of \(\ell^1\), and observe that \(m\) is precisely the vector measure of Example 3.7 in \cite{33}. According to Proposition 3.5 of \cite{33} we have \(L^1(m) = L^1(\mu)\), in which case \(L^1(\mu) = L^1(|m|)\) with equivalent norms \cite{38} Lemma 3.14. As noted in Example 3.7 of \cite{33} the integration map \(I_m : L^1(m) \to \ell^1\) is not compact. Since \(\ell^1\) has the Schur property, \(I_m\) also fails to be weakly compact, and hence is not 1-summing \cite{13} Theorem 2.17.

For \(2 \leq p < \infty\), let \(j^{(1,p)} : \ell^1 \to \ell^p\) be the natural embedding. Then \(m_p := j^{(1,p)} \circ m\) is an \(\ell^p\)-valued vector measure on \(\Sigma\). It is clear from (2.12) that \(m_p, m_p\) and \(\mu\) all have the same null sets. Since \(j^{(1,p)}\) is injective, it follows that \(L^1(m) \subseteq L^1(m_p)\) \cite{38} Lemma 3.27. Suppose that \(f \in L^1(m_p)\). For \(e^*_i := (1, 0, 0, \ldots) \in (\ell^p)^*\) we have \(\int_{\Omega} |f| \, d\mu = \int_{\Omega} |f| \, d\langle m_p, e^*_i \rangle < \infty\), and so \(f \in L^1(\mu) = L^1(m)\). Accordingly, \(L^1(m) = L^1(m_p) = L^1(\mu)\) with equivalence of norms, for each \(2 \leq p < \infty\).

Now, \(I_{m_2} : L^1(m_2) \to \ell^2\) is 1-summing by Grothendieck’s Theorem (see (i) above). Let \(j^{(2,p)} : \ell^2 \to \ell^p\) be the natural inclusion. Then \(I_{m_p} = j^{(2,p)} \circ I_{m_2}\), and so \(I_{m_p}\) is 1-summing for every \(2 \leq p < \infty\).

Observe that \(\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty} \subseteq \mathbb{B}[L^1(\mu)]\) and so \(\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty}\) is also contained in a multiple of \(\mathbb{B}[L^1(m_p)]\). It follows from the formula
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\[ I_{m_p}(f) = \left( \int f \, d\mu, \int f \, d\mu, \int f \, d\mu, \ldots \right), \quad \forall f \in L^1(m_p), \]

that \( I_{m_p}(\frac{1}{\mu(A(n))} \chi A(n)) = e_1 + e_{n+1} \) for \( n \in \mathbb{N} \), where \( \{e_n\}_{n=1}^{\infty} \) is the canonical basis of \( \ell^p \). Since \( \{e_1 + e_n\}_{n=1}^{\infty} \) is not relatively compact in \( \ell^p \), we conclude that \( I_{m_p} \) is not compact, for each \( 2 \leq p < \infty \).

**Example 2.9.** Let \((\Omega, \Sigma, \mu)\) be a finite positive measure space and \(\{A(n)\}_{n=1}^{\infty}\) be a partition of \(\Omega\) as in Example 2.8(i). Fix any \(2 < p < \infty\). We exhibit a vector measure \(m : \Sigma \to \ell^p\) such that

\[ I_m \in \left( \bigcap_{p<r<\infty} A_r \right) \setminus (A_p \cup A_c). \tag{2.13} \]

The canonical unit basis of \(\ell^1\) (resp. \(\ell^p\)) is denoted by \(\{f_n\}_{n=1}^{\infty}\) (resp. \(\{e_n\}_{n=1}^{\infty}\)). Select any operator \(T \in \mathcal{L}(\ell^1, \ell^p)\) which is not \(p\)-summing \([23]\) Lemma 4.1]; a concrete construction of such a \(T\) is presented in \([24], \S 4]\). Since \(\ell^p\) is a separable Banach space, there is a surjective operator \(Q \in \mathcal{L}(\ell^1, \ell^p)\) \([12, Corollary 24.6]\). The surjective linear operator \(Q \circ P \in \mathcal{L}(L^1(\mu), \ell^p)\) is \(r\)-summing for such \(r\) \([12] Corollary 24.6\]. The surjective linear operator \(P : L^1(\mu) \to \ell^1\) given by

\[ P(h) := \sum_{n=1}^{\infty} (\int_{A(n)} h \, d\mu) f_n \quad \text{for} \ h \in L^1(\mu) \]

is continuous, and hence \(Q \circ P \in \mathcal{L}(L^1(\mu), \ell^p)\) is \(r\)-summing for all \(p < r < \infty\). But, being a surjection, \(Q \circ P\) is not \(p\)-summing as \(T\) is not \(p\)-summing. On the other hand, \(Q \circ P\) is \(r\)-summing for all \(p < r < \infty\) because every operator in \(\mathcal{L}(\ell^1, \ell^p)\) is \(r\)-summing for such \(r\) \([12, Corollary 24.6]\]. Let \(R \in \mathcal{L}(\ell^p)\) denote the forward shift operator, i.e., \(R(\sum_{n=1}^{\infty} a_n e_n) := \sum_{n=1}^{\infty} a_n e_{n+1}\) for \(\sum_{n=1}^{\infty} a_n e_n \in \ell^p\). Since \(R \circ Q \circ P\) is continuous and \(L^1(\mu)\) has \(\sigma\)-order continuous norm, it follows that \(m : \Sigma \to \ell^p\) defined by

\[ m(A) := \mu(A) e_1 + (R \circ Q \circ P)(\chi_A), \quad \forall A \in \Sigma, \]

is a vector measure. Observing that the range of \(R \circ Q \circ P\) lies in \(\overline{\text{span}}(\{e_n\}_{n=2}^{\infty}) \subseteq \ell^p\), it follows that \(m \simeq \mu\). Moreover, \(L^1(\mu) \subseteq L^1(m)\) as

\[ \int_A f \, d\mu = \left( \int_A f \, d\mu \right) e_1 + (R \circ Q \circ P)(f \chi_A), \quad \forall f \in L^1(\mu), A \in \Sigma. \]

Let \(e_1^*\) denote the continuous linear functional \(\sum_{n=1}^{\infty} \psi(n) e_n \mapsto \psi(1)\) on \(\ell^p\). Then \(L^1(m) = L^1(\langle m, e_1^* \rangle) = L^1(\mu)\). Hence, \(L^1(m) = L^1(\mu)\) as isomorphic Banach spaces with

\[ I_m(f) = I_\mu(f) e_1 + (R \circ Q \circ P)(f), \quad \forall f \in L^1(m). \]

Since \(f \mapsto I_\mu(f) e_1\) is a rank-1 operator in \(\mathcal{L}(L^1(m), \ell^p)\) and \(R\) is a linear isometry onto its (closed) range \(\overline{\text{span}}(\{e_n\}_{n=2}^{\infty}) \subseteq \ell^p\), it follows that \(I_m\) is
r-summing for each \( p < r < \infty \), but \( I_m \) is neither \( p \)-summing nor is it compact. That is, (2.13) holds.

To conclude this section, let us point out that there exist vector measures \( m \) satisfying \( L^1(m) = L^1(|m|) \) and \( I_m \) is neither compact nor \( 1 \)-summing.

For \( 1 < r < \infty \), consider the Volterra vector measure \( m_r : \mathcal{B}([0,1]) \to L^r([0,1]) \) (see Section 1). Then \( m_r \) has finite variation with \( L^1(|m_r|) \subseteq L^1(m_r) \) \cite{38} Example 3.26. Moreover, \( I_{m_r} \) is not compact \cite{38} p. 154. Since \( I_{m_r} \) fails to be completely continuous \cite{38} Proposition 3.52, it fails to be \( p \)-summing for every \( 1 \leq p < \infty \) \cite{13} Theorem 2.17. Consider now \( r \in \{1, \infty\} \), in which case the Volterra measure \( m_r \) is defined by the same formula as for \( 1 < r < \infty \) given in Section 1. According to Example 3.26 in \cite{38} we have \( L^1(m_1) = L^1(|m_1|) \) and \( L^1(|m_\infty|) = L^1(m_\infty) \), which implies that \( I_{m_r} \in \mathcal{A}_{cc} \). Indeed, the range \( m_r(\mathcal{B}([0,1])) \) of \( m_r \) is relatively compact because the classical Volterra integral operator \( V_r \) from \( L^r([0,1]) \) into itself is compact \cite{38} pp. 113–114. Since \( L^1(m_1) = L^1(|m_1|) \), the complete continuity of \( I_{m_1} = V_r \) follows from \cite{38} Corollary 2.42. However, since both \( I_{m_1} \) and \( I_{m_\infty} \) fail to be weakly compact \cite{38} Example 3.49(iv), they also fail to be compact and \( f \)-summing for every \( 1 \leq p < \infty \) \cite{13} Theorem 2.17.

Or, let \( 0 < \alpha < 1 \) and consider the Sobolev vector measure \( m : \mathcal{B}([0,1]) \to L^\infty([0,1]) \) defined by

\[
(2.14) \quad m(A) : t \mapsto \int_0^1 \chi_A(s)s^{-\alpha} \, ds, \quad \forall t \in [0,1], \ A \in \mathcal{B}([0,1]).
\]

Then \( L^1(m) = L^1(|m|) \) \cite{11} Proposition 2.1, and the integration map \( I_m : L^1(m) \to L^\infty([0,1]) \) fails to be weakly compact \cite{11} Proposition 2.2. By an argument as for the Volterra measures with \( r \in \{1, \infty\} \) we can conclude that \( I_m \) is not compact and not \( p \)-summing, for every \( 1 \leq p < \infty \). Suppose that \( X \) is any rearrangement invariant Banach function space on \([0,1]\) which is not isomorphic to \( L^\infty([0,1]) \). Since \( L^\infty([0,1]) \) imbeds continuously into \( X \), the same formula (2.14) specifies an \( X \)-valued vector measure, denoted by \( m_X \), which has finite variation \cite{11} Proposition 3.1, and whose integration map \( I_{m_X} : L^1(m_X) \to X \) is not compact \cite{11} Proposition 3.6. If \( X \) happens to be a Lorentz \( \Lambda_\varphi \)-space with \( \varphi \) an increasing concave function on \([0,1]\) such that \( \varphi(0) = 0 \) (and with the norm \( \|f\|_{\Lambda_\varphi} := \int_0^1 f^+(s) \, d\varphi(s) \), where \( f^+ \) is the decreasing rearrangement of \( f \)), then actually \( L^1(m_{\Lambda_\varphi}) = L^1(|m_{\Lambda_\varphi}|) \) and \( I_{m_{\Lambda_\varphi}} \) fails to be weakly compact \cite{11} Proposition 3.8 & Corollary 4.3. Arguing as above we can conclude that \( I_{m_{\Lambda_\varphi}} \) is not \( p \)-summing, for every \( 1 \leq p < \infty \).

Finally, consider the vector measure \( m : \Sigma \to \ell^1 \) occurring in (2.12) in Example 2.8(ii), where it was observed that \( L^1(m) = L^1(|m|) \) and \( I_m : L^1(m) \to \ell^1 \) is not weakly compact. Hence, \( I_m \) is not compact and not \( p \)-summing, for every \( 1 \leq p < \infty \).
3. Proof of Theorem 1.2. The proof of Theorem 1.2 proceeds via a series of lemmata. We begin with a result from the realm of Banach space theory. Sometimes we will express sequences of scalars (i.e., elements of \( \mathbb{C}^n \)) as functions defined on \( \mathbb{N} \). Vectors from the finite-dimensional space \( \mathbb{R}^N \) are denoted by \( (\xi_n)_{n=1}^N \).

**Lemma 3.1.** Let \( X \) be a Banach space, \( K_n \subseteq \mathbb{B}[X] \) for \( n \in \mathbb{N} \) be a sequence of non-empty compact sets, and \( \delta > 0 \) be such that

\[
\delta \sum_{n=1}^N |a_n| \leq \sup \left\{ \left\| \sum_{n=1}^N a_n x_n \right\|_X : x_n \in K_n, n = 1, \ldots, N \right\}
\]

for all choices of \( N \in \mathbb{N} \) and \( \{a_n : n = 1, \ldots, N\} \subseteq \mathbb{C} \). Then there exists \( x_0^* \in \mathbb{B}[X^*] \) such that

\[
\limsup_{n \to \infty} (\sup_{x \in K_n} |\langle x, x_0^* \rangle|) > 0.
\]

**Proof.** Define a set of real sequences by

\[
W := \{ (\varepsilon_n |\langle x_n, x^* \rangle|)_{n=1}^\infty \in \mathbb{R}^N : x^* \in \mathbb{B}[X^*], \varepsilon_n \in \{-1, 1\}, x_n \in K_n, \forall n \in \mathbb{N}\}.
\]

Then \( W \subseteq \mathbb{B}[\ell^\infty] \) and the convex hull \( \text{co}(W) \) of \( W \) in \( \ell^\infty \) also consists of real sequences belonging to \( \mathbb{B}[\ell^\infty] \). Clearly \( \text{co}(W) \neq \emptyset \).

**Step 1.** For each \( N \in \mathbb{N} \), there exists \( \varphi_N \in \text{co}(W) \) satisfying

\[
\varphi_N(n) > \delta/2, \quad \forall n = 1, \ldots, N.
\]

To see this, fix \( N \in \mathbb{N} \), define an \( \mathbb{R} \)-linear map \( \Phi_N : \ell^\infty \to \mathbb{R}^N \) by \( \Phi_N(\psi) := (\text{Re}(\psi(n)))_{n=1}^N \in \mathbb{R}^N \) for \( \psi \in \ell^\infty \), and set \( U_N := \Phi(\text{co}(W)) \subseteq \mathbb{R}^N \). Let

\[
V_N := \{ (\xi_n)_{n=1}^N \in \mathbb{R}^N : \xi_n > \delta/2, \forall n = 1, \ldots, N\}.
\]

Now, suppose that the conclusion of Step 1 is not valid, i.e., for every \( \varphi \in \text{co}(W) \) we have \( \varphi(n) \leq \delta/2 \) for some \( n \in \{1, \ldots, N\} \), depending on \( \varphi \). Then \( U_N \cap V_N = \emptyset \). Since \( U_N \neq \emptyset \) is convex and \( V_N \neq \emptyset \) is convex and open, there exist \( (a_n)_{n=1}^N \in (\mathbb{R}^N)^* \) and \( r \in \mathbb{R} \) such that

\[
\sum_{n=1}^N a_n \xi_n < r < \sum_{n=1}^N a_n \eta_n, \quad \forall (\xi_n)_{n=1}^N \in U_N, (\eta_n)_{n=1}^N \in V_N.
\]

**First Separation Theorem, p. 130.** We claim that also

\[
\sum_{n=1}^N |a_n| \xi_n \leq r, \quad \forall (\xi_n)_{n=1}^N \in U_N.
\]

In fact, fix \( (\xi_n)_{n=1}^N \in U_N \) and select \( \varphi \in \text{co}(W) \) satisfying \( \varphi(n) = \xi_n \) for \( n = 1, \ldots, N \). Choose \( j_0 \in \mathbb{N} \), \( b_k \in [0, 1] \) with \( \sum_{k=1}^{j_0} b_k = 1 \), and \( \varphi_k \in W \) for \( k = 1, \ldots, j_0 \) such that \( \varphi = \sum_{k=1}^{j_0} b_k \varphi_k \). Furthermore, there exist \( \varepsilon_n \in \mathbb{C} \) for }
\{-1, 1\} for \( n = 1, \ldots, N \) satisfying \( \epsilon_n a_n = |a_n| \). For each \( k = 1, \ldots, j_0 \) define a function \( \psi_k \) on \( \mathbb{N} \) by

\[
\psi_k(i) := \begin{cases} 
\epsilon_i \varphi_k(i) & \text{for } i \in \{1, \ldots, N\}, \\
\varphi_k(i) & \text{for } i > N.
\end{cases}
\]

It is routine to check from the definition of \( W \) that \( \{\psi_1, \ldots, \psi_{j_0}\} \subseteq W \). Moreover, direct calculation yields \( \sum_{n=1}^{N} |a_n| \xi_n = \sum_{n=1}^{N} |a_n| (\sum_{k=1}^{j_0} b_k \psi_k)(n) \).

Since \( \psi := \sum_{k=1}^{j_0} b_k \psi_k \in \text{co}(W) \), it follows from the left inequality in (3.3), applied to \( (\psi(n))_{n=1}^{N} \in U_N \), that \( \sum_{n=1}^{N} |a_n| \xi_n \leq r \). This establishes (3.4).

Next we claim that

(3.5) \[ \sup \left\{ \left\| \sum_{n=1}^{N} a_n x_n \right\|_X : x_n \in K_n, \ n = 1, \ldots, N \right\} \leq r. \]

Indeed, fix any choice of \( x_n \in K_n \) for \( n = 1, \ldots, N \). For each \( x^* \in \mathbb{B}[X^*] \) it follows from the fact that \( |\langle x_n, x^* \rangle|_{n=1}^{N} \in U_N \) and (3.4) that \( \sum_{n=1}^{N} |a_n| \cdot |\langle x_n, x^* \rangle| \leq r \). Accordingly,

\[
\left\| \sum_{n=1}^{N} a_n x_n \right\|_X \leq \sup_{x^* \in \mathbb{B}[X^*]} \sum_{n=1}^{N} |a_n| \cdot |\langle x_n, x^* \rangle| \leq r.
\]

Since \( x_n \in K_n \) for \( n = 1, \ldots, N \) are arbitrary, this establishes (3.5).

It follows from (3.1) and (3.5) that \( \delta \sum_{n=1}^{N} |a_n| \leq r \). On the other hand, since \( (3\delta/4)(1, \ldots, 1) \in V_N \), from the right inequality in (3.3) we have

\[
r < \sum_{n=1}^{N} 3\delta a_n / 4 < \delta \sum_{n=1}^{N} |a_n|,
\]

which contradicts \( \delta \sum_{n=1}^{N} |a_n| \leq r \). Hence, Step 1 is established.

**Step 2.** Let \( \overline{W}^{\sigma^*} \) denote the closure of \( W \) in \( \ell^\infty \) with respect to the weak-* topology \( \sigma(\ell^\infty, \ell^1) \). Then \( (\overline{W}^{\sigma^*}) \setminus c_0 \) is non-empty, where \( c_0 \) is considered as a closed subspace of \( \ell^\infty \).

To establish Step 2, first observe that \( \mathbb{B}[\ell^\infty] \) is compact for the weak-* topology \([28\text{ Theorem 2.6.18}].\) Recalling that \( W \subseteq \mathbb{B}[\ell^\infty] \) it follows that \( \overline{W}^{\sigma^*} \subseteq \mathbb{B}[\ell^\infty] \), and hence \( \overline{W}^{\sigma^*} \) is also weak-* compact. Proceeding by contradiction, suppose that \( (\overline{W}^{\sigma^*}) \setminus c_0 = \emptyset \), i.e., \( \overline{W}^{\sigma^*} \subseteq c_0 \). Since the weak topology \( \sigma(c_0, \ell^1) \) on \( c_0 \) is that induced by the weak-* topology of \( \ell^\infty \), it follows that \( \overline{W}^{\sigma^*} \) is weakly compact in \( c_0 \), and hence so is its closed convex hull \( \overline{\text{co}}(\overline{W}^{\sigma^*}) \) \([28\text{ Theorem 2.8.14}].\) Choose a sequence \( \{\varphi_N\}_{N=1}^\infty \subseteq \text{co}(W) \subseteq \overline{\text{co}}(\overline{W}^{\sigma^*}) \) according to Step 1, which then admits a subsequence converging weakly in \( c_0 \) (hence also pointwise on \( \mathbb{N} \)) to some element \( \varphi \in c_0 \) \([28\text{ Theorem 2.8.6}].\)

It follows that \( \varphi(n) \geq \delta/2 \) for each \( n \in \mathbb{N} \) because \( \varphi_N(n) > \delta/2 \) for \( 1 \leq n \leq N \) whenever \( N \in \mathbb{N} \). This is impossible as \( \varphi \in c_0 \). So, we must have \( (\overline{W}^{\sigma^*}) \setminus c_0 \neq \emptyset \).
Step 3. There exist $c > 0$, a vector $x_0^* \in B[X^*]$ and an infinite subset $\Delta \subseteq \mathbb{N}$ such that

$$\sup_{x \in K_n} |\langle x, x_0^* \rangle| \geq c, \quad \forall n \in \Delta.$$

To see this, use Step 2 to select $\psi \in W^* \setminus c_0$. Since the Banach space $\ell^1$ is separable, its dual unit ball $B[\ell^\infty]$ is metrizable for the weak-* topology [28, Corollary 2.6.20]. Recalling that $W^* \subseteq B[\ell^\infty]$ enables us to choose a sequence $\{\psi_j\}_{j=1}^\infty \subseteq W$ which converges weak-* to $\psi$. As $\psi \in \ell^\infty \setminus c_0$, there is $c > 0$ such that $\Delta := \{n \in \mathbb{N} : |\psi(n)| > c\}$ is an infinite subset of $\mathbb{N}$. By the definition of $W$, given $j \in \mathbb{N}$ there exist $\varepsilon_{j,n} \in \{-1,1\}$, $x_{j,n} \in K_n$ and $x_{j,n}^* \in B[X^*]$ such that

$$\psi_j(n) = \varepsilon_{j,n}|\langle x_{j,n}, x_{j,n}^* \rangle|, \quad \forall n \in \mathbb{N}.$$ 

Now, the closed subspace $Y := \text{span}(\bigcup_{n=1}^\infty K_n)$ of $X$ is separable because each set $K_n$ for $n \in \mathbb{N}$ is compact; this follows routinely from [28, Theorem 1.12.15]. Accordingly, $B[Y^*]$ is compact and metrizable for the weak-* topology $\sigma(Y^*, Y)$ [28, Theorem 2.6.18 & Corollary 2.6.20]. Since the restrictions $y_j^* := x_{j,n}^*|_Y$ for $j \in \mathbb{N}$ belong to $B[Y^*]$, there is a subsequence $\{y_{j(k)}^*\}_{k=1}^\infty$ of $\{y_j^*\}_{j=1}^\infty$ which admits a weak-* limit $y_0^* \in B[Y^*]$. In particular, $\lim_{k \to \infty} \langle y_{j(k)}, y_{j(k)}^* \rangle = \langle y, y_0^* \rangle$ for all $y \in Y$. We claim that

$$\sup_{y \in K_n} |\langle y, y_0^* \rangle| \geq c, \quad \forall n \in \Delta. \quad (3.6)$$

Indeed, fix $n \in \Delta$. Then, $K_n$ being also compact in $Y$, the bounded sequence $\{y_{j(k)}^*\}_{k=1}^\infty \subseteq B[Y^*]$, which converges pointwise on $Y$ to $y_0^*$, also converges uniformly over $K_n$ to $y_0^*$ [22, Banach Steinhaus Theorem, p. 220]. In other words, the seminorm

$$p_n(y^*) := \sup_{y \in K_n} |\langle y, y^* \rangle|, \quad \forall y^* \in Y^*,$$

satisfies $\lim_{k \to \infty} p_n(y_{j(k)}^* - y_0^*) = 0$. On the other hand, since the subsequence $\{\psi_j(k)\}_{k=1}^\infty$ converges weak-* to $\psi$, it follows that $\lim_{k \to \infty} |\psi_j(k)(n)| = |\psi(n)| > c$ (as $n \in \Delta$). Choose $k_0 \in \mathbb{N}$ such that $|\psi_j(k)(n)| > c$ for all $k \geq k_0$. For such $k$ we have, as $x_{j(k),n} \in K_n \subseteq Y$,

$$p_n(y_{j(k)}^*) \geq |\langle x_{j(k),n}, y_{j(k)}^* \rangle| = |\langle x_{j(k),n}, x_{j(k)}^* \rangle| = |\psi_j(k)(n)| > 0.$$ 

This implies that

$$\sup_{y \in K_n} |\langle y, y_0^* \rangle| = p_n(y_0^*) = \lim_{k \to \infty} p_n(y_{j(k)}^*) \geq c.$$ 

Since $n \in \Delta$ is arbitrary, (3.6) holds.

Now, let $x_0^* \in B[X^*]$ be any continuous linear extension of $y_0^* \in B[Y^*]$ to $X$ [28, Theorem 1.9.6]. Then (3.6) establishes Step 3.
The proof of Lemma 3.1 is thereby complete, as Step 3 means precisely that (3.2) holds. ■

We now require further preparatory results from vector measure theory.

**Lemma 3.2.** Let \( m \) be a Banach-space-valued vector measure. For each \( f \in L^1(m) \), the subset of \( L^1(m) \) given by

\[
(3.7) \quad f\mathbb{B}[L^\infty(m)] := \{ f\psi : \psi \in \mathbb{B}[L^\infty(m)] \}
\]

is convex and weakly compact.

**Proof.** Convexity is clear.

The boundedness of \( f\mathbb{B}[L^\infty(m)] \) follows from the inequality (cf. (1.2))

\[
(3.8) \quad \| f\psi \|_{L^1(m)} \leq \| \psi \|_{L^\infty(m)} \| f \|_{L^1(m)}, \quad \forall \psi \in L^\infty(m).
\]

Next we show that \( f\mathbb{B}[L^\infty(m)] \) is weakly closed in \( L^1(m) \); by convexity it suffices to establish its norm-closedness. So, let \( \{ f\psi_n \}_{n=1}^\infty \subseteq f\mathbb{B}[L^\infty(m)] \) converge in \( L^1(m) \) to \( g \in L^1(m) \). In view of Proposition 2.2(ii) \& Theorem 3.7(iii) there is a subsequence \( \{ f\psi_{n(k)} \}_{k=1}^\infty \) such that \( f\psi_{n(k)} \to g \) pointwise \( m \)-a.e. as \( k \to \infty \). Since \( |f\psi_{n(k)}| \leq \| \psi_{n(k)} \|_{L^\infty(m)} |f| \leq |f| \) \((m\text{-a.e.)}\) for each \( k \in \mathbb{N} \), it follows that \( |g| \leq |f| \) \((m\text{-a.e.)}\). Define the measurable set \( A := \{ w : f(w) \neq 0 \} \), so \( f\chi_A = f \), and the function \( h \in \mathbb{B}[L^\infty(m)] \) by setting \( h := (g/f)\chi_A \). Since

\[
\| f\psi_n - fh \|_{L^1(m)} = \| f\psi_n\chi_A - g\chi_A \|_{L^1(m)} \leq \| \chi_A \|_{L^\infty(m)} \| f\psi_n - g \|_{L^1(m)}
\]

for each \( n \in \mathbb{N} \), we conclude that \( g = fh \) and so \( g \in f\mathbb{B}[L^\infty(m)] \). This shows that \( f\mathbb{B}[L^\infty(m)] \) is closed in \( L^1(m) \) and hence, as noted, its weak closedness follows.

Let \( \mu \) be a finite positive measure on \( \Sigma \) satisfying \( m \simeq \mu \). It follows from (3.8) and the fact that \( \chi_\Omega \in \mathbb{B}[L^\infty(m)] \) that

\[
\| f\chi_A \|_{L^1(m)} = \sup \{ \| f\chi_A\psi \|_{L^1(m)} : \psi \in \mathbb{B}[L^\infty(m)] \}, \quad \forall A \in \Sigma.
\]

But \( L^1(m) \) is a \( \sigma \)-order continuous Banach function space (relative to \((\Omega, \Sigma, \mu)) \) p. 23 \& Theorem 3.7(iii)], and so Lemma 2.37(ii) of \([38]\) yields

\[
(3.9) \quad \lim_{\mu(A) \to 0} \sup_{\psi \in \mathbb{B}[L^\infty(m)]} \| f\psi\chi_A \|_{L^1(m)} = \lim_{\mu(A) \to 0} \| f\chi_A \|_{L^1(m)} = 0,
\]

i.e., the bounded subset \( f\mathbb{B}[L^\infty(m)] \subseteq L^1(m) \) is uniformly \( \mu \)-absolutely continuous \([38]\) p. 56]. It then follows that \( f\mathbb{B}[L^\infty(m)] \) is a relatively weakly compact subset of \( L^1(m) \) \([38]\) Proposition 2.39(ii)]. Since \( f\mathbb{B}[L^\infty(m)] \) is weakly closed, it is actually a weakly compact subset of \( L^1(m) \). ■

**Lemma 3.3.** Let \((\Omega, \Sigma)\) be a measurable space and \( m : \Sigma \to X \) be a Banach-space-valued vector measure with relatively compact range. Fix any \( f \in L^1(m) \).
Moreover, for arbitrary choices of $m \simeq \mu$. As shown in the proof of Lemma 3.2, the set $f \mathbb{B}[L^\infty(m)]$ is bounded and uniformly $\mu$-absolutely continuous in $L^1(m)$. This implies that its image $I_m(f \mathbb{B}[L^\infty(m)])$ is relatively compact in $X$ because $\{I_m(\chi_A) : A \in \Sigma \} = m(\Sigma)$ is relatively compact (by assumption); see Proposition 2.41 of [28, Section 4.2]. Since $I_m : L^1(m) \to X$ is also continuous when both $L^1(m)$ and $X$ are equipped with their respective weak topologies, it follows that $I_m(f \mathbb{B}[L^\infty(m)])$ is weakly compact in $X$ and, in particular, norm-closed. Being relatively norm-compact in $X$, it is actually norm-compact.

(ii) The restriction of $\| \cdot \|_X : X \to [0, \infty)$ to the compact subset $I_m(f \mathbb{B}[L^\infty(m)]) \subseteq X$ attains its maximum at $f\psi_f$ for some $\psi_f \in \mathbb{B}[L^\infty(m)]$, i.e.,

$$||I_m(f\psi_f)||_X = \sup\{||I_m(\psi)||_X : \psi \in \mathbb{B}[L^\infty(m)] \} = ||f||_{L^1(m)},$$

where the second equality is known (see the identity (3.60) on p. 132 of [28]). 

We recall some facts about a Banach space $(X, \| \cdot \|_X)$ with an unconditional basis, say $\{e_n\}_{n=1}^\infty$ [1 Section 3.1], [28 Section 4.2]. Let $\{e^*_n\}_{n=1}^\infty \subseteq X^*$ denote the biorthogonal coordinate functionals associated with $\{e_n\}_{n=1}^\infty$, i.e.,

$x = \sum_{n=1}^\infty \langle x, e^*_n \rangle e_n$ for $x \in X$, with $\langle e_k, e^*_n \rangle = \delta_{k,n}$ for $k, n \in \mathbb{N}$ [28 Section 4.1 & Corollary 4.1.16]. Define

$$\|x\|_X := \sup\left\{ \left\| \sum_{n=1}^\infty e_n\langle x, e^*_n \rangle e_n \right\|_X : \langle e_n \rangle_{n=1}^\infty \in \mathbb{B}[\ell^\infty] \right\}, \quad \forall x \in X.$$

Then $\|e_n\|_X = \|e_n\|_X$ for $n \in \mathbb{N}$. The function $\| \cdot \|_X : X \to [0, \infty)$ is a norm on $X$ equivalent to $\| \cdot \|_X$, and $\|x\| \leq \|y\|$ whenever $x, y \in X$ satisfy $|\langle x, e^*_n \rangle| \leq |\langle y, e^*_n \rangle|$ for all $n \in \mathbb{N}$ [13 p. 344], [28 pp. 373–375]. It follows from the definition of $\| \cdot \|_X$ that

$$\|\sum_{n=1}^\infty \langle x, e^*_n \rangle e_n \|_X = \left\| \sum_{n=1}^\infty |\langle x, e^*_n \rangle| e_n \right\|_X, \quad \forall x \in X.$$ 

Moreover, for arbitrary choices of $\varepsilon_n \in \{0, 1\}$, for $n \in \mathbb{N}$, we have

$$\|\sum_{n=1}^\infty \varepsilon_n \langle x, e^*_n \rangle e_n \|_X \leq \left\| \sum_{n=1}^\infty \langle x, e^*_n \rangle e_n \right\|_X, \quad \forall x \in X.$$ 

Note that $\{(\|e_n\|_X)^{-1}e_n\}_{n=1}^\infty$ is a normalized unconditional basis for $(X, \| \cdot \|_X)$ [28 Corollary 4.2.13]. Henceforth, it is assumed that the norm of $X$ is chosen
to be $\| \cdot \|_X$ and that $\{e_n\}_{n=1}^\infty$ is a normalized unconditional basis relative to $\| \cdot \|_X$.

Fix $k \in \mathbb{N}$ and consider the $k$th natural projection

$$P_k : x \mapsto \sum_{n=1}^k \langle x, e_n^* \rangle e_n, \quad \forall x \in X,$$

necessarily continuous \[28\] Theorem 4.1.15], of $X$ onto the finite-dimensional subspace $\text{span}(\{e_n\}_{n=1}^k) \subseteq X$. Since $P_k$ is a non-zero projection we always have $\|P_k\|_{op} \geq 1$, whereas \[3.12\] then implies that actually $\|P_k\|_{op} = 1$. By a similar argument, also $\|Q_k\|_{op} = 1$ where $Q_k := I - P_k$, i.e.,

$$Q_k : x \mapsto \sum_{n=k+1}^\infty \langle x, e_n^* \rangle e_n, \quad \forall x \in X,$$

is the natural projection of $X$ onto its closed subspace $\overline{\text{span}}(\{e_n\}_{n=k+1}^\infty)$.

**Lemma 3.4.** Let $(X, \| \cdot \|_X)$ be a Banach space with a normalized unconditional basis $\{e_n\}_{n=1}^\infty$. Equip $X$ with the equivalent norm $\| \cdot \|_X$ given by \[3.10\] and let $P_k, Q_k$, for $k \in \mathbb{N}$, be the natural projections associated with $\{e_n\}_{n=1}^\infty$. Let $m$ be any $X$-valued vector measure, defined on a measurable space $(\Omega, \Sigma)$, whose range is relatively compact in $X$ and which has infinite variation.

(i) There exist a strictly increasing sequence $\{j(k)\}_{k=1}^\infty$ in $\mathbb{N}$ and a sequence $\{A(j)\}_{j=1}^\infty \subseteq \Sigma$ of non-$m$-null sets such that

$$\sup_{x \in K_j} \|P_{j-1}(x)\|_X \leq \frac{1}{2^j} \quad \text{and} \quad \sup_{x \in K_j} \|Q_{j-1}(x)\|_X \leq \frac{1}{2^j}, \quad \forall j \in \mathbb{N},$$

with $k(0) := 0$ and $P_k(0) := 0$, where the compact sets $K_j \neq \emptyset$ are given by

$$K_j := \{ \int_{\Omega} f_j \psi \, dm : \psi \in \mathcal{B}[L^\infty(m)] \} = I_m(f_j \mathcal{B}[L^\infty(m)]), \quad \forall j \in \mathbb{N},$$

with the corresponding non-negative functions $\{f_j\}_{j=1}^\infty \subseteq L^1(m)$ defined by

$$f_j := (\|\chi_{A(j)}\|_{L^1(m)})^{-1} \chi_{A(j)}, \quad \forall j \in \mathbb{N}. $$

(ii) There exists a sequence $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{B}[L^\infty(m)]$ satisfying

$$\|I_m(f_j \psi_j)\|_X = \|f_j\|_{L^1(m)} = 1, \quad \forall j \in \mathbb{N},$$

and also

$$\|I_m(f_j \psi_j) - I_m(f_q \psi_q)\|_X \geq 1/4, \quad \forall j, q \in \mathbb{N} \text{ with } j \neq q.$$

**Proof.** (i) Let $\mu : \Sigma \to [0, \infty)$ be a scalar measure satisfying $m \simeq \mu$. Set $A(1) := \Omega$ and then define $f_1$ by \[3.15\]. The subset $K_1 := I_m(f_1 \mathcal{B}[L^\infty(m)]) \subseteq X$ is compact by Lemma 3.3(i). Since $\sup_{j \in \mathbb{N}} \|Q_j\|_{op} = 1 < \infty$ and
\[\|Q_k(x)\|_X \to 0\] as \(k \to \infty\), for every \(x \in X\), it follows from the Banach–Steinhaus Theorem \cite[p. 220]{22}, that \(Q_k \to 0\) uniformly over the compact set \(K_1\), i.e., \(\sup_{x \in K_1} \|Q_k(x)\|_X \to 0\) as \(k \to \infty\). So, choose \(k(1) \in \mathbb{N}\) such that \(\sup_{x \in K_1} \|Q_k(1)(x)\|_X \leq 1/2\). Then (3.13) holds with \(j := 1\) as \(P_{k(0)} = 0\).

Now assume, for some fixed \(N \in \mathbb{N}\), that (3.13) holds for each \(j = 1, \ldots, N\). Since \(\|e_n\|_X = 1\), for \(n \in \mathbb{N}\) and for each \(x \in X\) we have \(\langle P_k(x), e_n^* \rangle = \langle x, e_n^* \rangle\) if \(1 \leq n \leq k(N)\) and 0 otherwise, it follows that

\[
(3.18) \quad \|P_k(x)\|_X = \left\| \sum_{n=1}^{k(N)} \langle P_k(x), e_n^* \rangle e_n \right\|_X \leq \sum_{n=1}^{k(N)} |\langle P_k(x), e_n^* \rangle|
\]

for each \(x \in X\). Let \(0 \leq \varphi \in L^1(\mu)\) be the Radon–Nikodým derivative of the non-negative scalar measure \(\sum_{n=1}^{k(N)} |\langle m, e_n^* \circ P_k \rangle|\) with respect to \(\mu\). Then there exists a set \(A(N + 1) \in \Sigma\) such that

\[
(3.19) \quad 2^{N+1} \int_{A(N+1)} \varphi \, d\mu < \|m(A(N + 1))\|_X \leq \|\chi_{A(N + 1)}\|_{L^1(\mu)}.
\]

Indeed, if the first inequality failed to hold for some \(A(N + 1) \in \Sigma\), then \(\|m(A)\|_X \leq 2^{N+1} \int_A \varphi \, d\mu\) for all \(A \in \Sigma\), which contradicts \(|m|_{\Omega} = \infty\). The inequality \(\|m(A)\|_X \leq \|\chi_A\|_{L^1(\mu)}\) always holds for every \(A \in \Sigma\) \cite[32]{38}, p. 112]. So, with \(A(N + 1)\) satisfying (3.19) we can define \(f_{N+1} \in L^1(\mu)\) by (3.15).

Given \(\psi \in B[L^\infty(m)]\), it follows from (3.18) with \(x := I_m(f_{N+1}\psi)\) there, the definition of \(f_{N+1}\) (cf. (3.15)), and (3.19) that

\[
\|P_k(\psi)\|_X \leq \sum_{n=1}^{k(N)} |\langle P_k(I_m(f_{N+1}\psi)), e_n^* \rangle|
\]

\[
= \sum_{n=1}^{k(N)} \left| \int_{\Omega} f_{N+1} \psi \, dm \right| e_n^* \circ P_k(\psi) \leq \sum_{n=1}^{k(N)} \int_{\Omega} f_{N+1} \psi \, d\mu \|m, e_n^* \circ P_k(\psi)\|
\]

\[
= \int_{\Omega} f_{N+1} \psi \, d\mu \leq \left( \int_{A(N+1)} \varphi \, d\mu \right) \|\chi_{A(N + 1)}\|_{L^1(\mu)} \leq 2^{-N(N+1)}.
\]

So, with \(K_{N+1} := I_m(f_{N+1}[\mathbb{B}[L^\infty(m)]]\) we have shown that

\[
\sup_{x \in K_{N+1}} \|P_k(x)\|_X \leq \frac{1}{2^{N+1}}.
\]

Next, since \(K_{N+1} \subseteq X\) is compact (cf. Lemma 3.3(i)), we can repeat the argument used to produce \(k(1)\) to find \(k(N + 1) \in \mathbb{N}\) with \(k(N + 1) > k(N)\) such that

\[
\sup_{x \in K_{N+1}} \|Q_k(x)\|_X \leq \frac{1}{2^{N+1}}.
\]

Accordingly, (3.13) holds for all \(j = 1, \ldots, N + 1\).
(ii) Given \( j \in \mathbb{N} \), apply Lemma 3.3(ii) to find \( \psi_j \in \mathbb{B}[L^\infty(m)] \) satisfying the first equality in (3.16). The second equality in (3.16) is clear from (3.15).

To verify (3.17), let \( j < q \). For ease of notation set \( x_j := I_m(f_j \psi_j) \in K_j \) and \( x_q := I_m(f_q \psi_q) \in K_q \). Then it follows from (3.13), from the identities \( \|P_{k(j)}\|_{op} = 1 \), \( P_{k(j)} + Q_{k(j)} = I \) and \( P_{k(j)} = P_{k(j)} \circ P_{k(q-1)} \) (as \( j < q \) implies that \( k(j) \leq k(q-1) \)), and from (3.16) which yields \( \|x_j\|_X = \|f_j\|_{L^1(m)} = 1 \), that

\[
\|x_j - x_q\|_X \geq \|P_{k(j)}(x_j - x_q)\|_X = \|x_j - Q_{k(j)}(x_j) - P_{k(j)}(x_q)\|_X \\
\geq \|x_j\|_X - \|Q_{k(j)}(x_j)\| - \|P_{k(j)} \circ P_{k(q-1)}(x_q)\|_X \\
\geq 1 - \frac{1}{2^j} - \|P_{k(q-1)}(x_q)\|_X \geq 1 - \frac{1}{2^j} - \frac{1}{2^q} \geq \frac{1}{4}.
\]

This is precisely (3.17). ■

Let us see that the hypotheses on \( X \) and \( m \) as required in Lemma 3.4 arise in many interesting settings.

**Example 3.5.** (i) Let \( X = \ell^p \) for \( 1 \leq p < 2 \), in which case \( X \) has an unconditional basis. Moreover, by a result of H. P. Rosenthal, every \( X \)-valued vector measure has relatively compact range (see Lemma 3.53(v) in [38] and its proof). So, for every \( X \)-valued vector measure \( m \) of infinite variation, all the hypotheses of Lemma 3.4 are satisfied. We point out that in every infinite-dimensional Banach space there always exist vector measures \( m \) of infinite variation, which can even be chosen to be either purely atomic or non-atomic: consider the vector measure \( m_f \) constructed in the proof of Proposition 4.4 below.

Or, let \( X \) be any infinite-dimensional Banach space with the Schur property. Then the range of every \( X \)-valued measure, being relatively weakly compact, is also relatively compact. If, in addition, \( X \) has an unconditional basis, then again for every \( X \)-valued vector measure \( m \) of infinite variation all the hypotheses of Lemma 3.4 are fulfilled. It is worth noting that such spaces \( X \) exist besides \( \ell^1 \). Indeed, for any sequence \( \{p_n\}_{n=1}^\infty \subset (1, \infty) \) define

\[
\ell^{(p_n)} := \left\{ (x_n)^\infty_{n=1} \in \mathbb{R}^\infty : \sum_{n=1}^\infty |tx_n|^{p_n} < \infty \text{ for some } t > 0 \right\},
\]

equipped with the norm

\[
\|(x_n)^\infty_{n=1}\| := \inf \left\{ t > 0 : \sum_{n=1}^\infty |x_n/t|^{p_n} \leq 1 \right\}, \quad \forall (x_n)^\infty_{n=1} \in \ell^{(p_n)}.
\]

Then \( \ell^{(p_n)} \) is a (real) Banach lattice [7], [18], [12], [13]. Moreover, the closed ideal \( \ell_d^{(p_n)} \) consisting of all absolutely continuous elements of \( \ell^{(p_n)} \) is
precisely
\[ \ell^p(a_n) = \left\{ (x_n)_{n=1}^\infty \in \ell^p : \sum_{n=1}^\infty |tx_n|p_n < \infty, \forall t \geq 0 \right\} \]

[42, p. 485]. A result of I. Halperin and H. Nakano states that \( \ell^p(a_n) \) has the Schur property iff \( \lim_{n \to \infty} p_n = 1 \) [15]; see also [43] pp. 1–3. In this case, also the closed subspace \( \ell^p(a_n) \) has the Schur property. Since spaces with the Schur property are hereditarily \( \ell^1 \) [43, p. 4], i.e., every infinite-dimensional closed subspace contains another closed subspace isomorphic to \( \ell^1 \), it follows that \( \ell^p(a_n) \) cannot contain a copy of \( c_0 \) [2 Theorem 14.21]. Accordingly, Theorem 3.5 of [15] with \( M_n(s) := s^{p_n} \) for \( s \in [0, \infty) \) and \( n \in \mathbb{N} \) (in which case the space \( c\{M_n\} \) given there is precisely \( \ell^p(a_n) \)) shows that \( \ell^p(a_n) = \ell^p(a) \) and that the canonical unit vectors form an unconditional basis of \( \ell^p(a_n) \). It follows that \( \ell^p(a_n) \) has the Schur property and possesses an unconditional basis whenever \( \lim_{n \to \infty} p_n = 1 \). If, in addition, we have \( \lim_{n \to \infty} p_n/(p_n - 1)\ln(n) = 0 \), then \( \ell^p(a_n) \) is not isomorphic to \( \ell^1 \) [12] Lemma 4. For instance, \( p_n := 1 + (\ln(n))^{-1/2} \) for \( n \geq 2 \) satisfies this condition. Actually, this same choice of \( \{p_n\} \) also satisfies the condition \( 1/p_{2n} - 1/p_n \leq a/\ln(n) \) for \( n \geq 2 \) (with \( a = 1 \)), and hence the canonical unit vectors are actually the only unconditional basis (up to equivalence) in \( \ell(p_n) \) [7 Theorem 5.8].

(ii) In the notation of Example 2.6 let \( G \) be any infinite compact abelian group with dual group \( \Gamma \) and normalized Haar measure \( \mu \). Recall the classical Banach algebra \( A(G) := \{f \in L^1(G) : \hat{f} \in \ell^1(\Gamma)\} \) under convolution and equipped with the norm
\[ \|f\|_{A(G)} := \|\hat{f}\|_{\ell^1(\Gamma)} := \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|, \quad \forall f \in A(G). \]

According to [19] Corollary 34.7, the Fourier transform map \( f \mapsto \hat{f} \) is an isometric isomorphism of \( (A(G), \| \cdot \|_{A(G)}) \) onto \( (\ell^1(\Gamma), \| \cdot \|_{\ell^1(\Gamma)}) \), and hence \( A(G) \) has the Schur property. Moreover, if \( G \) is also metrizable, then \( \Gamma \) is countable and so the characters \( \{(\cdot, \gamma) : \gamma \in \Gamma\} \) form an unconditional basis for \( A(G) \). Let \( \varphi \in L^2(G) \). Since \( L^2(G) \ast L^2(G) = A(G) \) [19 Corollary 34.16], we can define a finitely additive set function \( m_\varphi : \mathcal{B}(G) \to A(G) \) by \( m_\varphi : A \mapsto \chi_A \ast \varphi \) for \( A \in \mathcal{B}(G) \). It turns out that \( m_\varphi \) is actually \( \sigma \)-additive [35 Proposition 2.3 & Corollary 3.4]. As \( A(G) \) has the Schur property, \( m_\varphi \) necessarily has relatively compact range. However, \( m_\varphi \) has finite variation iff \( \varphi \in A(G) \) [35 Theorem 3.8], i.e., \( m_\varphi \) has infinite variation whenever \( \varphi \in L^2(G) \setminus A(G) \).

(iii) Let \( G \) be as in (ii) above (and metrizable). Recall that \( \mu \) is necessarily non-atomic (cf. Example 2.6). Since \( G \) is a Polish space, for each
1 < p < ∞ the Banach space $L^p(G)$ is isometrically isomorphic to $L^p([0,1])$ [11 p. 125]. But $L^p([0,1])$ has an unconditional basis [11 Theorem 6.1.6], and hence so does $L^p(G)$ [28 Proposition 4.2.14]. For each measure $\lambda \in M(G)$, let $m^{(p)}_{\lambda} : \mathcal{B}(G) \to L^p(G)$ be the vector measure defined in Example 2.6. If $M_0(G) := \{ \nu \in M(G) : \hat{\nu} \in c_0(\Gamma) \}$, where $\hat{\nu} : \Gamma \to \mathbb{C}$ is the Fourier–Stieltjes transform of $\nu$, i.e., $\hat{\nu}(\gamma) := \int_G \overline{(x,\gamma)} \, d\nu(x)$ for $\gamma \in \Gamma$, then it is known that the vector measure $m^{(p)}_{\lambda} \rho$ has relatively compact range in $L^p(G)$ iff $\lambda \in M_0(G)$ [38 Proposition 7.58]. On the other hand, $m^{(p)}_{\lambda} \rho$ has finite variation iff there exists $h \in L^p(G)$ such that $\lambda(A) = \int_A h \, d\mu$ for $A \in \mathcal{B}(G)$ [38 Theorem 7.67], i.e., $m^{(p)}_{\lambda} \rho$ has infinite variation whenever $\lambda \in M_0(G) \setminus L^p(G)$.

More generally, let $1 < p \leq 2$ and $\psi \in \ell^\infty(\Gamma)$ be any Fourier $p$-multiplier for $G$, i.e., there exists an operator $T^{(p)}_{\psi} : L^p(G)$, necessarily commuting with all translation operators, such that $(T^{(p)}_{\psi} f, \gamma) = \hat{f} \hat{\psi}$ for all $f \in L^p(G)$. The convolution operators $T^{(p)}_{\psi} = T^{(p)}_{\lambda}$ for $\lambda \in M(G)$ form a proper subclass of the Fourier $p$-multiplier operators. For each Fourier $p$-multiplier $\psi \neq 0$, the set function $m^{(p)}_{\psi} : A \mapsto T^{(p)}_{\psi}(\chi_A)$ for $A \in \mathcal{B}(G)$ is a vector measure with $m^{(p)}_{\psi} \approx \mu$ [32 Proposition 2.2]. It is known that $m^{(p)}_{\psi} \rho$ has relatively compact range in $L^p(G)$ precisely when $\psi \in c_0(\Gamma)$ (for $\psi := \hat{\lambda}$ with $\lambda \in M(G)$ this corresponds to $\lambda \in M_0(G)$) [32 Proposition 2.3], whereas $m^{(p)}_{\psi} \rho$ has finite variation iff $\psi = \hat{\lambda}$ for some $\lambda \in L^p(G)$, [32 Proposition 2.8]. For the circle group $G = \mathbb{T}$, we note (for every $1 < p \leq 2$) that there exist Fourier $p$-multipliers $\psi \in c_0(\mathbb{Z})$ which are not of the form $\hat{\lambda}$ for any $\lambda \in M_0(\mathbb{T})$ [32 Remark 2.6(ii)]. In particular, such a $p$-multiplier $\psi$ cannot be the Fourier–Stieltjes transform of any function from $L^p(\mathbb{T})$.

**Proposition 3.6.** Let $X$ be a Banach space with an unconditional basis. If there exists an $X$-valued vector measure $m$ having infinite variation and satisfying $I_m \in \mathcal{A}_{cc}$, then $\ell^1 \hookrightarrow X$.

**Proof.** Let $\{e_n\}_{n=1}^\infty$ be a normalized unconditional basis of $X$ and equip $X$ with the norm $\| \cdot \|_X$ as given by (3.10). Since $I_m \in \mathcal{A}_{cc}$, the range of $m$ is a relatively compact subset of $X$ [35 p. 153]. Let the sequence of non-empty compact sets $\{K_j\}_{j=1}^\infty \subseteq X$ be given by (3.14), the functions $\{f_j\}_{j=1}^\infty \subseteq L^1(m)$ be given by (3.15), and the sequence $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{B}[L^\infty(m)]$ be as in Lemma 3.4(ii).

**Step 1.** There exists a strictly increasing sequence $\{j(n)\}_{n=1}^\infty \subseteq \mathbb{N}$ such that $\{f_{j(n)} \psi_{j(n)}\}_{n=1}^\infty$ is a basic sequence in $L^1(m)$ which is equivalent to the
canonical basis of \( \ell^1 \). In particular, there exists \( \delta > 0 \) such that
\[
\delta \sum_{n=1}^{N} |a_n| \leq \left\| \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \right\|_{L^1(m)}
\]
for all choices of \( N \in \mathbb{N} \) and \( \{a_n : n = 1, \ldots, N\} \subseteq \mathbb{C} \).

To see this, first observe that (3.8) and (3.16) imply that \( \|f_j \psi_j\|_{L^1(m)} \leq 1 \) for all \( j \in \mathbb{N} \). Moreover, (3.17) shows that \( \{f_j \psi_j\}_{j=1}^{\infty} \subseteq L^1(m) \) cannot contain any weak Cauchy subsequences because the completely continuous operator \( I_m \) maps such subsequences of \( L^1(m) \) to norm-convergent sequences in \( X \). So, a result of H. P. Rosenthal [2, Theorem 14.24] establishes Step 1.

**Step 2.** With \( \{j(n)\}_{n=1}^{\infty} \subseteq \mathbb{N} \) as in Step 1, there exists \( x_0^* \in B[X^*] \) such that
\[
\varepsilon := \limsup_{n \to \infty} \sup_{x \in K_{j(n)}} |\langle x, x_0^* \rangle| > 0.
\]

Indeed, it follows from \( \|I_m\|_{op} = 1 \), the definition of \( K_{j(n)} \) (cf. (3.14)), and (3.16) that \( K_{j(n)} \subseteq B[X] \) for \( n \in \mathbb{N} \). Fix \( (a_n)_{n=1}^{\infty} \in \mathbb{C}^\mathbb{N} \). Given \( N \in \mathbb{N} \), Lemma 3.3(ii) with the \( m \)-integrable function \( \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \) in place of \( f \) guarantees the existence of \( \psi \in B[L^\infty(m)] \) satisfying
\[
\left\| \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \right\|_{L^1(m)} = \left\| I_m \left( \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \right) \right\|_{X}.
\]
These \( \psi_{j(n)} \)s satisfy (3.8) and (3.16) with \( m \)-integrable function \( \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \psi \) in place of \( f \). It then follows from (3.20) and (3.22) that
\[
\delta \sum_{n=1}^{N} |a_n| \leq \left\| \sum_{n=1}^{N} a_n I_m(f_{j(n)} \psi_{j(n)} \psi) \right\|_{X}
\]
\[
\leq \sup \left\{ \left\| \sum_{n=1}^{N} a_n x_n \right\|_{X} : x_n \in K_{j(n)}, n = 1, \ldots, N \right\}.
\]
This shows that (3.1) in the statement of Lemma 3.1 holds for the sequence \( \{K_{j(n)}\}_{n=1}^{\infty} \) of non-empty compact sets. Hence, (3.2) yields (3.22), i.e., Step 2 is valid.

**Step 3.** Let \( R_n := P_{k(j(n))} - P_{k(j(n))-1} \) for \( n \in \mathbb{N} \), with \( \{j(n)\}_{n=1}^{\infty} \) as in Step 1, and let \( x_0^* \in B[X^*] \) satisfy (3.21). Then there exist an infinite subset \( \Delta \subseteq \mathbb{N} \) and vectors \( y_n \in R_n(K_{j(n)}) \) for \( n \in \mathbb{N} \) such that
\[
\varepsilon/3 < |\langle y_n, x_0^* \rangle|, \quad \forall n \in \Delta.
\]
To see this first observe, for each \( n \in \mathbb{N} \), that
\[
(3.23) \quad R_n(x) = \sum_{i=k(j(n)-1)+1}^{k(j(n))} \langle x, e_i^* \rangle e_i, \quad \forall x \in X,
\]
and hence via (3.12) it follows that \( \|R_n\|_{op} = 1 \). Apply Step 2 to obtain an infinite subset \( \Delta_0 \subseteq \mathbb{N} \) such that
\[
\varepsilon/2 < \sup_{x \in K_{j(n)}} |\langle x, x_0^* \rangle|, \quad \forall n \in \Delta_0.
\]
Fix \( n \in \Delta_0 \). Select \( x_n \in K_{j(n)} \) such that \( |\langle x_n, x_0^* \rangle| > \varepsilon/2 \) and observe that
\[
x_n = P_{k(j(n)-1)}(x_n) + R_n(x_n) + Q_{k(j(n))}(x_n).
\]
So, apply (3.13) with \( j(n) \) in place of \( j \) to obtain
\[
|\langle x_n - R_n(x_n), x_0^* \rangle| \leq \|x_n - R_n(x_n)\| X
\]
\[
\leq \|P_{k(j(n)-1)}(x_n)\| X + \|Q_{k(j(n))}(x_n)\| X
\]
\[
\leq \frac{1}{2j(n)} + \frac{1}{2j(n)} = \frac{1}{2j(n)-1},
\]
which implies (as \( x_n \in K_{j(n)} \)) that
\[
|\langle R_n(x_n), x_0^* \rangle| \geq |\langle x_n, x_0^* \rangle| - |\langle x_n - R_n(x_n), x_0^* \rangle|
\]
\[
\geq |\langle x_n, x_0^* \rangle| - \frac{1}{2j(n)-1} > \varepsilon - \frac{1}{2j(n)-1}.
\]
In view of this inequality, which is valid for each \( n \in \Delta_0 \), there is an infinite subset \( \Delta \subseteq \Delta_0 \) such that
\[
(3.24) \quad \varepsilon/3 < |\langle R_n(x_n), x_0^* \rangle|, \quad \forall n \in \Delta.
\]
So, with \( y_n := R_n(x_n) \) for \( n \in \Delta \), we have established Step 3.

**Step 4.** Let \( \Delta \) be as in Step 3 and \( \{n(q) : q \in \mathbb{N}\} \) be an enumeration of \( \Delta \) with \( \{n(q)\}_{q=1}^\infty \) a strictly increasing sequence in \( \mathbb{N} \). Then \( \{y_n(q)\}_{q=1}^\infty \) is a basic sequence in \( X \) which is equivalent to the canonical basis of \( \ell^1 \).

Indeed, since (3.23) holds for \( n(q) \) in place of \( n \) with \( y_n(q) = R_n(q)(x_n(q)) \neq 0 \) (because of (3.24)), the vectors \( y_n(q) \) for \( q \in \mathbb{N} \) form a block basic sequence taken from \( \{e_n\}_{n=1}^\infty \) [28, Definition 4.3.15]. In particular, \( \{y_n(q)\}_{q=1}^\infty \) is an unconditional basic sequence in \( X \) [28, p. 398, Ex. 4.39], i.e., \( \{y_n(q)\}_{q=1}^\infty \) is an unconditional basis for the closed subspace \( Y := \text{span}(\{y_n(q)\}_{q=1}^\infty) \) of \( X \).

So, there exist positive constants \( \alpha, \beta \) and a norm \( \|\cdot\| \) in \( Y \) satisfying
\[
\alpha \|y\| \leq \|y\|_X \leq \beta \|y\|, \quad \forall y \in Y,
\]
and with the property that
\[
(3.25) \quad \left| \sum_{q=1}^\infty c_q y_n(q) \right| = \left| \sum_{q=1}^\infty c_q |y_n(q)| \right|, \quad \forall y = \sum_{q=1}^\infty c_q y_n(q) \in Y.
\]
Given $N \in \mathbb{N}$ and $\{a_q\}_{q=1}^N \subseteq \mathbb{C}$ we claim that

\begin{equation}
\frac{\varepsilon}{3} \sum_{q=1}^N |a_q| \leq \frac{\beta}{\alpha} \left\| \sum_{q=1}^N a_q y_n(q) \right\|_X \leq \frac{\beta}{\alpha} \sum_{q=1}^N |a_q|.
\end{equation}

In fact, for each $q = 1, \ldots, N$, we have $\langle y_n(q), x_0^* \rangle \neq 0$ (cf. (3.24)) and so we can define $b_q := |\langle y_n(q), x_0^* \rangle|/\langle y_n(q), x_0^* \rangle$, in which case $|b_q| = 1$ and $|\langle y_n(q), x_0^* \rangle| = \langle b_q y_n(q), x_0^* \rangle$. It then follows from (3.25) and Step 3 that

\[ \frac{\varepsilon}{3} \sum_{q=1}^N |a_q| \leq \sum_{q=1}^N |a_q| \cdot |\langle y_n(q), x_0^* \rangle| = \sum_{q=1}^N |a_q| |\langle b_q y_n(q), x_0^* \rangle| = \left( \sum_{q=1}^N |a_q| |b_q y_n(q)| \right) \leq \beta \left\| \sum_{q=1}^N |a_q| b_q y_n(q) \right\|_X = \beta \left\| \sum_{q=1}^N a_q y_n(q) \right\|_X. \]

So, the first inequality in (3.26) is valid. The second inequality in (3.26) is a consequence of $\left\| y_n(q) \right\|_X = \left\| R_n(q)(x_n(q)) \right\|_X \leq \left\| R_n(q) \right\|_{\text{op}} \left\| x_n(q) \right\|_X$ together with $\left\| R_n(q) \right\|_{\text{op}} = 1$ and $\left\| x_n(q) \right\|_X \leq 1$ as $x_n(q) \in K_j(n(q)) \subseteq \mathcal{B}[X]$, for $q \in \mathbb{N}$.

Step 4 is now immediate from (3.26) (see also [28, Theorem 4.3.6]) as $\left\| y_n(q) \right\|_X \leq 1$ for $q \in \mathbb{N}$.

Finally, Step 4 implies that $\ell^1 \hookrightarrow X$ [28, Theorem 4.3.17], which completes the proof of Proposition 3.6. \[ \square \]

**Proof of Theorem 1.2.** Let $X$ be a Banach space with an unconditional basis and such that $\ell^1 \hookrightarrow X$. If $m$ is any $X$-valued vector measure with $I_m \in \mathcal{A}_{cc}$, then Proposition 3.6 implies that $m$ must have finite variation, i.e., $X$ is $\mathcal{A}_{cc}$-variation admissible. By Proposition 1.1 we have $L^1(m) = L^1(|m|)$. \[ \square \]

As an application, for each $1 < r < \infty$ consider the Volterra vector measure $m_r : \mathcal{B}([0, 1]) \to L^r([0, 1])$ (see Section 2). As the Banach space $L^r([0, 1])$ is reflexive, we surely have $\ell^1 \hookrightarrow L^r([0, 1])$. Moreover, it was observed in Example 3.5(iii) that $L^r([0, 1])$ has an unconditional basis. Since $m_r$ has finite variation but $L^1(|m_r|) \not\subseteq L^1(m_r)$ (cf. Section 2), Theorem 1.2 implies that the integration map $I_{m_r}$ fails to be completely continuous. An alternative proof (rather non-trivial) of this fact can be found in [38, pp. 154–157].

We end this section with a

**QUESTION.** Does there exist a Banach space $X$ with $\ell^1 \hookrightarrow X$ such that $X$ is not $\mathcal{A}_{cc}$-variation admissible? Of course, $X$ could not have an unconditional basis.
4. Theorem 1.3 and related results. In this final section we establish Theorem 1.3 and present some related results and relevant examples.

Proof of Theorem 1.3. (i)⇒(ii). Let $X$ be a Banach space with the CRP and $m : \Sigma \to X$ be any vector measure satisfying $L^1(m) = L^1(|m|)$. Then $m$ has finite variation, and hence its range $m(\Sigma)$ is relatively compact in $X$. Since $\{I_m(\chi_A) : A \in \Sigma\} = m(\Sigma)$, it follows that $I_m \in \mathcal{A}_{cc}$ [38 Corollary 2.42].

(ii)⇒(i). Proceeding via a contrapositive argument suppose that $X$ fails the CRP, in which case there exists a vector measure $\nu : \Sigma \to X$ with finite variation such that $\nu(\Sigma)$ is not relatively compact in $X$.

Fix $u \in X \setminus \{0\}$ with $\|u\|_X = 1$ and choose $x^* \in X^*$ such that $\langle u, x^* \rangle = 1$. Then $X = \mathbb{C}u \oplus Y$ with $Y := \text{Ker}(x^*)$. Let $P$ be any continuous projection of $X$ onto $Y$, in which case $\eta := P \circ \nu$ is a $Y$-valued vector measure on $\Sigma$ whose range $\eta(\Sigma)$ is not relatively compact. Since

$$\|\eta(A)\|_Y \leq \|P\|_{\text{op}} \|\nu(A)\|_X \leq \|P\|_{\text{op}} |\nu|(A), \quad \forall A \in \Sigma,$$

it is clear that $\eta$ has finite variation and satisfies $|\eta|(A) \leq \|P\|_{\text{op}} |\nu|(A)$ for $A \in \Sigma$. Define the vector measure $m : \Sigma \to X$ by

$$m(A) := |\eta|(A)u + \eta(A), \quad \forall A \in \Sigma.$$  

(4.1)

Then $\|m(A)\|_X \leq 2|\eta|(A)$ for $A \in \Sigma$ implies that $m$ has finite variation and satisfies $|m|(A) \leq 2|\eta|(A)$ for $A \in \Sigma$. Moreover,

$$\langle m, x^* \rangle(A) = |\eta|(A)\langle u, x^* \rangle + \langle \eta(A), x^* \rangle = |\eta|(A), \quad \forall A \in \Sigma,$$

as $\eta(\Sigma) \subseteq Y$. Accordingly, $|m| \leq 2|\eta| = 2|\langle m, x^* \rangle|$ setwise on $\Sigma$ and so $x^*$ is a Rybakov functional for $m$ [14 Ch. IX, §2], [38 p. 108]. In particular, $L^1(m) = L^1(|m|)$ [38 Corollary 3.19(ii)].

In view of the fact that $I_m \in \mathcal{A}_{cc}$ iff $\{I_m(\chi_A) : A \in \Sigma\} = m(\Sigma) \subseteq X$ is relatively compact [38 Corollary 2.42], it remains to check that $m(\Sigma)$ is not relatively compact in $X$. But $\| |\eta|(A)u\|_X \leq |\eta|(\Omega)$ for all $A \in \Sigma$ and so $\{|\eta|(A)u : A \in \Sigma\}$ is contained in a compact subset of the 1-dimensional space $\mathbb{C}u$. It then follows from (4.1) and the fact that $\eta(\Sigma)$ is not relatively compact in $Y$ that $m(\Sigma)$ indeed fails to be relatively compact in $X$. ■

Remark 4.1. A Banach space $X$ has the CRP iff $\mathcal{L}(L^1([0,1]), X) \subseteq \mathcal{A}_{cc}$. This is stated in [11 Ch. 7]; a proof can be found in [16].

A local version of Theorem 1.3 is also available for an individual vector measure. Given an $X$-valued vector measure $m$ let $X_m$ denote the closed subspace of $X$ generated by the range of $m$. Since the simple functions are dense in $L^1(m)$, it follows that $X_m$ is also the closure in $X$ of the range $I_m(L^1(m))$ of $I_m$. Observe that $X_m$ is weakly compactly generated [14 Ch. I, Corollary 2.7], and hence $X_m$ has the WRNP iff it has the RNP (cf. Section 1).
Proposition 4.2. Let $X$ be a Banach space and $m : \Sigma \to X$ be a vector measure such that $L^1(m) = L^1(|m|)$. If $X_m$ has the CRP, then $I_m \in \mathcal{A}_{cc}$.

Proof. Let $\tilde{m} : \Sigma \to X_m$ be the vector measure $\tilde{m} : A \mapsto m(A)$ for $A \in \Sigma$, and $j : X_m \to X$ be the identity imbedding. It follows from [38] Theorem 3.5 that $L^1(m) = L^1(\tilde{m})$. Since $|\tilde{m}| = |m|$ is a finite measure, we can apply Theorem 1.3 to $\tilde{m}$ in $X_m$ to conclude that $I_{\tilde{m}} \in \mathcal{A}_{cc}$. Hence, also $I_m = j \circ I_{\tilde{m}} \in \mathcal{A}_{cc}$. 

Remark 4.3. (i) If $\ell^1 \hookrightarrow X$ and $m$ is any $X^*$-valued vector measure satisfying $L^1(m) = L^1(|m|)$, then $I_m \in \mathcal{A}_{cc}$. This follows from Theorem 1.3 and the fact that $X^*$ has the CRP (cf. Section 1).

(ii) Let $m$ be any purely atomic vector measure with finite variation. If $L^1(m) = L^1(|m|)$, then $I_m \in \mathcal{A}_{cc}$. Indeed, $m$ necessarily has compact range [21] Theorem 10, and so [38] Corollary 2.42 implies that $I_m \in \mathcal{A}_{cc}$.

The converse is false. To see this, let $X$ be any infinite-dimensional Banach space with the Schur property. Then $I_m \in \mathcal{A}_{cc}$ for every $X$-valued vector measure $m$. On the other hand, Proposition 4.4 below shows that there always exists a purely atomic, $X$-valued vector measure $m$ with finite variation such that $L^1(|m|) \subsetneq L^1(m)$. Many examples of Schur spaces (and their properties) occur in [43]; see also the references. Every Banach lattice with the Schur property has the RNP [13] Theorem 5. Of course, Banach spaces with the Schur property always have the CRP.

Proposition 4.4. In every infinite-dimensional Banach space there exists a vector measure $m$ with finite variation such that $L^1(|m|) \subsetneq L^1(m)$. Moreover, $m$ can be chosen to be purely atomic or to be non-atomic.

Proof. Let $X$ be an infinite-dimensional Banach space and $\sum_{n=1}^{\infty} x_n$ be any unconditionally convergent series in $X$ which is not absolutely convergent [13] Theorem 1.2. Let $\mu : \Sigma \to [0, \infty)$ be any measure for which there exists a sequence $\{A(n)\}_{n=1}^{\infty} \subseteq \Sigma$ of pairwise disjoint sets with $\mu(A(n)) > 0$ for $n \in \mathbb{N}$. Define $m : \Sigma \to X$ by

$$m(A) := \sum_{n=1}^{\infty} \mu(A \cap A(n)) x_n, \quad \forall A \in \Sigma.$$ 

The Vitali–Hahn–Saks Theorem [14] Ch. I, Corollary 5.6] ensures that $m$ is $\sigma$-additive. Moreover, $\|m(A)\|_X \leq (\sup_{n \in \mathbb{N}} \|x_n\|_X) \cdot \mu(A)$ for $A \in \Sigma$, so that $m$ has finite variation. Now, the function $f := \sum_{n=1}^{\infty} \frac{1}{\mu(A(n))} \chi_{A(n)}$ is $m$-integrable with

$$\int_A f \, dm = \sum_{n=1}^{\infty} \frac{\mu(A \cap A(n))}{\mu(A(n))} \cdot x_n, \quad \forall A \in \Sigma.$$
However, in the notation of [2.1], we have
\[ |m_f|\Omega(n) \geq \sum_{n=1}^{\infty} |m_f|A(n) \geq \sum_{n=1}^{\infty} \|m_f|A(n)\|X = \sum_{n=1}^{\infty} \|x_n\|X = \infty, \]
and hence \( f \in L^1(m) \setminus L^1(|m|) \) (see Lemma 2.1(i)).

Finally, \( m \) is purely atomic (resp. non-atomic) iff \( \mu \) is purely atomic (resp. non-atomic).

Recall that a Banach-space-valued vector measure \( m \) is called \( \sigma \)-decomposable if there exist (countably) infinitely many pairwise disjoint, non-\( m \)-null sets [38, p. 129].

**Proposition 4.5.** Let \( m \) be any Banach-space-valued vector measure.

(i) If \( I_m \in A_{cc} \), then \( c_0 \hookrightarrow L^1(m) \).

(ii) If \( m \) is \( \sigma \)-decomposable and \( I_m \in A_{cc} \), then \( \ell^1 \hookrightarrow L^1(m) \).

**Proof.** (i) According to [6, Theorem 3.6] and [38, Proposition 3.38(I)], \( I_m \in A_{cc} \) implies that the Banach lattice \( L^1(m) \) is weakly sequentially complete. Hence, \( c_0 \hookrightarrow L^1(m) \) [2, Theorem 14.12].

(ii) Since \( L^1(m) \) is weakly sequentially complete (cf. proof of part (i)), it follows from Rosenthal’s Theorem [2, p. 247] that either \( L^1(m) \) is reflexive or \( \ell^1 \hookrightarrow L^1(m) \).

Suppose that \( L^1(m) \) is reflexive. It then follows that \( I_m : L^1(m) \to X \), being already completely continuous, is compact, and so \( L^1(m) = L^1(|m|) \) (see Section 1). Let \( \{A(n)\}_{n=1}^{\infty} \) be any sequence of measurable, pairwise disjoint, non-\( m \)-null sets. Then each function \( \varphi_n := (|m|A(n))^{-1} \chi_{A(n)} \) belongs to \( L^1(|m|) \) with \( \|\varphi_n\|L^1(|m|) = 1 \) for \( n \in \mathbb{N} \). It is routine to check that the linear map \( u \mapsto \sum_{n=1}^{\infty} u_n\varphi_n \) for \( u = (u_n)_{n=1}^{\infty} \in \ell^1 \) is a bicontinuous linear isomorphism of \( \ell^1 \) onto a closed subspace of \( L^1(|m|) \). This contradicts the reflexivity of \( L^1(|m|) = L^1(m) \). Hence, \( L^1(m) \) is not reflexive, i.e., \( \ell^1 \not\hookrightarrow L^1(m) \).

**Remark 4.6.** (i) The analogue of Proposition 4.5(ii) with \( A_{cc} \) in place of \( A_{cc} \) is known [10, Claim, p. 3800].

(ii) Let \( m : \Sigma \to X \) be a \( \sigma \)-decomposable, Banach-space-valued vector measure. According to Proposition 4.5(ii), we have \( I_m \not\in A_{cc} \) whenever \( \ell^1 \not\hookrightarrow L^1(m) \) (equivalently, whenever the Banach lattice \( L^1(m) \) has the RNP; see Section 1). The condition \( \ell^1 \hookrightarrow L^1(m) \) has some useful consequences. For instance, it implies that the ideal in the dual Banach lattice \( L^1(m)^* \) which is generated by the family of continuous linear functionals \( \varphi_{x^*,A} : f \mapsto \int_A f d(m,x^*) \) for \( f \in L^1(m) \), for all \( x^* \in X^* \) and \( A \in \Sigma \) (cf. (1.2)), is dense in \( L^1(m)^* \). This in turn implies that weak convergence of bounded nets in \( L^1(m) \) is characterized by weak convergence (in \( X \)) of the integrals over arbitrary sets, i.e., if \( \sup_{\alpha} \|f_\alpha\|L^1(m) < \infty \), then \( \lim_{\alpha} f_\alpha = f \) weakly
in $L^1(m)$ iff $\lim_\alpha I_m(f_\alpha \chi_A) = I_m(f\chi_A)$ weakly in $X$ for every $A \in \Sigma$ \[9\] Theorem 4).

(iii) The converse of Proposition \[4.5\](ii) is false. Let $X = \ell^2$. Section 6 of \[10\] exhibits a vector measure $m : \Sigma \to X$ (denoted there by $\nu$) and a bounded basic sequence $\{f_n\}_{n=1}^\infty$ in $L^1(m)$, equivalent to the canonical basis of $\ell^1$, such that

\[
\lim_{n \to \infty} \varphi_{x^*,A}(f_n) = 0, \quad \forall x^* \in X^*, \ A \in \Sigma.
\]

In particular, $\ell^1 \hookrightarrow L^1(m)$ and $\{f_n\}_{n=1}^\infty$ is not weakly convergent to zero in $L^1(m)$. If $I_m \in \mathcal{A}_{cc}$, then $m(\Sigma)$ is relatively compact in $X$ \[38\] p. 153. Hence, \[4.2\] and \[31\] Proposition 17 imply that $\{f_n\}_{n=1}^\infty$ converges weakly to zero in $L^1(m)$; contradiction! So, $I_m \notin \mathcal{A}_{cc}$.

(iv) Since $L^1(m)$ is a Banach lattice with order continuous norm \[38\] Theorem 3.7(iii)], it is known that $\ell^1 \hookrightarrow L^1(m)$ iff $c_0 \hookrightarrow L^1(m)^*$ \[2\] p. 246, Ex. 13.

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Susumu Okada
112 Marconi Crescent
Kambah, ACT 2902, Australia
E-mail: okobrien@grapevine.com.au

Luis Rodríguez-Piazza
Facultad de Matemáticas
Universidad de Sevilla
Aptdo 1160
E-41080 Sevilla, Spain
E-mail: piazza@us.es

Werner J. Ricker
Math.-Geogr. Fakultät
Katholische Universität Eichstätt-Ingolstadt
D-85072 Eichstätt, Germany
E-mail: werner.ricker@ku-eichstaett.de

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