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The topological entropy versus level sets for interval maps

by

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Abstract. We answer affirmatively Coven's question [PC]: Suppose $f: I \to I$ is a continuous function of the interval such that every point has at least two preimages. Is it true that the topological entropy of f is greater than or equal to $\log 2$?

0. Introduction. The aim of this paper is to estimate the topological entropy of an interval map knowing the cardinalities of its level sets.

The topological entropy as a numerical measure for the complexity of a dynamical system is deeply studied in various contexts. One direction of possible research concerns the connection between entropy and level sets for a continuous (differentiable) map. Several interesting results have already been found—see for instance [Bo], [Ly], [MP], [MS]. However, many questions remain open even for interval maps. Our goal is to show one very particular result (Theorem 3.3), the proof of which is neither immediate nor easy. The solution is based on known strong results (Theorems 1.1 and 1.3) concerning the topological entropy and symbolic dynamics. We suppose that our proof can partially explain some difficulties that we meet when similar questions are considered in other topological dynamics.

Let $I = [a, b]$ be a closed real interval and let $L_2(I)$ be the set of all continuous functions mapping I into itself that satisfy the condition

$$\forall y \in I: \quad \operatorname{card} f^{-1}(y) \geq 2. \tag{1}$$

We show the following.

THEOREM 3.3. *The topological entropy of $f \in L_2(I)$ is greater than or equal to* $\log 2$.

The problem of entropy of maps from $L_2(I)$ has been stated by E. M. Coven in [PC]. The idea why the entropy of any map f from $L_2(I)$ should

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be at least $\log 2$ is not difficult and is based on a natural approach described below.

Definition 0.1. Let (X,ϱ) be a compact metric space, $f\colon X\to X$ be continuous and $S_0,S_1\subset X$ be closed. We say that the sets S_0,S_1 *create a 2-horseshoe* if $f(S_0)\cap f(S_1)\supset S_0\cup S_1$ and $S_0\cap S_1=\emptyset$.

As an easy consequence of the definition of topological entropy we obtain the following [DGS].

Proposition 0.2. *Let (X,ϱ) be a compact metric space, and $f\colon X\to X$ be continuous. If the sets $S_0,S_1\subset X$ create a 2-horseshoe then the topological entropy of f is greater than or equal to* $\log 2$.

Now, for $f\in L_2(I)$ define the set S_0, resp. S_1, as the closure of the minima, resp. maxima, of $f^{-1}(y)$, $y\in I$. Then $f(S_0)=f(S_1)=I$. If $S_0\cap S_1=\emptyset$ then in accordance with Definition 0.1 the sets S_0,S_1 create a 2-horseshoe, hence by Proposition 0.2, $\operatorname{ent}(f)\geq\log 2$. In general, for $f\in L_2(I)$ the intersection $S_0\cap S_1$ can be finite or even countable—we show this in Lemmas 2.1 and 2.2. Thus, the sets S_0,S_1 sometimes create an "almost" 2-horseshoe and one can assume the bound $\log 2$ again.

However, $S_0\cap S_1$ can contain the whole trajectory, hence all itineraries of one point taken with respect to S_0,S_1 give all 0-1 unilateral sequences. Then the corresponding shift map has entropy $\log 2$. At the same time it is not clear what is the entropy of f in this case. As the reader will see, the detailed analysis shows that there are sufficiently many points with different trajectories giving entropy $\log 2$ that are far from the intersection $S_0\cap S_1$.

The conclusion of Theorem 3.3 is rather delicate. One can easily find an interval map of zero entropy that does not satisfy (1) exactly for one point from I. Moreover, it is easy to see that a result analogous to our theorem does not hold for continuous circle maps.

The paper is organized as follows:

In Section 1 we give some basic notation, definitions and known results—Theorems 1.1 and 1.3.

Section 2 is devoted to the lemmas used throughout the paper.

In Section 3 we prove the key Lemma 3.1, Corollary 3.2 and Theorem 3.3.

1. Definitions and known results. We denote by $\mathbb{N}$ the set of positive integers.

We work with some *topological dynamics* (X,T), where X is a compact metric space and $T\colon X\to X$ is a continuous map. (X,T) is *minimal* if for each $x\in X$ the set $\{T^i(x)\colon\ i\in\mathbb{N}\}$ is dense in X. A subset M of X is *T-invariant* if $T(M)\subset M$. We say that $M\subset X$ is *minimal* (in X) if M is closed, T-invariant and $(M,T|M)$ is minimal.

Let ϱ be a metric on the space X. For the notion of *topological entropy* we use Bowen's definition [DGS]. A set $E \subset X$ is (n,ε)-*separated* (with respect to T) if, whenever $x, y \in E$, $x \neq y$ then $\max_{0\le i\le n-1} \varrho(T^i(x), T^i(y)) > \varepsilon$.

For a compact set $K \subset X$ we denote by $s(n,\varepsilon,K)$ the largest cardinality of any (n,ε)-separated subset of K. Put

$$\operatorname{ent}(T,K) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon,K)$$

and $\operatorname{ent}(T) = \operatorname{ent}(T,X)$. The quantity $\operatorname{ent}(T)$ is called the *topological entropy* of T.

A topological dynamics (Y,S) is a *factor* of (X,T) if there is a continuous surjective factor map $h: X \to Y$ such that $h \circ T = S \circ h$. The following theorem can be found in [Bo].

Theorem 1.1 ([Bo]). *If* (Y,S) *is a factor of* (X,T) *then*

$$\operatorname{ent}(S) \le \operatorname{ent}(T) \le \operatorname{ent}(S) + \sup_{y\in Y} \operatorname{ent}(T, h^{-1}(\{y\})).$$

As usual, the ω-*limit set* $\omega_T(x)$ of $x \in X$ consists of all the limit points of $\{T^i(x) \colon\ i \in \mathbb{N}\}$. A point $x \subset X$ is called *periodic* of period $n \in \mathbb{N}$ if $T^n(x) = x$ and $T^k(x) \neq x$ for $0 < k < n$. The set of all periodic points is denoted by $\operatorname{Per}(T)$. A normalized Borel measure μ on X is T-*invariant* if $\mu(T^{-1}(E)) = \mu(E)$ for each Borel set $E \subset X$.

Now we list several useful properties of ω-limit and minimal sets. As is well known they can be considered in any topological dynamics (X,T).

Lemma 1.2. (i) *For each* $x \in X$, *the* ω-*limit set* $\omega_T(x)$ *contains some minimal set.*

(ii) *Any minimal set in* X *is either finite and then a periodic orbit of* T, *or infinite and then uncountable.*

(iii) *If* (X,T) *is minimal and a measure* μ *on* X *is* T-*invariant then either* X *is finite and then* μ *is atomic*, *or* X *is infinite and then* μ *is nonatomic. In any case* $\operatorname{supp}\mu = X$.

(iv) *Let* $M \subset X$ *be minimal in* X. *If* M *is infinite then for each countable closed set* $C \subset M$ *and* $x \in M$,

$$\lim_{n\to\infty} \frac{\operatorname{card}\{0 \le i \le n-1 \colon\ T^i(x) \in C\}}{n} = 0.$$

(v) *Let* $\omega_T \subset X$ *be an* ω-*limit set. Then* $T(\omega_T) = \omega_T$ *and if for some open* G *we have* $\emptyset \neq G_1 = G \cap \omega_T \subsetneq \omega_T$ *then* $T(\overline{G}_1) \not\subseteq G_1$. *In particular*, $p \in \operatorname{Per}(T) \cap \omega_T$ *is not an isolated point in* ω_T.

Proof. See [DGS] for (i)–(iii) and [BC, Lemma 3, p. 70] for (v).

Let us prove (iv). Notice that by our assumption and (ii), M is uncountable. Suppose to the contrary that there is an increasing sequence $\{k_n\}_{n=1}^{\infty}$

such that

$$\lim_n \frac{C(k_n,x)}{k_n} = a \in (0,1],$$

where $C(n,x) = \operatorname{card}\{0 \le i \le n-1\colon\ T^i(x) \in C\}$. Then, using the standard method [DGS, Prop. 2.7], we can find an atomic T-invariant measure μ for which $\mu(C) > 0$ and $\operatorname{supp}\mu \subsetneq M$—a contradiction with (ii) and (iii). ■

We use the *symbolic dynamics* [DGS]. Consider $N_2 = \{0,1\}$ as a finite space with the discrete topology, and denote by Ω_2 the infinite product space $\prod_{i=0}^{\infty} X_i$, where $X_i = N_2$ for all i. The shift map $\sigma\colon \Omega_2 \to \Omega_2$ is defined by $(\sigma(\omega))_i = \omega_{i+1},\ i \in \mathbb{N}\cup\{0\}$. Obviously, the pair (Ω_2,σ) is a topological dynamics.

It is well known [DGS, Prop. 16.11] that for $\Omega \subset \Omega_2$ closed and σ-invariant,

$$\operatorname{ent}(\sigma,\Omega) = \lim_n \frac{1}{n} \log\operatorname{card}\Omega(n), \tag{2}$$

where $\Omega(n) = \{\omega(n) = (\omega_0,\ldots,\omega_{n-1})\colon\ \omega \in \Omega\}$.

The following result is known:

Theorem 1.3 ([G]). *For any positive ε there is a minimal set Γ in Ω_2 such that* $\operatorname{ent}(\sigma,\Gamma) > -\varepsilon + \log 2$.

The following easy lemma is needed in the proof of Theorem 3.3 in Section 3. Put $\Omega_{j,k} = \{\omega \in \Omega_2\colon\ \omega_{2i+j} = k \text{ for each } i \in \mathbb{N}\cup\{0\}\}$, $j,k \in N_2$, and

$$\Omega(M(\infty)) = \bigcup_{j,k\in N_2} \Omega_{j,k}. \tag{3}$$

Lemma 1.4. *The set $\Omega = \Omega(M(\infty))$ is closed σ-invariant in Ω_2 and* $\operatorname{ent}(\sigma,\Omega) = \frac{1}{2}\log 2$.

Proof. The closedness of Ω is clear. Since $\sigma(\Omega) \subset \Omega$, we can compute the entropy $\operatorname{ent}(\sigma,\Omega)$ using (2). Obviously, for each $n \in \mathbb{N}$ and $j,k \in N_2$ we have $\operatorname{card}\Omega_{j,k}(2n) = 2^n$, hence the conclusion follows. ■

2. Basic properties of maps from $L_2(I)$. This section is devoted to developing preliminary results. Statements 2.1–2.3 describe some simple properties of maps satisfying (1). The main results of this section are summarized in Lemma 2.4 and Corollary 2.5. These results show that it is sufficient to prove Theorem 3.3 for maps from a proper subset of $L_2(I)$ (see (5)).

In what follows we use the notation

$$\begin{aligned} B_1(f) &= \{x\in[a,b]\colon\ f(y)\ge f(x),\ a\le y\le x \ \&\ f(x)\ge f(y),\ x\le y\le b\},\\ B_2(f) &= \{x\in[a,b]\colon\ f(y)\le f(x),\ a\le y\le x \ \&\ f(x)\le f(y),\ x\le y\le b\}, \end{aligned} \tag{$\star$}$$

and $B(f) = B_1(f) \cup B_2(f)$. If there is no ambiguity we often write B_i, resp. B, instead of $B_i(f)$, resp. $B(f)$.

For $f \in L_2(I)$ and $y \in I$ we put $m_y = m_y(f) = \min f^{-1}(y)$ and $M_y = M_y(f) = \max f^{-1}(y)$. The closed sets $S_0 = S_0(f)$, $S_1 = S_1(f)$, $S_{01} = S_{01}(f)$ are defined as

$$S_0 = \overline{\{m_y\colon\ y \in I\}}, \qquad S_1 = \overline{\{M_y\colon\ y \in I\}}, \qquad S_{01} = S_0 \cap S_1.$$

Obviously $f(S_0) = f(S_1) = I$. As stated in Definition 0.1 and Proposition 0.2, if $S_{01} = \emptyset$ then the sets S_0, S_1 create a 2-horseshoe and $\operatorname{ent}(f) \geq \log 2$. That is why our attention is focused on the case when $S_{01} \neq \emptyset$.

An important lemma follows. For the sake of completeness we present its proof which is not difficult but contains a rather laborious calculation.

LEMMA 2.1. *Let $f \in L_2(I)$ and $S_{01} \neq \emptyset$. Then*:

(i) *Either B_1 or B_2 is empty, hence $B \in \{B_1, B_2\}$.*
(ii) $S_{01} \subset B \setminus \{a, b\}$.

Proof. (i) By ($\star$) the sets B_1, B_2 are closed. Suppose that $B_1 \neq \emptyset \neq B_2$.

If there exists an $\alpha \in B_1$ for which $f(\alpha) \in (a, b)$ then from ($\star$) we get $f([a, b]) \subset [a, f(\alpha)]$ if $\alpha < \beta$ for some $\beta \in B_2$, and $f([a, b]) \subset [f(\alpha), b]$ if $\alpha \geq \beta$ for some $\beta \in B_2$. But $f \in L_2(I)$ has to be surjective, hence $f(\alpha) \in \{a, b\}$ for each $\alpha \in B_1$ and by symmetry also $f(\beta) \in \{a, b\}$ for each $\beta \in B_2$.

If $f(B_1) = \{a, b\}$ then ($\star$) gives $B_2 = \emptyset$ and analogously the equality $f(B_2) = \{a, b\}$ implies $B_1 = \emptyset$. But we assume that both sets B_1, B_2 are nonempty. Thus, it remains to check the following four cases: (1) $f(B_1) = \{a\}$, $f(B_2) = \{b\}$; (2) $f(B_1) = \{a\}$, $f(B_2) = \{a\}$; (3) $f(B_1) = \{b\}$, $f(B_2) = \{b\}$; (4) $f(B_1) = \{b\}$, $f(B_2) = \{a\}$. Using ($\star$) the reader can verify by himself that (1) and (4) are impossible again.

Suppose that (2) holds and write $\beta = \max B_2$ and $\alpha = \min B_1$. From ($\star$) we obtain $f([a, \beta]) = \{a\}$ and $f([\alpha, b]) = \{a\}$, hence $\beta < \alpha$. Since $f \in L_2(I)$ we get $\operatorname{card} f^{-1}(b) \cap (\beta, \alpha) \geq 2$, hence $S_{01} = \emptyset$—a contradiction. The case (3) can be disproved similarly. We have shown that if $S_{01} \neq \emptyset$ then either B_1 or B_2 is empty. This proves (i).

(ii) As we have seen in (i) if $S_{01} \neq \emptyset$ then $B \in \{B_1, B_2\}$. We prove the conclusion for $B = B_1$.

Take $c, d \in B \cup \{a, b\}$ such that $(c, d) \cap B = \emptyset$. Under additional assumptions on c, d and $u \notin B \setminus \{a, b\}$ we obtain the following implications I_j, $j = 1, \ldots, 13$:

(1) $c, d \in B$ ($f([c, d]) = [f(d), f(c)]$) and $u \in (c, d)$; then by ($\star$) we get

$$I_1: \quad \min\{f(x)\colon\ x \in [c, u]\} < f(u) \ \Rightarrow\ u \notin S_0,$$
$$I_2: \quad \max\{f(x)\colon\ x \in [u, d]\} > f(u) \ \Rightarrow\ u \notin S_1.$$

(2) $c = a$, $d \in B$ ($f([c,d]) = [f(d), \cdot]$) and $u \in [c,d)$; then by ($\star$) we obtain

$$\begin{aligned}
I_3: &\quad \max\{f(x)\colon\ x \in [u,d]\} > f(u) \ \Rightarrow\ u \notin S_1,\\
I_4: &\quad (f(u) = b \,\&\, \exists x \in (u,d)\colon\ f(x) = b) \ \Rightarrow\ u \notin S_1,\\
I_5: &\quad ((\forall x \in [u,d]\colon\ f(x) \le f(u))\\
&\quad \& \,(\exists x, y \in [a,u)\colon\ f(x) < f(u) \le f(y) = b)) \ \Rightarrow\ u \notin S_0.
\end{aligned}$$

(3) $c \in B$, $d = b$ ($f([c,d]) = [\cdot, f(c)]$) and $u \in (c,d]$; the implications are analogous to those in (2).

(4) $B = \emptyset$ and $u \in [c = a, d = b]$; if $f(u) \in \{a,b\}$ then from ($\star$) we have

$$\begin{aligned}
I_6: &\quad (f(u) = a \,\&\, \min f^{-1}(a) < u) \ \Rightarrow\ u \notin S_0,\\
I_7: &\quad (f(u) = a \,\&\, \max f^{-1}(a) > u) \ \Rightarrow\ u \notin S_1,\\
I_8: &\quad (f(u) = b \,\&\, \min f^{-1}(b) < u) \ \Rightarrow\ u \notin S_0,\\
I_9: &\quad (f(u) = b \,\&\, \max f^{-1}(b) > u) \ \Rightarrow\ u \notin S_1;
\end{aligned}$$

similarly we obtain

$$\begin{aligned}
I_{10}: &\quad u = a \ \Rightarrow\ u \notin S_1,\\
I_{11}: &\quad u = b \ \Rightarrow\ u \notin S_0;
\end{aligned}$$

finally, if $\{u, f(u)\} \subset (a,b)$ then

$$\begin{aligned}
I_{12}: &\quad \exists x_1, x_2 \in (u,b]\colon\ f(x_1) < f(u) < f(x_2) \ \Rightarrow\ u \notin S_1,\\
I_{13}: &\quad \exists x_1, x_2 \in [a,u)\colon\ f(x_1) < f(u) < f(x_2) \ \Rightarrow\ u \notin S_0.
\end{aligned}$$

Since $u \notin B \setminus \{a,b\}$ and $f \in L_2(I)$ at least one of the hypotheses given in the implications I_j, $j = 1, \ldots, 13$, has to be satisfied. This proves (ii) for $B = B_1$.

In the case when $B = B_2$ we apply the above procedure to the map $\widetilde{f}(x) = f(-x + a + b)$, $x \in [a,b]$. ∎

By the proof of Lemma 2.1(i) (cases (2) and (3)) we have

Corollary. *If $f \in L_2(I)$ satisfies $B_1(f) \ne \emptyset \ne B_2(f)$ then either $f([a,a_1]) = \{a\}$ and $f([b_1,b]) = \{a\}$, or $f([a,a_1]) = \{b\}$ and $f([b_1,b]) = \{b\}$ for some $a \le a_1 < b_1 \le b$.*

In the next two lemmas we describe the structure of the set B for $f \in L_2(I)$ satisfying $S_{01}(f) \ne \emptyset$. We know from Lemma 2.1(i) that in this case $B \in \{B_1, B_2\}$. Since the conclusion can be easily seen, we omit its proof.

Lemma 2.2. *If $f \in L_2(I)$ satisfies $S_{01} \ne \emptyset$ then the closed set B can be expressed as a union ($n \ge 1$)*

$$\{b_n\}_{n<\mathfrak{K}} \cup \bigcup_{n<\mathfrak{L}} [b_n^-, b_n^+],$$

where $b_n^- < b_n^+$ *for each cardinal* n, $1 \le n < \mathfrak{L}$; *in the topology of* $I = [a,b]$, *the points* a, b *are not limit points of the set* $\{b_n\}_{n<\mathfrak{K}} \cup \bigcup_{n<\mathfrak{L}} \{b_n^-, b_n^+\}$ *and no point* $b_m \in \{b_n\}_{n<\mathfrak{K}}$ *is a two-sided limit point of that set*, *hence* $\mathfrak{K}$, $\mathfrak{L}$ *are at most countable cardinals.*

REMARK. The reader should notice that $a \in B_1$ ($a \in B_2$), resp. $b \in B_1$ ($b \in B_2$), if and only if $f(a) = b$ ($f(a) = a$), resp. $f(b) = a$ ($f(b) = b$). Since f is nonincreasing on B_1 we have $\operatorname{card}(B_1 \cap \operatorname{Fix}(f)) \le 1$.

The following lemma is an easy consequence of the fact that f is continuous, B_i is closed and $f|B_1$, resp. $f|B_2$, is nonincreasing, resp. nondecreasing.

LEMMA 2.3. *Let* $f \in L_2(I)$ *and* $S_{01} \ne \emptyset$. *Then*:

(i) *If* $\{f^n(x)\}_{n=0}^\infty \subset B$ *then* $\omega_f(x) \subset B$.

(ii) *If* $\omega_f \subset B_1$ *is an* ω*-limit set then either* $\omega_f = \{p\}$ *and* $p \in \operatorname{Fix}(f)$, *or* ω_f *is a cycle of period* 2.

(iii) *If* $\omega_f \subset B_2$ *is an* ω*-limit set then* $\omega_f = \{p\}$ *and* $p \in \operatorname{Fix}(f)$.

Proof. Since the set B is closed, property (i) is clear.

Let us show (ii). We use repeatedly the equality $f(\omega_f) = \omega_f$ and the fact that f is nonincreasing on B_1.

We are done if $\operatorname{card}\omega_f = 1$. Suppose that $\operatorname{card}\omega_f > 1$ and set $c = \min\omega_f < \max\omega_f = d$. Then $f(c) = d$, $f(d) = c$, hence we are done for $\operatorname{card}\omega_f = 2$. Finally, let $\operatorname{card}\omega_f > 2$. The set ω_f is countable and hence there is a point $e \in (c,d) \cap \omega_f$ isolated in ω_f. By Lemma 1.2(v), e is not periodic. Now, define an open set G by

$$G = \begin{cases} (e-\varepsilon, f(e)+\varepsilon), & e < f^2(e) < f(e), \\ (c-\varepsilon, e+\varepsilon) \cup (f(e)-\varepsilon, d+\varepsilon), & f^2(e) < e < f(e), \\ (f(e)-\varepsilon, e+\varepsilon), & f(e) < f^2(e) < e, \\ (c-\varepsilon, f(e)+\varepsilon) \cup (e-\varepsilon, d+\varepsilon), & f(e) < e < f^2(e). \end{cases}$$

For sufficiently small ε we get $\emptyset \ne G_1 = G \cap \omega_f \subsetneq \omega_f$ and $f(\overline{G}_1) \subset G_1$, which contradicts Lemma 2.1(v). Thus, $\operatorname{card}\omega_f \le 2$.

The proof of (iii) uses the fact that f is nondecreasing on B_2 and it is similar to that of (ii). ∎

We have seen that for $f \in L_2(I)$ if $\omega_f(x) \subset B$ then $\omega_f(x)$ has a simple structure. In fact it is a periodic orbit and $\operatorname{card}\omega_f(x) \le 2$. However, the number of different ω-limit sets that are subsets of B can be infinite. In what follows we show that for each $f \in L_2(I)$ there exists $g \in L_2(I)$ for which $\operatorname{ent}(f) \ge \operatorname{ent}(g)$ and $B(g)$ contains at most two ω-limit sets of g. A precise statement is given in Lemma 2.4.

Now we introduce some useful notation. For intervals $J = [\alpha, \beta] \subset I$, $K = [\gamma, \delta] \subset I$, where $a \le \alpha < \gamma \le \delta < \beta \le b$, the symbol $h(J,K)$ denotes a continuous nondecreasing piecewise linear map from I onto I that is constant on the intervals $[a,\alpha]$, K and $[\beta,b]$. If $P=(\alpha,\beta)$, $Q=(\gamma,\delta)$ are pairs of real numbers we write $P\prec Q$ if $\alpha<\gamma$.

Definition. Let $f\in L_2(I)$. Pairs $P=(\alpha,\beta)\prec Q=(\gamma,\delta)$ are *neighbouring* if there is a factor (I,g) of (I,f) with a factor map $h(J,K)$ such that $g\in L_2(I)$.

Remark. For $f\in L_2(I)$ satisfying $S_{01}\neq\emptyset\neq B_1$ put

(4) $$D=\{(x,f(x))\in B_1\times B_1\colon\ f^2(x)=x<f(x)\}\cup\{(a,b)\}.$$

If two pairs $P=(\alpha,\beta)\prec Q=(\gamma,\delta)$ are from D then there exists a factor (I,g) of (I,f) with a factor map $h(J,K)$. At the same time P,Q need not be neighbouring. In that case the requirement (1) is not satisfied for some $y\in\{h(\alpha)=a,h(\beta)=b,h(\gamma)=h(\delta)=c\}$, where $c\in\mathrm{Fix}(g)\cap B_1(g)$.

The main result of this section follows.

Lemma 2.4. *For each $f\in L_2(I)$ with $S_{01}(f)\neq\emptyset$ there is a factor (I,g) of (I,f) such that $g\in L_2(I)$ and one of the following possibilities is satisfied.*

(i) *$B_2(g)=\emptyset$ and if $\omega_g(x)\subset B_1(g)$ then either $\omega_g(x)=\{a,b\}$ or $\omega_g(x)=\mathrm{Fix}(g)\cap B_1(g)$.*

(ii) *$B_1(g)=\emptyset$ and if $\omega_g(x)\subset B_2(g)$ then $\omega_g(x)=\mathrm{Fix}(g)\cap\{a,b\}$.*

Proof. From Lemma 2.1(i), (ii) it follows that if $S_{01}(f)\neq\emptyset$ then $B(f)\in\{B_1(f),B_2(f)\}$ and $B(f)\neq\emptyset$. Without loss of generality we can assume that $B_1(f)\neq\emptyset$ and $B_2(f)=\emptyset$. We show that (i) holds in this case.

Consider the set D defined in (4). Since $\mathrm{card}(B_1\cap\mathrm{Fix}(f))\le 1$, by Lemma 2.3(ii) there is nothing to prove if $D=\{(a,b)\}$. In this case we put $g=f$.

For $(u,v)\in D$ we can consider a uniquely determined factor $(I,f_u^\star)$ of (I,f) with a factor map $h(J,K)$, where $\alpha=u$, $\beta=v$ and $\gamma=\delta$. Now, put

$$y=\max\{u\colon\ (u,v)\in D\ \&\ f_u^\star\in L_2(I)\};$$

the value y exists, otherwise f would not be from $L_2(I)$. Define

$$D_1=\{x>y\colon\ (x,f(x))\in D\}.$$

If $D_1=\emptyset$, we can put $g=f_y^\star$. In the case when $D_1\neq\emptyset$, the value $z=\min D_1$ exists, $z>y$ and the pairs $P=(y,\widetilde{y})\prec Q=(z,f(z))$ from D are neighbouring. This means that there is a factor (I,g) of (I,f) with a factor map $h(J,K)$, where $\alpha=y<\gamma=z<\delta=f(z)<\beta=\widetilde{y}$. Since $D_1\cap(y,z)=\emptyset$, the map g satisfies (i). Using the construction of g and the Corollary before Lemma 2.2 we get $B_2(g)=\emptyset$.

If $B_2(f)\neq\emptyset$ and $B_1(f)=\emptyset$ then the existence of $g\in L_2(I)$ satisfying (ii) can be shown similarly. ∎

Corollary 2.5. *Let $f,g\in L_2(I)$ be as in Lemma 2.4. Then* $\mathrm{ent}(f)\ge\mathrm{ent}(g)$ *and there is a positive integer $k_0=k_0(g)\ge 2$ such that for any*

$x \in B(g)$ we have $g^k(x) \in (I \setminus B(g)) \cup (\mathrm{Fix}(g) \cap B(g)) \cup \{a,b\}$ for some $k \le k_0$.

Proof. We know from Lemma 2.2 that the endpoints a, b are not limit points of $B(g)$. Similarly, if $\mathrm{Fix}(g) \cap B(g) \ne \emptyset$ then the fixed point from this set is not a two-sided limit point of $B(g)$. Now, from Lemma 2.4 we can see that for some $k_0 \ge 2$, the set $B(g) \setminus ((\mathrm{Fix}(g) \cap B(g)) \cup \{a,b\})$ contains at most k_0 consecutive iterates of any point of $B(g)$. ■

In what follows we use the following notation:

$$L_2^\star(I) = \{g \in L_2(I)\colon\ g \text{ satisfies (i) or (ii) of Lemma 2.4}\}. \tag{5}$$

3. The proof of the main result. In this section we prove our main result—Theorem 3.3.

As before, for $f \in L_2(I)$ we consider the closed sets $S_0(f), S_1(f), S_{01}(f)$ and also

$$S(f) = \bigcap_{i=0}^{\infty} f^{-i}(S_0 \cup S_1). \tag{6}$$

Fix $f \in L_2(I)$ and S given by (6). If $x \in S$ then by its *itinerary* with respect to S_0, S_1 we mean any $\omega \in \Omega_2$ such that $f^i(x) \in S_{\omega_i}$ for $i \in \mathbb{N} \cup \{0\}$. For $M \subset S$ we denote by $\Omega(M)$ the least closed σ-invariant subset of Ω_2 that contains all possible itineraries of points of M with respect to S_0, S_1. In particular, if $M = \mathrm{Fix}(f) \cap S_{01} \ne \emptyset$ then $\Omega(M) = \Omega_2$, hence $\mathrm{ent}(\sigma, \Omega(M)) = \log 2$.

Let $g \in L_2^\star(I)$ (see (5)), and consider the sets $S_0(g), S_1(g), S_{01}(g)$ and $S(g)$ given by (6) and the positive integer $k_0 = k_0(g) \ge 2$ described in Corollary 2.5. We have seen that for any $x \in B(g)$ we have $g^k(x) \in (I \setminus B) \cup (\mathrm{Fix}(g) \cap B) \cup \{a,b\}$ for some $k \le k_0$.

The key lemma follows.

Lemma 3.1. *Let g and $k_0 \ge 2$ be as above. If $M \subset S$, $M \subset I$ is minimal and $M \ne \mathrm{Fix}(g) \cap S_{01}$ then*

$$\mathrm{ent}(\sigma, \Omega(M)) \le \max\left(\mathrm{ent}(g, M), \frac{k_0 - 1}{k_0} \log 2\right). \tag{7}$$

Proof. Put $X = \{(x, \omega)\colon\ x \in M \ \&\ g^i(x) \in S_{\omega_i} \text{ for each } i \in \mathbb{N} \cup \{0\}\}$. The map $F = g \times \sigma$ defined by $F(x, \omega) = (g(x), \sigma(\omega))$ is continuous on the compact metric space X (with respect to the product metric). Moreover, the dynamical system (M, g), resp. $(\Omega(M), \sigma)$, is a factor of (X, F) given by the (factor map) projection $\Pi_1\colon X \to M$, resp. $\Pi_2\colon X \to \Omega(M)$. Using Theorem 1.1 we can see that

$$\mathrm{ent}(\sigma, \Omega(M)) \le \mathrm{ent}(F) \le \mathrm{ent}(g, M) + \sup_{x \in M} \mathrm{ent}(F, \Pi_1^{-1}(\{x\})). \tag{8}$$

Now we distinguish four possibilities.

I. $M \cap B = \emptyset$. Since M, B are closed we even have $\operatorname{dist}(M, B) > 0$. We know from Lemma 2.1(ii) that $S_{01} \subset B \setminus \{a, b\}$. This implies that for each $x \in M$ we have $\operatorname{card} \Pi_1^{-1}(\{x\}) = 1$, hence $\operatorname{ent}(F, \Pi_1^{-1}(\{x\})) = 0$. By (8), $\operatorname{ent}(\sigma, \Omega(M)) \le \operatorname{ent}(g, M)$, which proves (7) in this case.

II. $M \subset B$. By Lemma 2.4 and Corollary 2.5 this is possible for M with $\operatorname{card} M \in \{1, 2\}$. Obviously, $\operatorname{ent}(g, M) = 0$ and M is a periodic orbit. By Lemma 2.1(ii) we can see that $\{a, b\} \cap S_{01} = \emptyset$. Moreover by our assumption $M \ne \operatorname{Fix}(g) \cap S_{01}$, hence we immediately obtain $\Omega(M) \subset \Omega(M(\infty))$ for the set $\Omega(M(\infty))$ defined by (3). Since $k_0 \ge 2$, by Lemma 1.4 we get

$$\operatorname{ent}(\sigma, \Omega(M)) \le \frac{1}{2} \log 2 \le \frac{k_0 - 1}{k_0} \log 2.$$

Thus, (7) is also true in this case.

III. M is finite, $M \cap B \ne \emptyset$, $M \setminus B \ne \emptyset$. Then as above we have $\operatorname{ent}(g, M) = 0$ and we know from Lemma 1.2(ii) that M is a periodic orbit. Using Corollary 2.5, we can see that for each $n \in \mathbb{N}$ and assumed k_0 we have

$$\operatorname{card} \Omega(M)(n) \le \operatorname{card} M \cdot 2^{n(k_0 - 1)/k_0},$$

hence by (2), $\operatorname{ent}(\sigma, \Omega(M)) \le \frac{k_0 - 1}{k_0} \log 2$. Thus, (7) is true in this case.

IV. M is infinite, $M \cap B \ne \emptyset$. We know from Lemma 2.2 that B is countable and closed, hence this is also true for $M \cap B$. Since M is infinite, by Lemma 1.2(ii) the set $M \setminus B$ is uncountable. Using (8) it is sufficient to show that

$$\sup_{x \in M} \operatorname{ent}(F, \Pi_1^{-1}(\{x\})) = 0.$$

Fix $x \in M$ and put $C = M \cap B$. Then, as in the proof of Lemma 1.2(iv), we define $C(n, x) = \operatorname{card}\{0 \le i \le n-1 :\ g^i(x) \in C\}$. If $s(n, \varepsilon) = s(n, \varepsilon, \Pi_1^{-1}(\{x\}))$ is the maximal cardinality of an (n, ε)-separated subset of $\Pi_1^{-1}(\{x\})$ (with respect to F), by Lemma 2.1(ii) we have $s(n, \varepsilon) \le 2^{C(n,x)}$ for any sufficiently small ε. It follows from Lemma 1.2(iv) that

$$0 \le \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon) \le \lim_{n \to \infty} \frac{1}{n} \log 2^{C(n,x)} = 0,$$

hence $\operatorname{ent}(F, \Pi_1^{-1}(\{x\})) = 0$.

This proves the last part and also the whole lemma. ∎

Corollary 3.2. *Under the assumptions of Lemma* 3.1,

$$\operatorname{ent}(\sigma, \Omega(M)) \le \max\left(\operatorname{ent}(g), \frac{k_0 - 1}{k_0} \log 2\right).$$

Proof. By the definition, $\operatorname{ent}(g, M) \le \operatorname{ent}(g)$. Now the assertion follows directly from Lemma 3.1. ∎

Definition. Let $\Omega \subset \Omega_2$ and $j,k \in \mathbb{N}$, $j \le k$. We say that $\omega(k) \in \Omega(k)$ *contains* $\omega = (\omega_0, \dots, \omega_{j-1}) \in \{0,1\}^j$ if for some $l \in \{0, \dots, k-j\}$ and each $m \in \{0, \dots, j-1\}$,

$$\omega(k)_{l+m} = \omega_m.$$

Definition. Let $f \in L_2(I)$. We say that $\omega = (\omega_0, \dots, \omega_{j-1}) \in \{0,1\}^j$ is a *j-itinerary* of $x \in I$ if $f^i(x) \in S_{\omega_i}(f)$ for $i \in \{0, \dots, j-1\}$. We say that a *j-itinerary of x does not exist* if $\{x, \dots, f^{j-1}(x)\} \not\subseteq S_0(f) \cup S_1(f)$.

Combining Lemma 3.1 and Corollary 3.2 with the results of Sections 1 and 2 we obtain Theorem 3.3, which is the main result of this paper.

Theorem 3.3. *Let $f \in L_2(I)$. Then the topological entropy of f is greater than or equal to* $\log 2$.

Proof. There is nothing to prove if $S_{01}(f) = \emptyset$. In this case $S_0(f), S_1(f)$ create a 2-horseshoe and from Proposition 0.2 we obtain $\mathrm{ent}(f) \ge \log 2$.

Thus, let $f \in L_2(I)$ and $S_{01}(f) \ne \emptyset$. In order to simplify our proof we use Lemma 2.4 and Corollary 2.5. Using those statements, instead of f we can consider the map $g \in L_2^\star(I)$ (see (5)) such that $\mathrm{ent}(f) \ge \mathrm{ent}(g)$. Obviously it is sufficient to prove $\mathrm{ent}(g) \ge \log 2$.

In what follows all sets are taken with respect to g.

The relation $\mathrm{ent}(g) \ge \log 2$ is clear if $S_{01} = \emptyset$ since in this case the sets S_0, S_1 create a 2-horseshoe.

Suppose to the contrary that $S_{01} \ne \emptyset$ and $\mathrm{ent}(g) < \log 2$. From Lemma 2.1(i),(ii) we obtain $\emptyset \ne B \in \{B_1, B_2\}$. Let $k_0 \ge 2$ be as in Corollary 2.5. Using Theorem 1.3 we can consider a minimal set Γ in Ω_2 such that

$$\mathrm{ent}(\sigma, \Gamma) > \max\left(\mathrm{ent}(g), \frac{k_0 - 1}{k_0} \log 2\right). \tag{9}$$

As we know from Lemma 1.2(i), for each $x \in B$ there is a minimal set $M(x)$ in I such that $M(x) \subset \omega_g(x)$.

If we put $B_S = \{x \in B \cap S\colon\ M(x) \ne \mathrm{Fix}(g) \cap S_{01}\}$ (see (6) for S), we deduce from Lemma 3.1 that (7) is true for $M(x)$ and $\mathrm{ent}(\sigma, \Omega(M(x)))$ when $x \in B_S$. Hence by the minimality of Γ, Lemma 1.4 and (9) (for $x = \infty$ see (3)),

$$\forall x \in B_S \cup \{\infty\}\colon \quad \Omega(M(x)) \cap \Gamma = \emptyset.$$

Since Γ is σ-invariant we even see that for each $x \in B_S \cup \{\infty\}$ there is $n(x) \in \mathbb{N}$ such that

$$\text{no } \gamma \in \Gamma(m) \text{ contains } \omega(n(x)) \tag{10}$$

whenever $m \ge n(x)$ and $\omega(n(x)) \in \Omega(M(x)(n(x)))$.

Now we define an open cover $\{U(x)\}_{x\in B}$ of the set B in three steps:

(i) If $x \in B\setminus S$ and $g^{m(x)}(x) \notin S_0\cup S_1$, choose $U(x)$ in such a way that $g^{m(x)}(U(x))\cap(S_0\cup S_1)=\emptyset$.

(ii) If $x\in B_S$ then we can consider $m(x)\in\mathbb{N}$ such that for any itinerary ω of x, $\omega(m(x))$ contains some element of $\Omega(M(x))(n(x))$; now, using the continuity of g, choose the neighbourhood $U(x)$ of x in such a way that for any $y\in U(x)$ either the $m(x)$-itinerary does not exist, or for any itinerary ω of y, $\omega(m(x))$ contains some element of $\Omega(M(x))(n(x))$.

(iii) For $x\in B\cap S$ such that $M(x)=\mathrm{Fix}(g)\cap S_{01}=\{p\}$ we can consider $m(x)\in\mathbb{N}$ and $U(x)$ such that for any $y\in U(x)$ either $g^i(y)=p$ for some $i\le m(x)$, or the $m(x)$-itinerary does not exist, or for any itinerary ω of y, $\omega(m(x))$ contains some element of $\Omega(M(\infty))(n(\infty))$.

Obviously we have found the pairs $U(x), m(x)$, where $\{U(x)\}_{x\in B}$ is an open cover of the compact set B; let $\{U(x_1),\dots,U(x_k)\}$ be its finite subcover, and put

$$k^\star=\max\{m(x_1),\dots,m(x_k)\}.$$

In order to finish our proof we define the sets

$$R_0=S_0\setminus(\mathrm{Fix}(g)\cap S_{01}),\qquad R_1=S_1\setminus(\mathrm{Fix}(g)\cap S_{01}).$$

Since $g(R_0)\cap g(R_1)\supset R_0\cup R_1$, for each $m\in\mathbb{N}$ and $\gamma\in\Gamma(m)$ there is $x=x(\gamma)\in R_0\cup R_1$ such that for each $i\in\{0,1,\dots,m-1\}$ we have

$$g^i(x)\in R_{\gamma_i}\ \&\ g^i(x)\notin\mathrm{Fix}(g)\cap S_{01}. \tag{11}$$

It is clear that the sets $T_0=R_0\setminus\bigcup_{i=1}^k U(x_i)$, $T_1=R_1\setminus\bigcup_{i=1}^k U(x_i)$ are closed. Moreover, $\mathrm{dist}(T_0,T_1)=\delta>0$.

Suppose that for some $m>k^\star$, $\gamma\in\Gamma(m)$, $x(\gamma)$ and $i\in\{0,\dots\\ \dots,m-1-k^\star\}$ we have $g^i(x(\gamma))\in U(x_j)$. Then by definition of the cover $\{U(x)\}_{x\in B}$ either the $k^\star$-itinerary of $g^i(x(\gamma))$ does not exist, or γ contains some element of $\Omega(M(x_j))(n(x_j))$, which is impossible by (11) and (10). This implies that for any $m>k^\star$, $\gamma\in\Gamma(m)$ and $x(\gamma)$ we have $\{g^i(x(\gamma))\}_{i=0}^{m-1-k^\star}\subset T_0\cup T_1$.

Now, estimating the topological entropy of g we have, for an $\varepsilon<\delta$ and each $m>k^\star$,

$$s(m-1-k^\star,\varepsilon,I)\ge\operatorname{card}\Gamma(m)/2^{k^\star},$$

hence by (9) and (2), $\mathrm{ent}(g)\ge\mathrm{ent}(\sigma,\Gamma)>\mathrm{ent}(g)$—a contradiction. ∎

References

[BC] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math. 1513, Springer, Berlin, 1992.

[Bo] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer, 1976.

[G] C. Grillenberger, *Constructions of strictly ergodic systems*, Z. Wahrsch. Verw. Gebiete 25 (1973), 323–334.

[Ly] M. Yu. Lyubich, *Entropy of analytic endomorphisms of the Riemann sphere*, Funct. Anal. Appl. 15 (1981), 300–302.

[MP] M. Misiurewicz and F. Przytycki, *Topological entropy and degree of smooth mappings*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 25 (1977), 573–574.

[MS] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.

[PC] *Open Problem Session*, in: Proceedings of the Conference T.Y.A.S.T., Nonlinear Science Ser. B 8, World Sci., 1994, 41.

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