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Bilinear operators associated with Schrödinger operators

by

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Abstract. Let $L = -\Delta + V$ be a Schrödinger operator in $\mathbb{R}^d$ and $H^1_L(\mathbb{R}^d)$ be the Hardy type space associated to L. We investigate the bilinear operators T^+ and T^- defined by

$$T^{\pm}(f,g)(x) = (T_1 f)(x)(T_2 g)(x) \pm (T_2 f)(x)(T_1 g)(x),$$

where T_1 and T_2 are Calderón–Zygmund operators related to L. Under some general conditions, we prove that either T^+ or T^- is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $H^1_L(\mathbb{R}^d)$ for $1 < p, q < \infty$ with $1/p + 1/q = 1$. Several examples satisfying these conditions are given. We also give a counterexample for which the classical Hardy space estimate fails.

1. Introduction. Among other motivations, due to their close relations to the Cauchy integral along Lipschitz curves, Calderón commutators, and compensated compactness, bilinear (or multilinear) operators have attracted much attention. In [1, 2, 6] and references therein, Hardy space estimates of bilinear operators are extensively studied. In this article we consider bilinear operators related to a Schrödinger operator L. We establish an estimate for them with respect to a Hardy type space associated with the Schrödinger operator L, under some general conditions. Some examples satisfying these conditions are given. We also give a counterexample for which the classical Hardy space estimate fails.

Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian on $\mathbb{R}^d$, $d \geq 3$, and V belongs to a certain reverse Hölder class RH_q. A non-negative locally L^q integrable function V on $\mathbb{R}^d$ is said to belong to RH_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q\,dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \tag{1.1}$$

holds for every ball B in $\mathbb{R}^d$. Obviously, $RH_{q_2} \subset RH_{q_1}$ if $q_2 > q_1$. But it

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is important that the RH_q class has a property of "self-improvement": if $V \in RH_q$, then $V \in RH_{q+\varepsilon}$ for some $\varepsilon > 0$. Throughout this article, we always assume that $0 \not\equiv V \in RH_{d/2}$. Thus, $V \in RH_{q_0}$ for some $q_0 > d/2$.

Let $\{T_s\}_{s>0} = \{e^{s\Delta}\}_{s>0}$ be the heat semigroup with the kernel $H_s(x, y) = H_s(x - y)$. Because $V \geq 0$ and $V \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$, the Schrödinger operator $L = -\Delta + V$ generates a (C_0) contraction semigroup $\{T^L_s\}_{s>0} = \{e^{-sL}\}_{s>0}$. Let $K^L_s(x, y)$ denote the kernel of T^L_s. By the Trotter product formula (cf. [5]),

$$0 \leq K^L_s(x, y) \leq H_s(x, y) = (4\pi s)^{-d/2} e^{-|x-y|^2/(4s)}. \tag{1.2}$$

It is well-known that the maximal function

$$Mf(x) = \sup_{s>0} |T_s f(x)|$$

characterizes the Hardy space $H^1(\mathbb{R}^d)$, namely, $f \in H^1(\mathbb{R}^d)$ if and only if $Mf \in L^1(\mathbb{R}^d)$, and $\|f\|_{H^1} \sim \|Mf\|_{L^1}$. A Hardy type space $H^1_L(\mathbb{R}^d)$ associated with the Schrödinger operator L was introduced by Dziubański and Zienkiewicz [4]. The maximal function with respect to the semigroup $\{T^L_s\}_{s>0}$ is defined by

$$M^L f(x) = \sup_{s>0} |T^L_s f(x)|.$$

A function $f \in L^1(\mathbb{R}^d)$ is said to be in $H^1_L(\mathbb{R}^d)$ if $M^L f \in L^1(\mathbb{R}^d)$. The norm of such a function is defined by $\|f\|_{H^1_L} = \|M^L f\|_{L^1}$. It is visible from (1.2) that $H^1(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$, by the atomic decomposition of $H^1_L(\mathbb{R}^d)$ (see [4]).

Following [7], we define the auxiliary function $\rho(x, V) = \rho(x)$ by

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy \leq 1 \right\}, \qquad x \in \mathbb{R}^d.$$

The auxiliary function $\rho(x)$ plays an important role in studying the boundedness of singular integral operators related to the Schrödinger operator L as well as the atomic decomposition of $H^1_L(\mathbb{R}^d)$ (see [4, 7]).

In this article we consider the bilinear operators

$$T^{\pm}(f, g)(x) := (T_1 f)(x)(T_2 g)(x) \pm (T_2 f)(x)(T_1 g)(x),$$

where $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ with $1 < p, q < \infty$ and $1/p + 1/q = 1$, and T_i $(i = 1, 2)$ are Calderón–Zygmund operators related to the Schrödinger operator L and satisfying the following two conditions:

(i) There exist parallel Calderón–Zygmund operators $\widetilde{T}_i$ related to the Laplacian Δ and a constant $\delta > 0$ such that

$$|T_i(x, y) - \widetilde{T}_i(x, y)| \leq \frac{C}{\rho(y)^\delta |x - y|^{d-\delta}}, \qquad x \neq y, \tag{1.3}$$

where $T_i(x,y)$ and $\widetilde{T}_i(x,y)$ denote the kernels of T_i and $\widetilde{T}_i$, respectively.

(ii) One of the parallel bilinear operators

$$\widetilde{T}^{\pm}(f,g)(x) := (\widetilde{T}_1 f)(x)(\widetilde{T}_2 g)(x) \pm (\widetilde{T}_2 f)(x)(\widetilde{T}_1 g)(x)$$

has the vanishing moment; that is, either $\widetilde{T}^{+}$ or $\widetilde{T}^{-}$ satisfies

$$\int_{\mathbb{R}^d} \widetilde{T}^{\pm}(f,g)(x)\,dx = 0 \quad \text{for all } f,g \in C_c^{\infty}(\mathbb{R}^d). \tag{1.4}$$

We will show that either T^+ or T^- is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $H_L^1(\mathbb{R}^d)$.

Theorem 1.1. *Suppose that the bilinear operators $T^{\pm}$ are defined as above. Let $1 < p,q < \infty$ and $1/p + 1/q = 1$. Then either T^+ or T^- (but not both), which corresponds to the parallel bilinear operator satisfying* (1.4)*, maps $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $H_L^1(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that*

$$\|T^{\pm}(f,g)\|_{H_L^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

The proof of Theorem 1.1 will be given in the next section. In Section 3, we give some examples which satisfy the conditions of Theorem 1.1, and also a counterexample for which the standard Hardy space estimate fails.

We will use C to denote a positive constant, which is not necessarily the same at each occurrence and may depend on the dimension d and the constant in (1.1). By $A \sim D$, we mean that there exists a constant $C > 1$ such that $1/C \le A/D \le C$. For a given ball B, we denote by B^* the concentric ball with twice the radius, and $B^{**} = (B^*)^*$.

2. The proof of Theorem 1.1. Firstly, we recall some useful facts about the auxiliary function $\rho(x)$. It is known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^d$ (cf. [7]). Therefore,

$$\mathbb{R}^d = \bigcup_{k=-\infty}^{\infty} \Omega_k,$$

where

$$\Omega_k = \{x \in \mathbb{R}^d : 2^{-k-1} \le \rho(x) < 2^{-k}\}.$$

Lemma 2.1 ([4]). *There exists a constant $N = N(V)$ and a sequence $\{x_{(k,\alpha)} \in \Omega_k : k,\alpha \in \mathbb{Z}\}$ of points such that the family $\{B_{(k,\alpha)}\}$ of critical balls defined by $B_{(k,\alpha)} = \{x \in \mathbb{R}^d : |x - x_{(k,\alpha)}| < \rho(x_{(k,\alpha)})\}$ satisfies*

(i) $\bigcup_{(k,\alpha)} B_{(k,\alpha)} = \mathbb{R}^d$;

(ii) $\#\{(k',\alpha') : B^{**}_{(k',\alpha')} \cap B^{**}_{(k,\alpha)} \ne \emptyset\} \le N$ *for every* (k,α).

Furthermore, there exists a family $\{\xi_{(k,\alpha)}\}_{k,\alpha\in\mathbb{Z}}$ of C^{∞} functions such that

(iii) $\operatorname{supp} \xi_{(k,\alpha)} \subseteq B^*_{(k,\alpha)}$;
(iv) $0 \le \xi_{(k,\alpha)}(x) \le 1$ *and* $\sum_{(k,\alpha)} \xi_{(k,\alpha)}(x) = 1$ *for all* $x \in \mathbb{R}^d$;
(v) $\|\nabla \xi_{(k,\alpha)}\|_\infty \le C/\rho(x_{(k,\alpha)})$.

Lemma 2.2 ([7]). *There exists* $l_0 > 0$ *such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l_0} \le \frac{\rho(y)}{\rho(x)} \le C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{l_0}{l_0+1}}, \qquad \forall x, y \in \mathbb{R}^d.$$

In particular, $\rho(x) \sim \rho(y)$ *if* $|x-y| < C\rho(x)$.

To prove Theorem 1.1, we need the following estimates for the kernel $K^L_s(x,y)$:

Lemma 2.3 ([3, 4]). *For any* $l > 0$, *there exists* $C_l > 0$ *such that*

$$K^L_s(x,y) \le C_l s^{-d/2} e^{-|x-y|^2/(5s)} \left(1 + \frac{\sqrt{s}}{\rho(x)} + \frac{\sqrt{s}}{\rho(y)}\right)^{-l}. \tag{2.1}$$

Also, there exists a constant $C > 0$ *such that*

$$|K^L_s(x,y) - H_s(x,y)| \le \frac{C}{\rho(x)^\sigma |x-y|^{d-\sigma}}, \tag{2.2}$$

where $\sigma = 2 - d/q_0 > 0$.

We also need the following classical result about bilinear operators, which is a special case of [6, Theorem I].

Lemma 2.4. *Suppose that* $\{T^1_i\}$ *and* $\{T^2_i\}$, $i = 1, \dots, N$ $(N \ge 2)$, *are Calderón–Zygmund operators on* $\mathbb{R}^d$. *Let*

$$S(f,g) = \sum_{i=1}^N (T^1_i f)(T^2_i g).$$

If

$$\int_{\mathbb{R}^d} S(f,g)(x)\,dx = 0 \quad \text{for all } f, g \in C^\infty_c(\mathbb{R}^d),$$

then, for $1 < p, q < \infty$ *with* $1/p + 1/q = 1$, S *maps* $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ *into* $H^1(\mathbb{R}^d)$ *and there exists a constant* $C > 0$ *such that*

$$\|S(f,g)\|_{H^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

Remark 2.5. Although Grafakos [6] proves his results for convolution operators, Lemma 2.4 can be proved for (generalized) Calderón–Zygmund operators by the same argument.

We are ready to show Theorem 1.1.

Proof of Theorem 1.1. We give the proof for the bilinear operator T^- only; the proof for T^+ is similar. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Assume that the parallel Calderón–Zygmund operators $\widetilde{T}_i$ $(i = 1, 2)$ and the parallel bilinear operator $\widetilde{T}^-$ satisfy (1.3) and (1.4). We split $T^-(f,g)$ into three parts,

$$T^-(f,g)(x) = T_1^-(f,g)(x) + T_2^-(f,g)(x) + \widetilde{T}^-(f,g)(x), \tag{2.3}$$

where

$$T_1^-(f,g) = (T_1 f - \widetilde{T}_1 f)(T_2 g) - (T_2 f - \widetilde{T}_2 f)(T_1 g),$$
$$T_2^-(f,g) = (\widetilde{T}_1 f)(T_2 g - \widetilde{T}_2 g) - (\widetilde{T}_2 f)(T_1 g - \widetilde{T}_1 g).$$

It follows from Lemma 2.4 that $\widetilde{T}^-(f,g) \in H^1(\mathbb{R}^d)$ and

$$\|\widetilde{T}^-(f,g)\|_{H^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

Since $H^1(\mathbb{R}^d) \subset H^1_L(\mathbb{R}^d)$, we have $\widetilde{T}^-(f,g) \in H^1_L(\mathbb{R}^d)$ and

$$\|\widetilde{T}^-(f,g)\|_{H^1_L(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{2.4}$$

It suffices to show $T_1^-(f,g) + T_2^-(f,g) \in H^1_L(\mathbb{R}^d)$, that is,

$$M^L(T_1^-(f,g) + T_2^-(f,g))(x) = \sup_{s>0} |T_s^L(T_1^-(f,g) + T_2^-(f,g))(x)| \in L^1(\mathbb{R}^d).$$

For simplicity we write $F(f,g)(x) = T_1^-(f,g)(x) + T_2^-(f,g)(x)$. Choose a sequence $\{x_{(k,\alpha)}\}$ of points and a family $\{\xi_{(k,\alpha)}\}$ of functions as in Lemma 2.1. Then

$$\begin{aligned} \|T_1^-(f,g) + T_2^-(f,g)\|_{H^1_L(\mathbb{R}^d)} &\le \sum_{(k,\alpha)} \Big(\int_{(B^{**}_{(k,\alpha)})^c} + \int_{B^{**}_{(k,\alpha)}} \Big) \sup_{s>0} |T_s^L(\xi_{(k,\alpha)} F(f,g))(x)|\,dx \\ &=: \sum_{(k,\alpha)} I_{(k,\alpha)} + \sum_{(k,\alpha)} J_{(k,\alpha)}. \end{aligned} \tag{2.5}$$

By (2.1),

$$\begin{aligned} I_{(k,\alpha)} &= \int_{(B^{**}_{(k,\alpha)})^c} \sup_{s>0} \Big| \int_{\mathbb{R}^d} K_s^L(x,y)\xi_{(k,\alpha)}(y)F(f,g)(y)dy \Big|\,dx \\ &\le C_l \int_{(B^{**}_{(k,\alpha)})^c} \sup_{s>0} \int_{\mathbb{R}^d} \rho(y)^l s^{-(d+l)/2} e^{-|x-y|^2/(5s)} |\xi_{(k,\alpha)}(y)F(f,g)(y)|\,dy\,dx \\ &\le C_l \int_{(B^{**}_{(k,\alpha)})^c} \int_{\mathbb{R}^d} \frac{\rho(y)^l}{|x-y|^{d+l}} |\xi_{(k,\alpha)}(y)F(f,g)(y)|\,dy\,dx. \end{aligned}$$

For $x \notin B^{**}_{(k,\alpha)}$ and $y \in \operatorname{supp} \xi_{(k,\alpha)} \subset B^*_{(k,\alpha)}$, Lemma 2.2 yields $\rho(y) \sim \rho(x_{(k,\alpha)})$ and

$$|x-y| \ge |x - x_{(k,\alpha)}| - |x_{(k,\alpha)} - y| \ge \tfrac{1}{2}|x - x_{(k,\alpha)}|.$$

Thus,

$$I_{(k,\alpha)} \le C_l \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)} \int\limits_{(B^{**}_{(k,\alpha)})^c} \frac{\rho(x_{(k,\alpha)})^l}{|x - x_{(k,\alpha)}|^{d+l}}\, dx$$
$$\le C_l \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)},$$

and hence

$$\sum_{(k,\alpha)} I_{(k,\alpha)} \le C \sum_{(k,\alpha)} \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)} = C\|F(f,g)\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{2.6}$$

To estimate $\sum_{(k,\alpha)} J_{(k,\alpha)}$, we write

$$\sum_{(k,\alpha)} J_{(k,\alpha)} \le \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{s \ge \rho(x_{(k,\alpha)})^2} |T^L_s(\xi_{(k,\alpha)} F(f,g))(x)|\, dx + \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0 < s < \rho(x_{(k,\alpha)})^2} |T^L_s(\xi_{(k,\alpha)} F(f,g))(x)|\, dx =: J_1 + J_2. \tag{2.7}$$

For J_1, we apply (2.1) and Lemma 2.2 to obtain

$$\int\limits_{B^{**}_{(k,\alpha)}} \sup_{s \ge \rho(x_{(k,\alpha)})^2} |T^L_s(\xi_{(k,\alpha)} F(f,g))(x)|\, dx$$
$$\le C_l \int\limits_{B^{**}_{(k,\alpha)}} \sup_{s \ge \rho(x_{(k,\alpha)})^2} \int\limits_{\mathbb{R}^d} \rho(y)^l s^{-(d+l)/2} e^{-|x-y|^2/(5s)} |\xi_{(k,\alpha)}(y) F(f,g)(y)|\, dy\, dx$$
$$\le C_l \int\limits_{B^{**}_{(k,\alpha)}} \sup_{s \ge \rho(x_{(k,\alpha)})^2} \int\limits_{\mathbb{R}^d} s^{-(d+l)/2} \rho(x_{(k,\alpha)})^l |\xi_{(k,\alpha)}(y) F(f,g)(y)|\, dy\, dx$$
$$\le C_l \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)} \int\limits_{B^{**}_{(k,\alpha)}} \rho(x_{(k,\alpha)})^{-d}\, dx \le C_l \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)}$$

and hence

$$J_1 \le C \sum_{(k,\alpha)} \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{2.8}$$

To estimate J_2, we decompose

$$J_2 \le \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0 < s < \rho(x_{(k,\alpha)})^2} |(T^L_s - T_s)(\xi_{(k,\alpha)} F(f,g))(x)|\, dx + \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0 < s < \rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T^-_1(f,g) + \xi_{(k,\alpha)} T^-_2(f,g))(x)|\, dx =: J_{21} + J_{22}. \tag{2.9}$$

For $x \in B^{**}_{(k,\alpha)}$ and $y \in \operatorname{supp} \xi_{(k,\alpha)} \subset B^{*}_{(k,\alpha)}$, by Lemma 2.2 we have $|x-y| < 6\rho(x_{(k,\alpha)}) \sim \rho(x)$. Then the estimate (2.2) gives

$$\begin{aligned}\int_{B^{**}_{(k,\alpha)}} &\sup_{0<s<\rho(x_{(k,\alpha)})^2} |(T^L_s - T_s)(\xi_{(k,\alpha)} F(f,g))(x)|\,dx \\ &\le C \int_{B^{**}_{(k,\alpha)}} \int_{\mathbb{R}^d} \frac{1}{\rho(x)^{\sigma}|x-y|^{d-\sigma}}\,|\xi_{(k,\alpha)}(y) F(f,g)(y)|\,dy\,dx \\ &\le C \int_{\mathbb{R}^d} |\xi_{(k,\alpha)} F(f,g)|\,\rho(x_{(k,\alpha)})^{-\sigma}\Big(\int_{|x|<6\rho(x_{(k,\alpha)})} \frac{dx}{|x|^{d-\sigma}}\Big)\,dy \\ &\le C\|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

which yields

$$J_{21} \le C \sum_{(k,\alpha)} \|\xi_{(k,\alpha)} F(f,g)\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}. \tag{2.10}$$

We still have to estimate J_{22}. For $s>0$, let $\Gamma_0 = \{y \in \mathbb{R}^d : |x-y| < \sqrt{s}\}$ and $\Gamma_n = \{y \in \mathbb{R}^d : 2^{n-1}\sqrt{s} \le |x-y| < 2^n\sqrt{s}\}$, $n \in \mathbb{N}$. Then

$$T_s(\xi_{(k,\alpha)} T^-_m(f,g))(x) = \sum_{n=0}^{\infty} T_s(\xi_{(k,\alpha)} T^-_m(f,g)\chi_{\Gamma_n})(x), \qquad m=1,2,$$

where χ_{Γ_n} denotes the characteristic function of the set Γ_n. Let $\eta \in C^\infty_0(\mathbb{R}^d)$ with $0 \le \eta \le 1$ satisfy $\eta(y)=1$ for $|y|<2$ and $\eta(y)=0$ for $|y|>4$. Set $\eta^n_0(y) = \eta\big(\frac{x-y}{2^n\sqrt{s}}\big)$ and $\eta^n_1(y) = 1-\eta^n_0(y)$, $n \in \mathbb{N}\cup\{0\}$. We split the operator $T^-_m(f,g)$ into four parts:

$$\begin{aligned}&T^-_m(f,g) \\ &\quad = T^-_m(f,\eta^n_1 g) + T^-_m(\eta^n_1 f, g) - T^-_m(\eta^n_1 f, \eta^n_1 g) + T^-_m(\eta^n_0 f, \eta^n_0 g), \qquad m=1,2.\end{aligned}$$

We first consider

$$\int_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T^-_1(f,\eta^n_1 g)\chi_{\Gamma_n})(x)|\,dx.$$

Write

$$T^-_1(f,\eta^n_1 g) = (U^1_1 f)\,U^2_1(\eta^n_1 g) - (U^1_2 f)\,U^2_2(\eta^n_1 g),$$

where

$$U^1_1 = T_1 - \widetilde{T}_1, \quad U^1_2 = T_2 - \widetilde{T}_2, \quad U^2_1 = T_2, \quad U^2_2 = T_2.$$

Then

$$\begin{aligned}
&|T_s(\xi_{(k,\alpha)}T_1^-(f,\eta_1^n g)\chi_{\Gamma_n})(x)| \\
&\quad\le \sum_{l=1}^{2}\int_{\Gamma_n} H_s(x-y)|U_l^1 f(y)|\,|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)|\,dy \\
&\qquad+\sum_{l=1}^{2}\int_{\Gamma_n} H_s(x-y)|U_l^1 f(y)|\,|U_l^2(\eta_1^n g)(x)|\,dy \\
&\quad\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\frac{1}{(2^n\sqrt{s})^d}\int_{|x-y|<2^n\sqrt{s}}|U_l^1 f(y)|\,|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)|\,dy \\
&\qquad+C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\frac{1}{(2^n\sqrt{s})^d}\int_{|x-y|<2^n\sqrt{s}}|U_l^1 f(y)|\,|U_l^2(\eta_1^n g)(x)|\,dy,
\end{aligned}$$

where we have used (1.2) in the last inequality. Because U_l^j $(j,l=1,2)$ are Calderón–Zygmund operators, their kernels satisfy the standard kernel estimate for some $\delta>0$. For $|x-y|\le 2^n\sqrt{s}$,

$$\begin{aligned}
|U_l^2(\eta_1^n g)(y)-U_l^2(\eta_1^n g)(x)| &\le \int_{|x-z|\ge 2^{n+1}\sqrt{s}}\frac{|x-y|^\delta}{|x-z|^{d+\delta}}|(\eta_1^n g)(z)|\,dz \\
&\le CM(\eta_1^n g)(x),
\end{aligned}$$

where M is the Hardy–Littlewood maximal operator. Thus,

$$\begin{aligned}
&|T_s(\xi_{(k,\alpha)}T_1^-(f,\eta_1^n g)\chi_{\Gamma_n})(x)| \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\big(M(U_l^1 f)(x)M(\eta_1^n g)(x)+M(U_l^1 f)(x)|U_l^2(\eta_1^n g)(x)|\big)
\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<s<\rho(x_{(k,\alpha)})^2}|T_s(\xi_{(k,\alpha)}T_1^-(f,\eta_1^n g)\chi_{\Gamma_n})(x)|\,dx \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\int_{\mathbb{R}^d}\sum_{l=1}^{2}M(U_l^1 f)(x)(M(\eta_1^n g)(x)+|U_l^2(\eta_1^n g)(x)|)\,dx \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\sum_{l=1}^{2}\|M(U_l^1 f)\|_{L^p(\mathbb{R}^d)}(\|M(g)\|_{L^q(\mathbb{R}^d)}+\|U_l^2 g\|_{L^q(\mathbb{R}^d)}) \\
&\qquad\le C2^{nd}e^{-2^{2n-4}}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.
\end{aligned}$$

A similar argument shows

$$\begin{aligned}
&\sum_{(k,\alpha)}\int_{B^{**}_{(k,\alpha)}}\sup_{0<s<\rho(x_{(k,\alpha)})^2}|T_s(\xi_{(k,\alpha)}T_1^-(\eta_1^n f,g)\chi_{\Gamma_n})(x)|\,dx \\
&\qquad\qquad\le C2^{nd}e^{-2^{2n-4}}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}
\end{aligned}$$

and

$$\sum_{(k,\alpha)} \int_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T_1^-(\eta_1^n f, \eta_1^n g)\chi_{\Gamma_n})(x)|\,dx \\ \le C2^{nd} e^{-2^{2n-4}} \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Finally we consider the term $T_1^-(\eta_0^n f, \eta_0^n g)$. Using (1.2) and (1.3), we obtain

$$\begin{aligned}
&\sup_{0<s<\rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T_1^-(\eta_0^n f, \eta_0^n g)\chi_{\Gamma_n})(x)| \\
&\le \sum_{l=1}^{2} \sup_{0<s<\rho(x_{(k,\alpha)})^2} \int_{\Gamma_n} H_s(x-y)|U_l^1(\eta_0^n f)(y)|\,|\xi_{(k,\alpha)}(y)|\,|U_l^2(\eta_0^n g)(y)|\,dy \\
&\le \sup_{0<s<\rho(x_{(k,\alpha)})^2} \int_{\Gamma_n} H_s(x-y)\Big(\int_{\mathbb{R}^d} |T_1(y,z)-\widetilde{T}_1(y,z)|\,|(\eta_0^n f)(z)|\,dz\Big) \\
&\qquad\qquad \times |T_2(\eta_0^n g)(y)|\,dy \\
&\quad + \sup_{0<s<\rho(x_{(k,\alpha)})^2} \int_{\Gamma_n} H_s(x-y)\Big(\int_{\mathbb{R}^d} |T_2(y,z)-\widetilde{T}_2(y,z)|\,|(\eta_0^n f)(z)|\,dz\Big) \\
&\qquad\qquad \times |T_1(\eta_0^n g)(y)|\,dy \\
&\le Ce^{-2^{2n-4}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} s^{-d/2} \|\eta_0^n g\|_{L^{p_1}(\mathbb{R}^d)} \\
&\quad \times \bigg\{\int_{\Gamma_n}\bigg(\int_{\mathbb{R}^d} \frac{|(\eta_0^n f)(z)|}{\rho(z)^\delta |y-z|^{d-\delta}}\,dz\bigg)^{p_1'} dy\bigg\}^{1/p_1'},
\end{aligned}$$

where $1/p_1' + 1/p_1 = 1$. For $z \in \operatorname{supp} \eta_0^n = \{y \in \mathbb{R}^d : |x-y| \le 2^{n+2}\sqrt{s}\}$, $x \in B^{**}_{(k,\alpha)}$, and $s < \rho(x_{(k,\alpha)})^2$, we have $|z - x_{(k,\alpha)}| \le 2^{n+3}\rho(x_{(k,\alpha)})$. It follows from Lemma 2.2 that $\rho(z)^{-\delta} \le C2^{n\delta l_0}\rho(x_{(k,\alpha)})^{-\delta}$ for a fixed constant $l_0 > 0$. Hence, a well-known result for fractional integrals gives

$$\begin{aligned}
&\sup_{0<s<\rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T_1^-(\eta_0^n f, \eta_0^n g)\chi_{\Gamma_n})(x)| \\
&\qquad \le C2^{n\delta l_0} e^{-2^{2n-4}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} s^{-d/2} \rho(x_{(k,\alpha)})^{-\delta} \bigg\| |\eta_0^n f| * \frac{1}{|\cdot|^{d-\delta}} \bigg\|_{L^{p_1'}(\mathbb{R}^d)} \\
&\qquad\quad \times \|\eta_0^n g\|_{L^{p_1}(\mathbb{R}^d)} \\
&\qquad \le C2^{n\delta l_0} e^{-2^{2n-4}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} \frac{(2^n\sqrt{s})^{d+\delta}}{\sqrt{s^d}\rho(x_{(k,\alpha)})^\delta} \frac{1}{(2^n\sqrt{s})^{d+\delta}} \|\eta_0^n f\|_{L^{q_1}(\mathbb{R}^d)} \\
&\qquad\quad \times \|\eta_0^n g\|_{L^{p_1}(\mathbb{R}^d)} \\
&\qquad \le C2^{n(d+\delta+\delta l_0)} e^{-2^{2n-4}} M(|f|^{q_1})^{1/q_1}(x) M(|g|^{p_1})^{1/p_1}(x),
\end{aligned}$$

where $1/p_1' = 1/q_1 - \delta/d$. Since $1/p_1 + 1/q_1 = 1 + \delta/d > 1/p + 1/q$, we are always able to choose p_1 and q_1 such that $1 < p_1 < q$ and $1 < q_1 < p$. Then

$$\begin{aligned}\sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} &|T_s(\xi_{(k,\alpha)} T_1^-(\eta_0^n f, \eta_0^n g)\chi_{\Gamma_n})(x)|\,dx \\ &\le C2^{n(d+\delta+\delta l_0)} e^{-2^{2n-4}} \int\limits_{\mathbb{R}^d} M(|f|^{q_1})^{1/q_1}(x) M(|g|^{p_1})^{1/p_1}(x)\,dx \\ &\le C2^{n(d+\delta+\delta l_0)} e^{-2^{2n-4}} \\ &\quad \times \Big(\int\limits_{\mathbb{R}^d} M(|f|^{q_1})^{p/q_1}(x)\,dx\Big)^{1/p} \Big(\int\limits_{\mathbb{R}^d} M(|g|^{p_1})^{q/p_1}(x)\,dx\Big)^{1/q} \\ &\le C2^{n(d+\delta+\delta l_0)} e^{-2^{2n-4}} \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Therefore,

$$\begin{aligned}(2.11)\qquad &\sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} |T_s(\xi_{(k,\alpha)} T_1^-(f, g))(x)|\,dx \\ &\le \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} \sum_{n=0}^{\infty} |T_s(\xi_{(k,\alpha)} T_1^-(f, g)\chi_{\Gamma_n})(x)|\,dx \\ &\le C\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \sum_{n=0}^{\infty} 2^{n(d+\delta+\delta l_0)} e^{-2^{2n-4}} \le C\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Similarly,

$$\begin{aligned}(2.12)\qquad \sum_{(k,\alpha)} \int\limits_{B^{**}_{(k,\alpha)}} \sup_{0<s<\rho(x_{(k,\alpha)})^2} &|T_s(\xi_{(k,\alpha)} T_2^-(f, g))(x)|\,dx \\ &\le C\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Both (2.11) and (2.12) imply

$$(2.13)\qquad J_{22} \le C\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Combining (2.3)–(2.10) and (2.13), we complete the proof for the bilinear operator T^-. ∎

3. Examples. In this section we give some examples to illustrate our result.

EXAMPLE 3.1. Let $T_1 = L^{i\gamma_1}$, $T_2 = L^{i\gamma_2}$, $\gamma_1, \gamma_2 \in \mathbb{R}$. Then T_1, T_2 are Calderón–Zygmund operators (cf. [7, Theorem 0.4]) and the parallel operators $\widetilde{T}_1 = (-\Delta)^{i\gamma_1}, \widetilde{T}_2 = (-\Delta)^{i\gamma_2}$ satisfy the condition (1.3) with $\delta = 2 - d/q_0$ (cf. [7, (4.11)]). It is easy to verify that $\widetilde{T}^-(f,g) = (\widetilde{T}_1 f)(\widetilde{T}_2 g) - (\widetilde{T}_2 f)(\widetilde{T}_1 g)$

has the vanishing moment. Therefore, the bilinear operator $T^-(f,g) = (T_1 f)(T_2 g) - (T_2 f)(T_1 g)$ satisfies the assumptions of Theorem 1.1.

EXAMPLE 3.2. Suppose $V \in RH_{q_0}$ for some $q_0 > d$. Let $T_1 = L^{-1/2}\partial/\partial x_j$, $T_2 = L^{-1/2}\partial/\partial x_k$, $1 \le j,k \le d$. Then T_1, T_2 are Calderón–Zygmund operators (cf. [7, Theorem 0.8]) and the parallel operators $\widetilde{T}_1 = (-\Delta)^{-1/2}\partial/\partial x_j = R_j$, $\widetilde{T}_2 = (-\Delta)^{-1/2}\partial/\partial x_k = R_k$ are the classical Riesz transforms, which satisfy the condition (1.3) with $\delta = 2 - d/q_0$ (cf. [7, (5.9)]). Because $\widetilde{T}^-(f,g) = (\widetilde{T}_1 f)(\widetilde{T}_2 g) - (\widetilde{T}_2 f)(\widetilde{T}_1 g)$ has the vanishing moment, the bilinear operator $T^-(f,g) = (T_1 f)(T_2 g) - (T_2 f)(T_1 g)$ satisfies the assumptions of Theorem 1.1.

EXAMPLE 3.3. Suppose $V \in RH_{q_0}$ for some $q_0 > d$. Let $T_1 = (\partial/\partial x_j) \circ L^{-1}\partial/\partial x_k$, $T_2 = (\partial/\partial x_l)L^{-1}\partial/\partial x_m$, $1 \le j,k,l,m \le d$. Then T_1, T_2 are Calderón–Zygmund operators (cf. [7, Theorem 0.8]). The parallel operators $\widetilde{T}_1 = R_j R_k$, $\widetilde{T}_2 = R_l R_m$ also satisfy the condition (1.3) with $\delta = 2 - d/q_0$. This will be proved in Lemma 3.7 below. As in Examples 3.1 and 3.2, $\widetilde{T}^-(f,g) = (\widetilde{T}_1 f)(\widetilde{T}_2 g) - (\widetilde{T}_2 f)(\widetilde{T}_1 g)$ has the vanishing moment. Thus the bilinear operator $T^-(f,g) = (T_1 f)(T_2 g) - (T_2 f)(T_1 g)$ satisfies the assumptions of Theorem 1.1.

EXAMPLE 3.4. Suppose $V \in RH_{q_0}$ for some $q_0 > d$. Let $T_1 = L^{-1/2}\partial/\partial x_j$, $T_2 = (\partial/\partial x_k)L^{-1}\partial/\partial x_l$, $1 \le j,k,l \le d$. Then the parallel bilinear operator $\widetilde{T}^+(f,g) = (\widetilde{T}_1 f)(\widetilde{T}_2 g) + (\widetilde{T}_2 f)(\widetilde{T}_1 g)$ has the vanishing moment. It follows that the bilinear operator $T^+(f,g) = (T_1 f)(T_2 g) + (T_2 f)(T_1 g)$ satisfies the assumptions of Theorem 1.1.

EXAMPLE 3.5. Suppose $V \in RH_{q_0}$ for some $q_0 > d$. Let $T_1 = L^{i\gamma}$, $\gamma \in \mathbb{R}$, $T_2 = L^{-1/2}\partial/\partial x_j$, $1 \le j \le d$. As in Example 3.4, the parallel operator $\widetilde{T}^+(f,g) = (\widetilde{T}_1 f)(\widetilde{T}_2 g) + (\widetilde{T}_2 f)(\widetilde{T}_1 g)$ has the vanishing moment. It follows that the bilinear operator $T^+(f,g) = (T_1 f)(T_2 g) + (T_2 f)(T_1 g)$ satisfies the assumptions of Theorem 1.1.

We also give a counterexample for which the classical Hardy space estimate fails.

EXAMPLE 3.6. Consider the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^3$. We have

$$H = \frac{1}{2}\sum_{j=1}^{3}(A_j A_j^* + A_j^* A_j),$$

where

$$A_j = -\frac{\partial}{\partial x_j} + x_j, \quad A_j^* = \frac{\partial}{\partial x_j} + x_j, \quad j = 1,2,3.$$

Let

$$T_j = H^{-1/2}\frac{\partial}{\partial x_j} = \frac{1}{2}H^{-1/2}(A_j^* - A_j),$$

and $f, g \in L^2(\mathbb{R}^3)$. In view of Example 3.2, the bilinear operator $T^-(f,g) = (T_1 f)(T_2 g) - (T_2 f)(T_1 g)$ belongs to the Hardy type space $H^1_H(\mathbb{R}^d)$ associated with the Hermite operator H.

The 3-dimensional Hermite functions $\Phi_\mu(x)$ are the eigenfunctions of H and form an orthonormal basis for $L^2(\mathbb{R}^3)$. Moreover,

$$A_j\Phi_\mu = (2\mu_j + 2)^{1/2}\Phi_{\mu+e_j}, \quad A_j^*\Phi_\mu = (2\mu_j)^{1/2}\Phi_{\mu-e_j},$$

where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ (cf. [8, pp. 5–6]), and

$$H^{-1/2}\Phi_\mu = (2|\mu| + 3)^{-1/2}\Phi_\mu.$$

Let $f(x) = \Phi_{e_1}(x)$ and $g(x) = \Phi_{e_2}(x)$. We have

$$T_1 f(x) = \frac{1}{2}H^{-1/2}(A_1^* - A_1)\Phi_{e_1}(x) = \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{3}}\Phi_{(0,0,0)}(x) - \frac{2}{\sqrt{7}}\Phi_{(2,0,0)}(x)\right),$$

$$T_2 f(x) = \frac{1}{2}H^{-1/2}(A_2^* - A_2)\Phi_{e_1}(x) = \frac{1}{2}\left(0 - \frac{\sqrt{2}}{\sqrt{7}}\Phi_{(1,1,0)}(x)\right),$$

$$T_1 g(x) = \frac{1}{2}H^{-1/2}(A_1^* - A_1)\Phi_{e_2}(x) = \frac{1}{2}\left(0 - \frac{\sqrt{2}}{\sqrt{7}}\Phi_{(1,1,0)}(x)\right),$$

$$T_2 g(x) = \frac{1}{2}H^{-1/2}(A_2^* - A_2)\Phi_{e_2}(x) = \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{3}}\Phi_{(0,0,0)}(x) - \frac{2}{\sqrt{7}}\Phi_{(0,2,0)}(x)\right).$$

It follows from the orthogonality of $\{\Phi_\mu\}$ that

$$\begin{aligned}\int_{\mathbb{R}^3} T^-(f,g)(x)\,dx &= \int_{\mathbb{R}^3} (T_1 f(x) T_2 g(x) - T_2 f(x) T_1 g(x))\,dx \\ &= \frac{1}{6} - \frac{1}{14} = \frac{2}{21}.\end{aligned}$$

Thus $T^-(f,g)$ has a nonzero integral, which implies $T^-(f,g) \notin H^1(\mathbb{R}^3)$.

Finally, we prove that the parallel operators in Example 3.3 satisfy the condition (1.3). Let $R^L_{j,k}(x,y)$ and $R_{j,k}(x,y)$ denote the kernels of the operators $(\partial/\partial x_j)L^{-1}\partial/\partial x_k$ and $R_j R_k$ for $1 \le j,k \le d$, respectively.

LEMMA 3.7. *Suppose $V \in RH_{q_0}$, $q_0 > d$, and $\delta = 2 - d/q_0$. Then, for $1 \le j,k \le d$,*

$$|R^L_{j,k}(x,y) - R_{j,k}(x,y)| \le \frac{C}{\rho(y)^\delta |x-y|^{d-\delta}}, \quad x \ne y. \tag{3.1}$$

To prove Lemma 3.7, we need the following three lemmas (see [7]).

Lemma 3.8. *Suppose $V \in RH_{q_0}$, $q_0 > d/2$. For $0 < r < R < \infty$,*

$$\frac{1}{r^{d-2}} \int\limits_{B(x,r)} V(y)\,dy \le C\left(\frac{r}{R}\right)^{2-d/q_0} \frac{1}{R^{d-2}} \int\limits_{B(x,R)} V(y)\,dy.$$

Lemma 3.9. *Suppose $V \in RH_{q_0}$, $q_0 > d/2$. Then*

$$\frac{1}{r^{d-2}} \int\limits_{B(x,r)} V(y)\,dy \sim 1 \quad \text{if and only if} \quad r \sim \rho(x).$$

Lemma 3.10. *Suppose $V \in RH_{q_0}$, $q_0 > d$. There exist $C > 0$ and $l_0 > 0$ such that*

$$\frac{1}{R^{d-2}} \int\limits_{B(x,R)} V(y)\,dy \le C\left(1 + \frac{R}{\rho(x)}\right)^{l_0}$$

and

$$\int\limits_{B(x,R)} \frac{V(y)}{|x-y|^{d-1}}\,dy \le \frac{C}{R^{d-1}} \int\limits_{B(x,R)} V(y)\,dy.$$

Proof of Lemma 3.7. If $|x-y| > \rho(y)$, the estimate (3.1) is trivial. Suppose $|x-y| \le \rho(y)$. Let $\Gamma^L(x,y)$ and $\Gamma(x,y)$ be the fundamental solutions for L and $-\Delta$, respectively. Note that

$$-\Delta_x(\Gamma^L(x,y) - \Gamma(x,y)) = -V(x)\Gamma^L(x,y),$$

where Δ_x denotes the Laplacian in the variable x. We have

$$\Gamma^L(x,y) - \Gamma(x,y) = -\int\limits_{\mathbb{R}^d} \Gamma(x,z)V(z)\Gamma^L(z,y)\,dz.$$

Thus,

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma^L(x,y) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma(x,y) = -\int\limits_{\mathbb{R}^d} \frac{\partial}{\partial x_j}\Gamma(x,z)V(z)\frac{\partial}{\partial y_k}\Gamma^L(z,y)\,dz.$$

Let $R = |x-y|/4$. By [7, (6.1)], for any $l > 0$, there exists C_l such that

$$\begin{aligned}
&\left|\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma^L(x,y) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma(x,y)\right| \\
&\qquad \le \int\limits_{\mathbb{R}^d} \frac{C_l}{|x-z|^{d-1}} \cdot \frac{V(z)\,dz}{(1+|y-z|\,\rho(h)^{-1})^l |y-z|^{d-1}} \\
&\qquad = \int\limits_{|x-z|<R/2} + \int\limits_{|y-z|<R/2} + \int\limits_{|x-z|\ge R/2,\,|y-z|\ge R/2} \\
&\qquad = I_1 + I_2 + I_3.
\end{aligned}$$

Using Lemmas 3.10, 3.8, 3.9 and 2.2, we obtain

$$I_1 \le \frac{C_l}{R^{d-1}} \int\limits_{B(x,R/2)} \frac{V(z)\,dz}{|x-z|^{d-1}} \le \frac{C_l}{R^{2d-2}} \int\limits_{B(x,R/2)} V(z)\,dz \le \frac{C_l}{\rho(y)^\delta R^{d-\delta}}.$$

Similarly,

$$I_2 \le \frac{C_l}{\rho(y)^\delta R^{d-\delta}}.$$

Note that $|x-z| \sim |y-z|$ when $|x-z| \ge R/2$ and $|y-z| \ge R/2$. This yields

$$\begin{aligned} I_3 &\le C_l \int\limits_{|y-z|\ge R/2} \frac{V(z)\,dz}{(1+|y-z|\,\rho(y)^{-1})^l |y-z|^{2d-2}} \\ &\le C_l\Bigg(\int\limits_{\rho(y)>|y-z|\ge R/2} \frac{V(z)\,dz}{|y-z|^{2d-2}} + \rho(y)^l \int\limits_{|y-z|\ge\rho(y)} \frac{V(z)\,dz}{|y-z|^{2d-2+l}}\Bigg). \end{aligned}$$

By Hölder's inequality, the RH_{q_0} condition, and Lemma 3.9,

$$\begin{aligned} &\int\limits_{\rho(y)>|y-z|\ge R/2} \frac{V(z)\,dz}{|y-z|^{2d-2}} \\ &\qquad\le C\Bigg(\int\limits_{B(y,\rho(y))} V(z)^{q_0}\,dz\Bigg)^{1/q_0} \Bigg(\int_{R/2}^{\rho(y)} t^{-(2d-2)q_0'+d-1}\,dt\Bigg)^{1/q_0'} \\ &\qquad\le C\rho(y)^{d/q_0-2} R^{-(2d-2)+d/q_0'} = \frac{C}{\rho(y)^\delta R^{d-\delta}}, \end{aligned}$$

where $d-(2d-2)q_0' < 0$ is valid for $d \ge 3$. Using Lemma 3.10 and taking l sufficiently large, we obtain

$$\begin{aligned} &\rho(y)^l \int\limits_{|y-z|\ge\rho(y)} \frac{V(z)\,dz}{|y-z|^{2d-2+l}} \\ &\qquad\le C_l\,\rho(y)^l \sum_{m=1}^{\infty} (2^m\rho(y))^{-2d+2-l} \int\limits_{B(y,2^m\rho(y))} V(z)\,dz \\ &\qquad\le \frac{C_l}{\rho(y)^d} \sum_{m=1}^{\infty} 2^{-(l-l_0+d)m} = \frac{C_l}{\rho(y)^d} \le \frac{C_l}{\rho(y)^\delta R^{d-\delta}}, \end{aligned}$$

where we have used $R \le \rho(y)$ and $d > \delta$ in the last inequality. Thus we obtain

$$\begin{aligned} |R^L_{j,k}(x,y) - R_{j,k}(x,y)| &= \left| \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma^L(x,y) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k}\Gamma(x,y)\right| \\ &\le \frac{C}{\rho(y)^\delta |x-y|^{d-\delta}} \end{aligned}$$

and Lemma 3.7 follows. ∎

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