STUDIA MATHEMATICA 193 (3) (2009)

On the trigonometric conjugate to the general Franklin system

by

Gegham G. Gevorkyan (Yerevan) and Anna Kamont (Sopot)

Abstract. We investigate when the trigonometric conjugate to the periodic general Franklin system is a basis in $C(\mathbb{T})$. For this, we find some necessary and some sufficient conditions.

1. INTRODUCTION

The current paper is a next step in the study of general Franklin systems. The classical Franklin system is an orthonormal system consisting of piecewise linear continuous functions with dyadic knots. It was constructed by Ph. Franklin [8] in 1928 as an example of a complete orthonormal system which is a basis in $C[0,1]$. Since then, it has been studied by several authors from different points of view and it has been used to answer various questions in functional and harmonic analysis. The crucial property which allowed the study of the Franklin system are the exponential estimates obtained by Z. Ciesielski [6] in 1966. It is well known that the classical Franklin system is a basis in $C[0,1]$ and $L_p[0,1]$, $1 \le p < \infty$, unconditional when $1 < p < \infty$ (S. V. Bochkarev [2]), a basis in $H_p[0,1]$, $1/2 \le p \le 1$, unconditional for $1/2 < p \le 1$ (P. Wojtaszczyk [16], P. Sjölin and J. O. Strömberg [15]). Moreover, it was used by S. V. Bochkarev [1] to construct a basis in the disc algebra. Later, S. V. Bochkarev [3] also proved that the conjugate to the periodic Franklin system is a basis in $C(\mathbb{T})$. (See also [4].)

By a general Franklin system we mean an orthonormal system consisting of piecewise linear functions with an arbitrary sequence of knots. A systematic study of properties of general Franklin systems started with the papers of Z. Ciesielski and A. Kamont [7] and G. G. Gevorkyan and A. Kamont [9]. The following general problem is considered: how the properties of a general Franklin system depend on the regularity of the corresponding sequence of knots? It turns out that some properties, like boundedness

2000 *Mathematics Subject Classification*: 42C10, 42C15, 46E15.
Key words and phrases: general Franklin system, trigonometric conjugate, basis in $C(\mathbb{T})$.

DOI: 10.4064/sm193-3-1

of partial sums in $L_1[0,1]$, maximal inequalities, convergence a.e. or unconditionality in $L_p[0,1]$, $1 < p < \infty$, hold without any conditions on the sequence of knots (cf. Z. Ciesielski [5], Z. Ciesielski and A. Kamont [7], and G. G. Gevorkyan and A. Kamont [10]). For other properties, being a basis or unconditional basis in $H_1[0,1]$, we have obtained simple geometric conditions, necessary and sufficient for these properties (cf. G. G. Gevorkyan and A. Kamont [11]). Analogous results for the periodic general Franklin system have been obtained by K. Keryan [13] and K. Keryan and M. Pogosyan [14].

In the current paper we address the question when the conjugate system to a periodic general Franklin system is a basis in $C(\mathbb{T})$. For this, we obtain some necessary and some sufficient condition. In contrast to the cases mentioned above, these conditions do not have a simple geometric form. A summary of these results, without proofs, has been announced in G. G. Gevorkyan and A. Kamont [12].

The paper is organized as follows. In Section 2 we recall the definition of general periodic Franklin systems and formulate the main results, Theorems 2.1 and 2.2. Section 3 is devoted to their proofs. In Section 3.1 we recall estimates and properties of general periodic Franklin system needed in this paper. Necessary conditions for the conjugate system to the general Franklin system to be a basis in $C(\mathbb{T})$ are discussed in Sections 3.2 and 3.4, and sufficient conditions are discussed in Section 3.3. In particular, we prove some estimates from above and from below for the Lebesgue functions of the conjugate to the general periodic Franklin system. These estimates are the content of Theorems 3.3 and 3.5. Finally, in Section 4 some particular examples are discussed, and Corollary 4.2 contains a characterization of quasi-dyadic sequences for which the conjugate to the corresponding general periodic Franklin system is a basis in $C(\mathbb{T})$.

2. DEFINITIONS AND NOTATION

Set $\mathbb{T} = [-\pi, \pi)$. In this paper, we consider the space $C(\mathbb{T})$ of continuous 2π-periodic real-valued functions on $\mathbb{R}$.

For $x, y \in \mathbb{T}$ the periodic distance between x, y is denoted by $\operatorname{dist}(x,y) = \min(|x-y|, 2\pi - |x-y|)$. Moreover, we use the notation $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$. Writing $a \sim b$ means that there are positive constants C_1, C_2 such that $C_1 a \le b \le C_2 a$, and we write $a \sim_\gamma b$ to emphasise that these constants may depend only on the parameter γ.

2.1. Periodic general Franklin system. Let $n \in \mathbb{N}$ and $\sigma = \{s_j, 1 \le j \le n\}$ be a sequence of simple knots from $[-\pi, \pi)$, with $-\pi = s_1 < \cdots < s_n$. These knots are 2π-periodically extended to $\mathbb{R}$, i.e. we put $s_m = s_j + 2k\pi$ for $m = kn + j$ with $k \in \mathbb{Z}$ and $1 \le j \le n$. We denote by

$\mathcal{S}_\sigma$ the space of continuous piecewise linear periodic functions with knots σ. For given σ, we put $I_j = [s_j, s_{j+1}]$ and $\lambda_j = |I_j| = s_{j+1} - s_j$; in particular, we have $\lambda_{j+n} = \lambda_j$. We let P_σ denote the $L_2(\mathbb{T})$-orthogonal projection onto $\mathcal{S}_\sigma$, and by K_σ we denote the Dirichlet kernel of P_σ.

Analogous definitions make sense when double knots are allowed in σ, i.e. $\sigma = \{s_j, 1 \le j \le n\}$ with $-\pi = s_1 \le \cdots \le s_n$ and $s_i < s_{i+2}$. Then $\mathcal{S}_\sigma$ consists of the periodic piecewise linear functions, continuous at simple knots (i.e. at s_i such that $s_{i-1} < s_i < s_{i+1}$), and discontinuous, but left continuous and having limits from the right at double knots (i.e. when $s_i = s_{i+1}$).

Next, let $\mathcal{T} = \{t_n, n \ge 1\}$ be a sequence of at most double knots in $\mathbb{T} = [-\pi, \pi)$, with $t_1 = -\pi$, and dense in $\mathbb{T}$. Let

$$\mathcal{T}_n = \{t_j, 1 \le j \le n\} = \{t_{n,j}, 1 \le j \le n\}$$

with $-\pi = t_{n,1} \le \cdots \le t_{n,n}$, $t_{n,j} < t_{n,j+2}$. By $\mathcal{S}_n$, $I_{n,j}$, $\lambda_{n,j}$, P_n, K_n we mean the space $\mathcal{S}_{\mathcal{T}_n}$ and $I_j, \lambda_j, P_\sigma, K_\sigma$ coresponding to $\mathcal{T}_n$.

For each $n \ge 2$, there is a unique function $f_n \in \mathcal{S}_n$, orthogonal to $\mathcal{S}_{n-1}$ in $L_2(\mathbb{T})$, with $\|f_n\|_2 = 1$ and $f_n(t_n) > 0$. Put also $f_1(t) = 1/\sqrt{2\pi}$. The collection of functions $\{f_n, n \ge 1\}$ is called the *general general periodic Franklin system corresponding to the sequence $\mathcal{T}$ of knots*.

Clearly, $K_n(t,s) = \sum_{j=1}^n f_j(t) f_j(s)$.

2.2. Regularity conditions. Now, we recall two notions of regularity of a sequence $\mathcal{T} = \{t_n, n \ge 1\}$ of knots in $\mathbb{T}$. We say that $\mathcal{T}$ is *admissible* if it consists of at most double knots and is dense in $\mathbb{T}$.

Definition 2.1. Let $\gamma > 1$ and let $\mathcal{T} = \{t_n, n \ge 1\}$ be an admissible sequence of knots in $\mathbb{T}$. Then $\mathcal{T}$ is said to satisfy the *strong periodic regularity condition for pairs with parameter* γ if for each $n \ge 2$ and $1 \le j \le n$,

$$\frac{1}{\gamma} \le \frac{\lambda_{n,j-1} + \lambda_{n,j}}{\lambda_{n,j} + \lambda_{n,j+1}} \le \gamma.$$

Definition 2.2. Let $\gamma > 1$ and let $\mathcal{T} = \{t_n, n \ge 1\}$ be an admissible sequence of knots in $\mathbb{T}$. Then $\mathcal{T}$ is said to satisfy the *strong periodic regularity condition with parameter* γ if for each $n \ge 2$ and $1 \le j \le n$,

$$\frac{1}{\gamma} \le \frac{\lambda_{n,j}}{\lambda_{n,j+1}} \le \gamma.$$

Remark. The strong periodic regularity condition is the same as the periodic version of the bounded local mesh ratio condition.

Observe that strong periodic regularity implies that all knots in $\mathcal{T}$ are simple.

Recall that for sequences of knots satisfying these regularity conditions the corresponding general periodic Franklin system is a basis or an unconditional basis in $\operatorname{Re} H_1(\mathbb{T})$ (cf. [14]); a non-periodic version of this result can be found in [11].

In the current paper, a different type of regularity conditions will be considered. For a sequence of points σ, $k \in \mathbb{N}$ and $t \in \mathbb{T}$, define

$$\mathcal{J}_\sigma^{(k)}(t) = \sum_{j=1}^{n} \left(\frac{\lambda_j}{\lambda_j + \operatorname{dist}(t, I_j)} \right)^k.$$

For given $\mathcal{T}$ and n, we denote by $\mathcal{J}_n^{(k)}$ the function $\mathcal{J}_\sigma^{(k)}$ corresponding to $\mathcal{T}_n$, i.e.

$$\mathcal{J}_n^{(k)}(t) = \sum_{j=1}^{n} \left(\frac{\lambda_{n,j}}{\lambda_{n,j} + \operatorname{dist}(t, I_{n,j})} \right)^k.$$

Regularity conditions discussed in this paper relate to boundedness or unboundedness of the collection of functions $\{\mathcal{J}_n^{(k)}(\cdot), n \geq 2\}$, for appropriate k.

2.3. The trigonometric conjugate to the periodic general Franklin system. The *trigonometric conjugate* to a periodic function f is denoted by $\widetilde{f}$, i.e.

$$\widetilde{f}(t) = \frac{-1}{2\pi} \lim_{\varepsilon \to 0} \int_{|u|=\varepsilon}^{\pi} f(t+u) \cot \frac{u}{2} \, du. \tag{2.1}$$

Let $\{f_n, n \geq 1\}$ be a periodic general Franklin system. By the *trigonometric conjugate* to $\{f_n, n \geq 1\}$ we mean the system $\{\widetilde{f}_n, n \geq 1\}$, where $\widetilde{f}_1(t) = 1/\sqrt{2\pi}$, and $\widetilde{f}_n$ for $n \geq 2$ is defined by (2.1) with $f = f_n$.

For convenience, we introduce the following notation. Let $\widetilde{k}_\sigma$ denote the conjugate to K_σ with respect to both variables, and set

$$\widetilde{K}_\sigma = \frac{1}{2\pi} + \widetilde{k}_\sigma, \qquad \widetilde{L}_\sigma(t) = \int_{\mathbb{T}} |\widetilde{K}_\sigma(t,s)| \, ds.$$

As usual, $\widetilde{K}_n$ and $\widetilde{L}_n$ are $\widetilde{K}_\sigma$ and $\widetilde{L}_\sigma$ corresponding to $\mathcal{T}_n$, i.e.

$$\widetilde{K}_n(t,s) = \sum_{j=1}^{n} \widetilde{f}_j(t) \widetilde{f}_j(s), \qquad \widetilde{L}_n(t) = \int_{\mathbb{T}} |\widetilde{K}_n(t,s)| \, ds.$$

2.4. The main results. We are interested in the following question: when is the trigonometric conjugate to a general periodic Franklin system a basis in $C(\mathbb{T})$? Let $\mathcal{T}$ be a sequence of knots in $\mathbb{T}$. First, note that all knots in $\mathcal{T}$ must be simple: if some knots are double, then the corresponding Franklin functions are piecewise linear, but discontinuous at double knots, and then the conjugate functions are not continuous, even not bounded at the points of discontinuity of the Franklin functions. So we assume that all knots in $\mathcal{T}$ are simple. Let $\mathrm{lip}(\mathbb{T}, \alpha)$ be the space of functions from $C(\mathbb{T})$ satisfying $\omega(f, \delta) = o(\delta^\alpha)$, where $0 < \alpha < 1$ and $\omega(f, \delta)$ is the modulus of smoothness of f (of the first order, in the uniform norm). Then density of $\mathcal{T}$ in $\mathbb{T}$ implies that the piecewise linear functions with knots $\mathcal{T}$ are dense in $\mathrm{lip}(\mathbb{T}, \alpha)$ in the corresponding Hölder norm. Hence the periodic general Franklin system corresponding to $\mathcal{T}$ is linearly dense in $\mathrm{lip}(\mathbb{T}, \alpha)$. As the trigonometric conjugate operator is an isomorphism of $\mathrm{lip}(\mathbb{T}, \alpha)$ (cf. e.g. A. Zygmund [17, Chapter III, Theorems 13.29 and 13.30]), this implies that the conjugate system is linearly dense in $\mathrm{lip}(\mathbb{T}, \alpha)$, and consequently also in $C(\mathbb{T})$. Thus, the question whether $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $C(\mathbb{T})$ or not is a question of the boundedness of the corresponding partial sum operators and Lebesgue functions. The first necessary condition is the following:

THEOREM 2.1. *Let $\mathcal{T}$ be an admissible sequence of knots such that the conjugate to the corresponding general Franklin system is a basis in $C(\mathbb{T})$. Then $\mathcal{T}$ satisfies the strong periodic regularity condition.*

Therefore, in further considerations we assume the strong periodic regularity condition. Under this condition we get the following result:

THEOREM 2.2. *Let $\mathcal{T}$ be an admissible sequence of knots satisfying the strong periodic regularity condition with the corresponding periodic Franklin system $\{f_n, n \geq 1\}$. Then*:

- *If* $\sup_{n\geq 1} \sup_{t\in\mathbb{T}} \mathcal{J}_n^{(3)}(t) < \infty$, *then* $\{\widetilde{f}_n, n \geq 1\}$ *is a basis in* $C(\mathbb{T})$.
- *If* $\sup_{n\geq 1} \sup_{t\in\mathbb{T}} \mathcal{J}_n^{(4)}(t) = \infty$, *then* $\{\widetilde{f}_n, n \geq 1\}$ *is not a basis in* $C(\mathbb{T})$.

For the proof of Theorem 2.2 we estimate the Lebesgue functions $\widetilde{L}_\sigma(t)$ from above and from below by $\mathcal{J}_\sigma^{(3)}(t)$ and $\mathcal{J}_\sigma^{(4)}(t)$, respectively. These estimates are formulated and proved as Theorems 3.4 and 3.5, respectively.

3. PROOFS OF THE THEOREMS

3.1. Estimates for K_n and f_n. To prove the main results, we need some estimates for K_n and f_n. In the non-periodic case such estimates can be found in [9] or [10]. The periodic versions have been obtained by K. Keryan [13].

Note that if $\mathcal{T}$ contains at least one double knot, then the knot is a point of discontinuity of all but finitely many functions f_n corresponding to $\mathcal{T}$, and then the estimates reduce to the non-periodic case. Therefore, in the periodic case we are interested only in the situation when all knots are simple.

We first estimate the kernel K_σ, where $\sigma = \{s_j, 1 \le j \le n\}$ is a sequence of simple knots in $\mathbb{T}$. Let N_j, $1 \le j \le n$, be a B-spline basis in $\mathcal{S}_\sigma$, i.e. N_j is a piecewise linear periodic function with knots σ such that $N_j(s_j) = 1$ and $N_j(s_i) = 0$ for $1 \le i \ne j \le n$. Observe that $\operatorname{supp} N_j$ is the periodic interval $[t_{j-1}, t_{j+1}]$ containing t_j, and $|\operatorname{supp} N_j| = \lambda_{j-1} + \lambda_j$.

Let $G = [(N_i, N_j), 1 \le i, j \le n]$ be the Gram matrix of the system N_j, $1 \le j \le n$, and $A = [a_{i,j}, 1 \le i, j \le n] = G^{-1}$. Clearly,

$$K_\sigma(t,s) = \sum_{i,j=1}^{n} a_{i,j} N_i(t) N_j(s) \quad \text{and} \quad K_\sigma(s_i, s_j) = a_{i,j}. \tag{3.1}$$

We need uniform boundedness of the norms $\|P_\sigma\|_\infty$ proved by Z. Ciesielski [5]:

$$\|P_\sigma\|_\infty = \sup_{t\in\mathbb{T}} \int_{\mathbb{T}} |K_\sigma(t,s)|\, ds \le 3. \tag{3.2}$$

In fact, the proof in [5] concerns the non-periodic case, but the same argument applies in the periodic case.

We need the following properties of the matrix A (see Property 1 in [13]).

Theorem 3.1. *Let $\sigma = \{s_j, 1 \le j \le n\}$ be a sequence of knots in $\mathbb{T}$. Then*

$$a_{i,i} > 0, \qquad 2.4 \le a_{i,i} |\operatorname{supp} N_i| \le 4,$$
$$|a_{i,j}| \le \frac{4}{2^{|i-j|_n}} \frac{1}{|\operatorname{supp} N_i| \vee |\operatorname{supp} N_j|},$$

where $|i-j|_n = \min(|i-j|, n - |i-j|)$.

Combination of Theorem 3.1 and formula (3.1) implies pointwise estimates for $|K_\sigma(t,s)|$ for $t \in I_i$, $s \in I_j$; we will need them only when σ satisfies $\lambda_k \sim_\gamma \lambda_{k+1}$. In that case we have

$$|K_\sigma(t,s)| \le \frac{C_\gamma}{2^{|i-j|_n}} \frac{1}{\lambda_i \vee \lambda_j}.$$

Now we formulate estimates for periodic general Franklin functions. We quote them in a simplified version, for sequences of simple knots satisfying the strong periodic regularity condition for pairs. In [13] these estimates have been proved for general sequences of points, but for the purpose of this paper the special case is enough.

Let $\mathcal{T} = \{t_j, j \ge 1\}$ be a sequence of simple knots. Recall that $\mathcal{T}_m = \{t_j, 1 \le j \le m\} = \{t_{j,1} < \cdots < t_{j,m}\}$. Since $\mathcal{T}_n$ is obtained by adding a single point t_n to $\mathcal{T}_{n-1}$, for $n \ge 3$ there is a unique index j, $1 \le j \le n-1$, such that $t_{n-1,j} < t_n < t_{n-1,j+1}$. For convenience we set $t_n^- = t_{n-1,j}$ and $t_n^+ = t_{n-1,j+1}$. The points t_n^-, t_n^+ define two periodic intervals on $\mathbb{T}$; exactly one of them contains t_n, and we denote it by $\mathcal{I}_n$. Let i_n, $1 \le i_n \le n$, be such that

$$|f(t_{i_n})| = \min\{|f_n(t_k)| : 1 \le k \le n\}.$$

Next, let $t \in \mathbb{T}$, $t \notin \mathcal{I}_n$. Then t and t_n^- define two periodic intervals on $\mathbb{T}$; exactly one of them contains neither t_n nor t_n^+, and we denote it by $\alpha_n(t)$. Analogously, the interval $\beta_n(t)$ is defined as the periodic interval induced by t and t_n^+ which does not contain t_n^-, t_n. If $t \ne t_{i_n}$, then exactly one of $\alpha_n(t)$, $\beta_n(t)$ does not contain t_{i_n}, and we denote this interval by $\Lambda_n(t)$. Finally, let

$$d_n(t) = \begin{cases} 0 & \text{when } t \in \mathcal{I}_n, \\ \#\{k : t_k \in \Lambda_n(t),\ 1 \le k \le n\} & \text{when } t \notin \mathcal{I}_n. \end{cases}$$

In addition, set $\nu_n = |\mathcal{I}_n|$.

The following estimates of general periodic Franklin functions are taken from Section 4.1 of [13].

THEOREM 3.2. *Let $\mathcal{T} = \{t_j, j \ge 1\}$ be a sequence of simple knots satisfying the strong periodic regularity condition for pairs with parameter γ. Let $\{f_n, n \ge 1\}$ be the corresponding general periodic Franklin system. Then for $n \ge 5$,*

$$f_n(t_n) > 0, \qquad f_n(t_n^-) < 0, \qquad f_n(t_n^+) < 0. \tag{3.3}$$

Moreover,

$$|f_n(t_n)| \sim_\gamma |f_n(t_n^-)| \sim_\gamma |f_n(t_n^+)| \sim_\gamma \nu_n^{-1/2}, \tag{3.4}$$

and the equivalence constants depend only on γ. Moreover,

$$|f_n(t)| \le C_\gamma (1/2)^{d_n(t)} |f_n(t_n)|. \tag{3.5}$$

An immediate consequence of (3.5) is

$$\sum_{k=1}^{n} \max_{t \in I_{n,k}} |f_n(t)| \le C_\gamma |f_n(t_n)|. \tag{3.6}$$

We also need a characterization of sequences $\mathcal{T}$ for which the corresponding general periodic Franklin system is a basis in $\operatorname{Re} H_1$. The following is Theorem 1 of [14]:

THEOREM 3.3. *Let $\mathcal{T}$ be an admissible sequence of simple knots in $\mathbb{T}$ with the corresponding general periodic Franklin system $\{f_n, n \ge 1\}$. Then $\{f_n, n \ge 1\}$ is a basis in $\operatorname{Re} H_1$ if and only if $\mathcal{T}$ satisfies the strong periodic regularity condition for pairs with some parameter $\gamma > 1$.*

REMARK. The proof in [14] concerns in fact the atomic H_1 space on $[0, 1]$. It should be noted that the atoms considered in [14] are in fact periodic atoms, i.e. with supports contained in periodic intervals. It is well known that atomic decomposition with periodic atoms characterizes the space $\operatorname{Re} H_1$.

3.2. Necessity of strong regularity

Proof of Theorem 2.1. Suppose that $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $C(\mathbb{T})$. First, note that continuity of all $\widetilde{f}_n$'s implies that all knots of $\mathcal{T}$ must be simple. Moreover, as $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $C(\mathbb{T})$, it is also a basis in $L_1(\mathbb{T})$. Now, let $f \in L_1(\mathbb{T})$ be such that also $\widetilde{f} \in L_1(\mathbb{T})$. Then $f \in \operatorname{Re} H_1(\mathbb{T})$. As $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $L^1(\mathbb{T})$ we have

$$f = \sum_{n=1}^{\infty} (f, \widetilde{f}_n)\widetilde{f}_n, \qquad \widetilde{f} = \sum_{n=1}^{\infty} (\widetilde{f}, \widetilde{f}_n)\widetilde{f}_n,$$

with convergence of both series in $L_1(\mathbb{T})$. Since $\|f\|_{\operatorname{Re} H_1} \sim \|f\|_1 + \|\widetilde{f}\|_1$, this implies that $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $\operatorname{Re} H_1(\mathbb{T})$. Theorem 3.3 now implies that $\mathcal{T}$ satisfies the strong periodic regularity condition for pairs.

Now, suppose that $\mathcal{T}$ satisfies the strong periodic regularity condition for pairs, but it does not satisfy the strong periodic regularity condition. Let γ be the pair-regularity parameter for $\mathcal{T}$. We will show that the Lebesgue functions of $\{\widetilde{f}_n, n \geq 1\}$ are not bounded; more precisely, for M large enough there is N such that

$$\int_{\mathbb{T}} |\widetilde{f}_N(t_N)\widetilde{f}_N(u)|\, du \geq C_\gamma \log M, \tag{3.7}$$

where C_γ depends only on γ.

Fix $M > 2\gamma$. As $\mathcal{T}$ does not satisfy the strong regularity condition, there is a partition $\mathcal{T}_N$ containing two neighbouring intervals Δ_1 and Δ_2 such that $|\Delta_1| \geq M|\Delta_2|$; to simplify the writing, assume that $|\Delta_1| = M|\Delta_2|$. Let Δ_3 be the other neighbour of Δ_2 in $\mathcal{T}_N$. By strong regularity for pairs, $|\Delta_1| + |\Delta_2| \sim_\gamma |\Delta_2| + |\Delta_3|$, and consequently $|\Delta_1| \sim_\gamma |\Delta_3|$. Moreover, we can assume that

$$N = \min\{n : \text{both endpoints of } \Delta_2 \text{ are in } \mathcal{T}_n\}.$$

This means that t_N is an endpoint of Δ_2. To fix ideas, assume that t_N is the common endpoint of Δ_1 and Δ_2, t_N^- is the other endpoint of Δ_1, and t_N^+ is the other endpoint of Δ_2. To prove (3.7), we need to estimate $|\widetilde{f}_N(t_N)|$ and $\|\widetilde{f}_N\|_1$ from below. For this, let ξ_i be the value of f_N' on Δ_i. The estimates for f_N from Theorem 3.2 and the fact that $|\Delta_1| > |\Delta_2|$ imply that

$$|f_N(t_N^-)| \sim_\gamma |f_N(t_N)| \sim_\gamma |f_N(t_N^+)| \sim_\gamma |\Delta_1|^{-1/2}, \tag{3.8}$$

$$|\xi_1| \sim_\gamma |\xi_3| \sim_\gamma |\Delta_1|^{-3/2}, \qquad |\xi_2| \sim_\gamma |\Delta_1|^{-1/2}|\Delta_2|. \tag{3.9}$$

Since f_N is linear on Δ_1 and $|f_N(t_N^-)| \sim_\gamma |f_N(t_N)|$, there is $L_\gamma > 0$, depending only on γ, such that $f_N(t) \geq \frac{1}{2} f_N(t_N)$ on $[t_N - L_\gamma|\Delta_1|, t_N] \subset \Delta_1$; clearly, we can assume $L_\gamma \leq 1/3$.

Now, let k be such that $[t_N-(k+1)|\Delta_2|, t_N-k|\Delta_2|] \subset [t_N-L_\gamma|\Delta_1|, t_N]$. By the choice of Δ_1 and Δ_2, this is possible for $0 \le k \le ML_\gamma - 1$. Let us estimate $|\widetilde{f}_N(t)|$ from below for $t \in [t_N-(k+1)|\Delta_2|, t_N-k|\Delta_2|]$. Let

$$\kappa = \max\{\delta > 0 : [t-\delta, t+\delta] \subset \Delta_1 \cup \Delta_2 \cup \Delta_3\}.$$

For M large enough, we have $\kappa \sim_\gamma |\Delta_1|$, and $\Delta_2 \subset [t-\kappa, t+\kappa]$. Now, the integral defining $\widetilde{f}_N(t)$ is split into three parts:

$$\begin{aligned}\widetilde{f}_N(t) &= \frac{-1}{2\pi}\int_0^\pi (f_N(t+u) - f_N(t-u))\cot\frac{u}{2}\,du \\ &= \frac{-1}{2\pi}\Big(\int_0^{t_N^+-t} + \int_{t_N^+-t}^{\kappa} + \int_\kappa^\pi\Big) = Q_1 + Q_2 + Q_3.\end{aligned}$$

Let us start with the estimate of Q_3. We have

$$\Big|\int_\kappa^\pi (f_N(t+u) - f_N(t-u))\cot\frac{u}{2}\,du\Big| \le C\int_\kappa^\pi \Big((|f_N(t+u)| + |f_N(t-u)|)\frac{1}{u}\,du\Big).$$

To estimate these integrals, we consider $f_N(t+\cdot)$ on its intervals of linearity. Let $[t_{N,j}-t, t_{N,j+1}-t]$ be a (periodic) interval of linearity of $f_N(t+\cdot)$ disjoint from $[-\kappa,\kappa]$. Then

$$\max_{u\in[t_{N,j}-t, t_{N,j+1}-t]} |f_N(t+u)| = \max_{s\in[t_{N,j}, t_{N,j+1}]} |f_N(s)|.$$

By strong regularity for pairs (and the fact that $\kappa \sim_\gamma |\Delta_1|$) we have $1 \le (t_{N,j+1}-t)/(t_{N,j}-t) \le C_\gamma$, which implies

$$\int_{t_{N,j}-t}^{t_{N,j+1}-t} \frac{1}{u}\,du = \log\frac{s_{i+1}-t}{s_i-t} \le C_\gamma.$$

Therefore

$$\int_{t_{N,j}-t}^{t_{N,j+1}-t} |f_N(t+u)|\,\frac{1}{u}\,du \le C_\gamma \max_{s\in I_{N,j}} |f_N(s)|. \tag{3.10}$$

If $[t_{N,j}-t, t_{N,j+1}-t]$ is a periodic interval not included in but not disjoint from $[-\kappa,\kappa]$ (there are at most two such intervals), we find by an analogous argument (and applying (3.5)) that

$$\int_{[t_{N,j}-t, t_{N,j+1}-t]\cap\{u:|u|\ge\kappa\}} |f_N(t+u)|\,\frac{1}{u}\,du \le C_\gamma |f_N(t_N)|.$$

Combining (3.10) with (3.6) and the last inequality we obtain

$$\int_\kappa^\pi |f_N(t+u)|\,\frac{1}{u}\,du \le C_\gamma |f_N(t_N)|.$$

By an analogous argument,

$$\int_{\kappa}^{\pi} |f_N(t-u)|\,\frac{1}{u}\,du \le C_\gamma |f_N(t_N)|,$$

so we get

$$|Q_3| \le C_\gamma |f_N(t_N)|. \tag{3.11}$$

To estimate Q_1 we use the Lipschitz condition for f_N on Δ_1 and Δ_2, and estimates (3.9):

$$\begin{aligned}
|Q_1| &= \frac{1}{2\pi}\Big|\int_0^{t_N^+-t} (f_N(t+u)-f_N(t-u))\cot\frac{u}{2}\,du\Big| \\
&\le C\Big(\int_0^{t_N-t} |f_N(t+u)-f_N(t-u)|\,\frac{1}{u}\,du \\
&\quad + \int_{t_N-t}^{t_N^+-t} |f_N(t+u)-f_N(t-u)|\,\frac{1}{u}\,du\Big) \\
&\le C_\gamma\Big(\int_0^{t_N-t} u\cdot|\xi_1|\cdot\frac{1}{u}\,du + \int_{t_N-t}^{t_N^+-t} u\cdot(|\xi_1|+|\xi_2|)\cdot\frac{1}{u}\,du\Big) \\
&\le C_\gamma(|t_N-t|\cdot|\Delta_1|^{-3/2} + |\Delta_2|(|\Delta_1|^{-3/2} + |\Delta_1|^{-1/2}|\Delta_2|^{-1})) \\
&\le C_\gamma|\Delta_1|^{-1/2},
\end{aligned}$$

which in combination with (3.8) gives

$$|Q_1| \le C_\gamma |f_N(t_N)|. \tag{3.12}$$

Now, we turn to estimating Q_2. For this, we split the integral defining Q_2 into the sum of

$$Q_{2,1} = \frac{-1}{2\pi}\int_{t_N^+-t}^{\kappa} (f_N(t+u)-f_N(t_N^+))\cot\frac{u}{2}\,du,$$

$$Q_{2,2} = \frac{-1}{2\pi}\int_{t_N^+-t}^{\kappa} (f_N(t_N^+)-f_N(t))\cot\frac{u}{2}\,du,$$

$$Q_{2,3} = \frac{-1}{2\pi}\int_{t_N^+-t}^{\kappa} (f_N(t)-f_N(t-u))\cot\frac{u}{2}\,du.$$

Using the Lipschitz condition for f_N on Δ_3 and estimate (3.9) for $|\xi_3|$ we find

$$|Q_{2,1}| \le C\int_{t_N^+-t}^{\kappa} \frac{t+u-t_N^+}{u}\cdot|\xi_3|\,du \le C_\gamma\kappa|\Delta_1|^{-3/2} \le C_\gamma|\Delta_1|^{-1/2}.$$

Combining this with (3.8) we get

$$|Q_{2,1}| \le C_\gamma |f_N(t_N)|. \tag{3.13}$$

By a similar argument, using the Lipschitz condition for f_N on Δ_1 in combination with (3.9), (3.8) we find

$$|Q_{2,3}| \le C_\gamma |f_N(t_N)|. \tag{3.14}$$

Finally, we estimate $|Q_{2,2}|$ from below. Since $f_N(t) \ge \frac{1}{2} f_N(t_N)$ and $\operatorname{sgn} f_N(t_N^+) = -\operatorname{sgn} f_N(t_N)$, we have $|f_N(t) - f_N(t_N^+)| \sim_\gamma |f_N(t_N)|$. Moreover,

$$\int_{t_N^+ - t}^{\kappa} \cot \frac{u}{2}\, du \sim \int_{t_N^+ - t}^{\kappa} \frac{1}{u}\, du = \log \kappa - \log(t_N^+ - t).$$

Set $w_k = \log \frac{\kappa}{(k+1)|\Delta_2|}$. Since $t \in [t_N - (k+1)|\Delta_2|, t_N - k|\Delta_2|]$, these considerations imply

$$|Q_{2,2}| \ge D_\gamma w_k |f_N(t_N)|. \tag{3.15}$$

Summarizing (3.11)–(3.14) we have

$$|Q_1| + |Q_{2,1}| + |Q_{2,3}| + |Q_3| \le B_\gamma |f_N(t_N)|. \tag{3.16}$$

Now, we need $\frac{1}{2} D_\gamma w_k \ge B_\gamma$. Let R_γ, A_γ be constants, depending only on γ, such that $\kappa \ge (1/A_\gamma)|\Delta_1|$ and $\log R_\gamma = 2B_\gamma / D_\gamma$. Observe that if $k + 1 \le M/(R_\gamma \cdot A_\gamma)$ then $\frac{1}{2} D_\gamma w_k \ge B_\gamma$, and consequently

$$|\widetilde{f}_N(t)| \ge |Q_{2,2}| - (|Q_1| + |Q_{2,1}| + |Q_{2,3}| + |Q_3|) \ge \frac{1}{2} D_\gamma w_k |f_N(t_N)|. \tag{3.17}$$

We are ready to estimate $\int_{\mathbb{T}} |\widetilde{f}_N(t)|\, dt$. Setting $\varrho_\gamma = \min(L_\gamma, 1/(A_\gamma R_\gamma))$ we have

$$\int_{\mathbb{T}} |\widetilde{f}_N(t)|\, dt \ge \sum_{k=0}^{M\varrho_\gamma - 1} \int_{t_N - (k+1)|\Delta_2|}^{t_N - k|\Delta_2|} |\widetilde{f}_N(t)|\, dt \ge \sum_{k=0}^{M\varrho_\gamma - 1} \frac{1}{2} D_\gamma w_k |\Delta_2|\, |f_N(t_N)|.$$

For $0 \le k \le M\varrho_\gamma - 1$ we have

$$\begin{aligned} w_k = \log \frac{\kappa}{(k+1)|\Delta_2|} &\ge \log \frac{\frac{1}{A_\gamma}|\Delta_1|}{M\varrho_\gamma |\Delta_2|} \\ &\ge \log \frac{1}{A_\gamma \varrho_\gamma} \ge \log R_\gamma = \frac{2B_\gamma}{D_\gamma} > 0. \end{aligned}$$

Putting together these estimates we get

$$\int_{\mathbb{T}} |\widetilde{f}_N(t)|\, dt \ge C_\gamma M \varrho_\gamma |\Delta_2|\, |f_N(t_N)| \ge C_\gamma |\Delta_1|\, |f_N(t_N)|.$$

Since $|f_N(t_N)| \sim_\gamma |\Delta_1|^{-1/2}$, we have obtained

$$\int_{\mathbb{T}} |\widetilde{f}_N(t)|\, dt \ge C_\gamma |\Delta_1|^{1/2}. \tag{3.18}$$

Moreover, applying (3.17) to t_N (with $k = 0$) we find

$$|\widetilde{f}_N(t_N)| \ge C_\gamma |\Delta_1|^{-1/2} \log M. \tag{3.19}$$

Now, (3.18) and (3.19) give (3.7). Define $S_n(t,s) = \sum_{i=1}^n \widetilde{f}_i(t)\widetilde{f}_i(s)$. It follows from (3.7) that

$$\max(\|S_N(t_N,\cdot)\|_1, \|S_{N-1}(t_N,\cdot)\|_1) \ge \frac{1}{2}\int_{\mathbb{T}} |\widetilde{f}(t_N)|\,|\widetilde{f}_N(t)|\,dt \ge C_\gamma \log M.$$

As M is arbitrary, this means that the Lebesgue functions of $\{\widetilde{f}_n, n \ge 1\}$ are not bounded. Consequently, $\{\widetilde{f}_n, n \ge 1\}$ is not a basis in $C(\mathbb{T})$. ∎

3.3. Estimate of $\widetilde{L}_\sigma(t)$ from above by $\mathcal{J}_\sigma^{(3)}(t)$. Now, we turn to the sufficiency part. By Theorem 2.1, we assume the strong regularity condition. We prove the following upper estimate for the Lebesgue functions of the conjugate of the general Franklin system:

THEOREM 3.4. *Let $\gamma > 1$. Let $\sigma = \{s_i, 1 \le i \le n\}$ be a sequence of simple knots in $\mathbb{T}$ such that*

$$1/\gamma \le \lambda_i/\lambda_{i+1} \le \gamma \quad \textit{for all } 1 \le i \le n.$$

Then there is a constant C_γ, depending only on γ, such that for $t \in \mathbb{T}$,

$$\widetilde{L}_\sigma(t) \le C_\gamma \mathcal{J}_\sigma^{(3)}(t).$$

Proof. Following S. V. Bochkarev (cf. the proof of Theorem 3 in [4]), for fixed $\varepsilon > 0$, decompose $\varphi(u) = \cot(u/2) = \varphi_{1,\varepsilon}(u) + \varphi_{2,\varepsilon}(u)$, with

$$\begin{aligned} \varphi_{1,\varepsilon}(u) &= \begin{cases} \cot(u/2) & \text{for } |u| \le \varepsilon, \\ 0 & \text{for } \varepsilon \le |u| \le \pi, \end{cases} \\ \varphi_{2,\varepsilon}(u) &= \begin{cases} 0 & \text{for } |u| \le \varepsilon, \\ \cot(u/2) & \text{for } \varepsilon \le |u| \le \pi, \end{cases} \end{aligned} \tag{3.20}$$

and extend both $\varphi_{1,\varepsilon}$, $\varphi_{2,\varepsilon}$ 2π-periodically to $\mathbb{R}$.

Then we have the following decomposition of the kernel $\widetilde{K}$:

$$\begin{aligned} \widetilde{K}_\sigma(t,s) &= \frac{1}{2\pi} + \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \Delta_{u,v}K_\sigma(t,s)\cot\frac{u}{2}\cot\frac{v}{2}\,du\,dv \\ &= \frac{1}{2\pi} + V_1(t,s) + V_2(t,s) + V_3(t,s), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \Delta_{u,v}K_\sigma(t,s) = K_\sigma(t+u,s+v) &- K_\sigma(t-u,s+v) \\ &- K_\sigma(t+u,s-v) + K_\sigma(t-u,s-v), \end{aligned}$$

and

$$V_1(t,s) = \frac{1}{4\pi^2}\int_0^\pi \Big(\int_0^\pi \Delta_{u,v} K_\sigma(t,s)\cot\frac{u}{2}\,du\Big)\cot\frac{v}{2}\,dv$$
$$- \frac{1}{4\pi^2}\int_0^\pi (\varphi_{2,\varepsilon}(s-t+v) - \varphi_{2,\varepsilon}(s-t-v))\cot\frac{v}{2}\,dv,$$

$$V_2(t,s) = \frac{1}{4\pi^2}\int_0^\pi ((\varphi_{2,\varepsilon}(s-t+v) - \varphi_{2,\varepsilon}(s-t-v)))\varphi_{2,\varepsilon}(v)\,dv,$$

$$V_3(t,s) = \frac{1}{4\pi^2}\int_0^\pi ((\varphi_{2,\varepsilon}(s-t+v) - \varphi_{2,\varepsilon}(s-t-v)))\varphi_{1,\varepsilon}(v)\,dv.$$

We need to estimate $\widetilde{L}_\sigma(t) = \int_{\mathbb{T}} |\widetilde{K}_\sigma(t,s)|\,ds$. First, recall that (see the estimate for $|1/2\pi + I_3|$ in the proof of Theorem 3 in [4])

$$\Big|\frac{1}{2\pi} + V_2(t,s)\Big| \le C\min\Big(\frac{1}{\varepsilon}, \frac{\varepsilon}{(\operatorname{dist}(t,s))^2}\Big),$$

where C does not depend on ε, which implies

$$\int_{\mathbb{T}} \Big|\frac{1}{2\pi} + V_2(t,s)\Big|\,ds \le C. \tag{3.22}$$

Next, we turn to the part corresponding to $V_3(t,s)$. The way of estimating V_3 depends on $\operatorname{dist}(t,s)$.

First, consider the case $\operatorname{dist}(t,s) > 2\varepsilon$. In this case if $|v| \le \varepsilon$ then $\operatorname{dist}(s-t,\pm v) \ge \varepsilon$ and $\frac{1}{2}\operatorname{dist}(t,s) \le \operatorname{dist}(s-t,\pm v) \le \frac{3}{2}\operatorname{dist}(t,s)$. Therefore, by the mean value theorem,

$$|\varphi_{2,\varepsilon}(s-t\pm v) - \varphi_{2,\varepsilon}(s-t)|$$
$$= \Big|\cot\frac{s-t\pm v}{2} - \cot\frac{s-t}{2}\Big| \le C\,\frac{|v|}{(\operatorname{dist}(t,s))^2}.$$

Consequently,

$$|V_3(t,s)| = \frac{1}{4\pi^2}\Big|\int_0^\varepsilon (\varphi_{2,\varepsilon}(s-t+v) - \varphi_{2,\varepsilon}(s-t))\cot\frac{v}{2}\,dv\Big|$$
$$+ \frac{1}{4\pi^2}\Big|\int_0^\varepsilon (\varphi_{2,\varepsilon}(s-t-v) - \varphi_{2,\varepsilon}(s-t))\cot\frac{v}{2}\,dv\Big|$$
$$\le C\int_0^\varepsilon \frac{|v|}{(\operatorname{dist}(t,s))^2}\,\frac{1}{|v|}\,dv \le C\,\frac{\varepsilon}{(\operatorname{dist}(t,s))^2}.$$

The last inequality implies that

$$\int\limits_{\{s\in\mathbb{T}\,:\,\mathrm{dist}(t,s)>2\varepsilon\}} |V_3(t,s)|\,ds \le C\int\limits_{2\varepsilon}^{\infty} \frac{\varepsilon}{x^2}\,dx \le C. \tag{3.23}$$

When $\varepsilon < \mathrm{dist}(t,s) \le 2\varepsilon$, we write

$$\begin{aligned} V_3(t,s) &= \frac{1}{4\pi^2}\int\limits_0^{\mathrm{dist}(t,s)-\varepsilon} (\varphi_{2,\varepsilon}(s-t+v)-\varphi_{2,\varepsilon}(s-t-v))\cot\frac{v}{2}\,dv \\ &\quad+\frac{1}{4\pi^2}\int\limits_{\mathrm{dist}(t,s)-\varepsilon}^{\varepsilon} (\varphi_{2,\varepsilon}(s-t+v)-\varphi_{2,\varepsilon}(s-t-v))\cot\frac{v}{2}\,dv \\ &= V_{3,1}(t,s)+V_{3,2}(t,s). \end{aligned}$$

For $0\le v\le \mathrm{dist}(t,s)-\varepsilon$ we have $\varepsilon\le \mathrm{dist}(s-t,\pm v)\le 3\varepsilon$, so (again by the mean value theorem)

$$|\varphi_{2,\varepsilon}(s-t+v)-\varphi_{2,\varepsilon}(s-t-v)| \le C\,\frac{v}{\varepsilon^2}.$$

This implies

$$|V_{3,1}(t,s)| \le C\int\limits_0^{\mathrm{dist}(t,s)-\varepsilon} \frac{v}{\varepsilon^2}\,\frac{1}{v}\,dv \le C\,\frac{\mathrm{dist}(t,s)-\varepsilon}{\varepsilon^2}. \tag{3.24}$$

For $\mathrm{dist}(t,s)-\varepsilon\le v\le\varepsilon$, we note that $|\varphi_{2,\varepsilon}(\cdot)|\le C/\varepsilon$ and so

$$|V_{3,2}(t,s)| \le C\int\limits_{\mathrm{dist}(t,s)-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon}\,\frac{1}{v}\,dv = C\,\frac{1}{\varepsilon}\log\frac{\varepsilon}{\mathrm{dist}(t,s)-\varepsilon}.$$

Using the last inequality and (3.24), and setting $x=\mathrm{dist}(t,s)-\varepsilon$, we find

$$\begin{aligned} \int\limits_{\{s\in\mathbb{T}\,:\,\varepsilon<\mathrm{dist}(t,s)\le 2\varepsilon\}} |V_3(t,s)|\,ds &\le \int\limits_{\{s\in\mathbb{T}\,:\,\varepsilon<\mathrm{dist}(t,s)\le 2\varepsilon\}} |V_{3,1}(t,s)|\,ds \\ &\quad+ \int\limits_{\{s\in\mathbb{T}\,:\,\varepsilon<\mathrm{dist}(t,s)\le 2\varepsilon\}} |V_{3,2}(t,s)|\,ds \\ &\le C\int\limits_0^{\varepsilon}\Big(\frac{x}{\varepsilon^2}+\frac{1}{\varepsilon}\log\frac{\varepsilon}{x}\Big)\,dx = C\int\limits_0^1\Big(y+\log\frac{1}{y}\Big)\,dy. \end{aligned}$$

As $\int_0^1\big(y+\log\frac{1}{y}\big)\,dy<\infty$, we have obtained

$$\int\limits_{\{s\in\mathbb{T}\,:\,\varepsilon<\mathrm{dist}(t,s)\le 2\varepsilon\}} |V_3(t,s)|\,ds \le C. \tag{3.25}$$

Finally, let $\mathrm{dist}(t,s)\le\varepsilon$. Note that for $|v|<\varepsilon-\mathrm{dist}(t,s)$ we have $\mathrm{dist}(s-t,\pm v)<\varepsilon$, and consequently $\varphi_{2,\varepsilon}(s-t\pm v)=0$. Combining this with $|\varphi_{2,\varepsilon}(\cdot)| \le C/\varepsilon$ we get

$$|V_3(t,s)| \le \int\limits_{\varepsilon-\mathrm{dist}(t,s)}^{\varepsilon} |\varphi_{2,\varepsilon}(s-t+v)-\varphi_{2,\varepsilon}(s-t-v)|\left|\cot\frac{v}{2}\right| dv$$
$$\le C\int\limits_{\varepsilon-\mathrm{dist}(t,s)}^{\varepsilon} \frac{1}{\varepsilon}\frac{1}{v}\,dv = C\,\frac{1}{\varepsilon}\log\frac{\varepsilon}{\varepsilon-\mathrm{dist}(t,s)}.$$

This implies (now setting $x=\mathrm{dist}(t,s)$)

$$\int\limits_{\{s\in\mathbb{T}\,:\,\mathrm{dist}(t,s)\le\varepsilon\}} |V_3(t,s)|\,ds \le C\int\limits_0^{\varepsilon}\frac{1}{\varepsilon}\log\frac{\varepsilon}{\varepsilon-x}\,dx = C\int\limits_0^1\log\frac{1}{x}\,dx = C. \tag{3.26}$$

Putting together (3.23), (3.25) and (3.26) we get

$$\int\limits_{\mathbb{T}} |V_3(t,s)|\,ds \le C. \tag{3.27}$$

Now, we turn to estimating $\int_{\mathbb{T}}|V_1(t,s)|\,ds$. Let $m=m(t)$ be such that $t\in I_m$. Set

$$\varepsilon_t = \frac{|I_{m-1}|\wedge|I_{m+1}|}{2}. \tag{3.28}$$

Observe that

$$\{s\in\mathbb{T} : \mathrm{dist}(t,s)<\varepsilon_t\}\subset I_{m-1}\cup I_m\cup I_{m+1}.$$

By strong regularity with parameter γ we have $\varepsilon_t\sim_\gamma |I_m|$. Define

$$\Phi_t(s) = \int\limits_0^{\pi}(K_\sigma(t+u,s)-K_\sigma(t-u,s))\cot\frac{u}{2}\,du - \varphi_{2,\varepsilon_t}(s-t). \tag{3.29}$$

Observe that $V_1(t,s)=\frac{-1}{2\pi}\widetilde{\Phi}_t(s)$, which implies

$$\int\limits_{\mathbb{T}}|V_1(t,s)|\,ds = \frac{1}{2\pi}\int\limits_{\mathbb{T}}|\widetilde{\Phi}_t(s)|\,ds \le \frac{1}{2\pi}\,(\|\Phi_t\|_1+\|\widetilde{\Phi}_t\|_1)\le C\|\Phi_t\|_{\mathrm{Re}\,H^1}. \tag{3.30}$$

Therefore, the next step is to prove that $\Phi_t\in\mathrm{Re}\,H^1$ and

$$\|\Phi_t\|_{\mathrm{Re}\,H^1}\le C_\gamma\mathcal{J}_\sigma^{(3)}(t). \tag{3.31}$$

Then by (3.31) and (3.30) we get

$$\int\limits_{\mathbb{T}}|V_1(t,s)|\,ds \le C_\gamma\mathcal{J}_\sigma^{(3)}(t). \tag{3.32}$$

To prove (3.31), we find a suitable atomic decomposition of Φ_t. More precisely, we show that

$$\Phi_t = \sum_{i=1}^{n}\Phi_t N_i, \tag{3.33}$$

after a suitable normalization, yields such a decomposition.

Define

$$\Theta_t(s) = \int_0^\pi (K_\sigma(t+u,s) - K_\sigma(t-u,s)) \cot\frac{u}{2}\,du.$$

This is well defined, because $K_\sigma(\cdot,s)$ is a Lipschitz function. We show that

$$|\Theta_t(s)| \le C_\gamma/\varepsilon_t. \tag{3.34}$$

Indeed, $K_\sigma(v,s) = \sum_{i,j=1}^n a_{i,j}N_i(v)N_j(s)$. If $0 \le u \le \varepsilon_t$ then $t \pm u \in I_{m-1} \cup I_m \cup I_{m+1}$. Therefore, applying the estimates for $|a_{ij}|$ from Theorem 3.1 and strong regularity with parameter γ, we have, for $s \in I_k$,

$$\begin{aligned}|K_\sigma(t+u,s) - K_\sigma(t-u,s)| &\le \sum_{|i-m|\le 1} |N_i(t+u) - N_i(t-u)| \Big|\sum_{j=1}^n a_{i,j}N_j(s)\Big| \\ &\le C_\gamma \frac{u}{|I_m|} \sum_{|i-m|\le 1} \Big|\sum_{j=k}^{k+1} a_{i,j}N_j(s)\Big| \\ &\le C_\gamma \frac{u}{|I_m|}\, \frac{1}{2^{|m-k|_n}}\, \frac{1}{\lambda_m \vee \lambda_k}.\end{aligned}$$

Applying this estimate, we find, for $s \in I_k$,

$$\int_0^{\varepsilon_t} |K_\sigma(t+u,s) - K_\sigma(t-u,s)| \Big|\cot\frac{u}{2}\Big|\,du \le C_\gamma \frac{1}{2^{|m-k|_n}}\, \frac{1}{\lambda_m \vee \lambda_k}. \tag{3.35}$$

If $\varepsilon_t \le u \le \pi$, then $|\cot(u/2)| \le C/\varepsilon_t$, and consequently, using (3.2), we get

$$\int_{\varepsilon_t}^\pi |K_\sigma(t+u,s) - K_\sigma(t-u,s)| \Big|\cot\frac{u}{2}\Big|\,du \le \frac{C}{\varepsilon_t}\int_{\mathbb{T}} |K_\sigma(v,s)|\,dv \le \frac{C}{\varepsilon_t}.$$

Combining the last two inequalities we get (3.34). In fact, we have proved that

$$\int_0^\pi \Big|(K_\sigma(t+u,s) - K_\sigma(t-u,s)) \cot\frac{u}{2}\Big|\,du \le \frac{C}{\varepsilon_t}. \tag{3.36}$$

This allows us to use Fubini's theorem, which implies

$$\int_{\mathbb{T}} \Theta_t(s)\,ds = \int_0^\pi \int_{\mathbb{T}} (K_\sigma(t+u,s) - K_\sigma(t-u,s)) \cot\frac{u}{2}\,ds\,du = 0.$$

In particular, the last equality and the definition of $\varphi_{2,\varepsilon}$ (cf. (3.20)) imply

$$\int_{\mathbb{T}} \Phi_t(s)\,ds = 0. \tag{3.37}$$

The definition of ε_t guarantees that for $i \ne m-1, m, m+1$,

$$\{s \in \mathbb{T} : \operatorname{dist}(s, \operatorname{supp} N_i) < \varepsilon_t\} \cap \{s \in \mathbb{T} : \operatorname{dist}(t,s) < \varepsilon_t\} = \emptyset. \tag{3.38}$$

Let $A_i = N_i \cdot \Phi_t$ for $i \neq m-1, m, m+1$, and $A = \Phi_t - \sum_{i:|m-i|>1} A_i$. Note that by strong regularity with parameter γ,

$$|\operatorname{supp} A_i| \le C_\gamma \lambda_i, \qquad |\operatorname{supp} A| \le C_\gamma \lambda_m. \tag{3.39}$$

Now, using again Fubini's theorem (which is allowed because of (3.36)), the fact that P_σ is the orthogonal projection onto $\mathcal{S}_\sigma$, the definition of $\varphi_{2,\varepsilon_t}$ and (3.38) we find

$$\begin{aligned}\int_{\mathbb{T}} A_i(s)\,ds &= \int_0^\pi \int_{\mathbb{T}} (K_\sigma(t+u,s) - K_\sigma(t-u,s))N_i(s)\cot\frac{u}{2}\,ds\,du \\ &\quad - \int_{\mathbb{T}} \varphi_{2,\varepsilon_t}(s-t)N_i(s)\,ds \\ &= \int_0^\pi (N_i(t+u) - N_i(t-u))\cot\frac{u}{2}\,du - \int_{\mathbb{T}} \varphi_{2,\varepsilon_t}(s-t)N_i(s)\,ds = 0.\end{aligned}$$

Clearly, from these equalities and (3.37) we also have $\int_{\mathbb{T}} A(s)\,ds = 0$.

Put $\alpha = \|A\|_\infty \cdot \lambda_m$, $B = (1/\alpha)A$, $\alpha_i = \|A_i\|_\infty \cdot \lambda_i$, $B_i = (1/\alpha_i)A_i$. Now, by (3.39), to complete the proof of (3.31), it is enough to show that

$$\alpha + \sum_{i:|m-i|>1} \alpha_i \le C_\gamma \mathcal{J}_\sigma^{(3)}(t). \tag{3.40}$$

First, (3.34) and the definition of $\varphi_{2,\varepsilon}$ yield $|\Phi_t| \le C_\gamma/\varepsilon_t$. Since $\varepsilon_t \sim_\gamma 1/\lambda_m$, this implies that $\alpha \le C_\gamma$.

Now, we turn to estimating α_j. Let Λ be a piecewise linear function interpolating $\varphi_{2,\varepsilon_t}(\cdot - t)$ at the knots $\sigma' = \sigma \setminus \{s_m, s_{m+1}\}$. First, we estimate $|\Lambda(s) - \varphi_{2,\varepsilon_t}(s-t)|$ for $s \in I_j$. For this, we use the simple and well-known estimate

$$\max_{x\in[a,b]} |f(x) - l(x)| \le (b-a)^2 \max_{x\in[a,b]} |f''(x)|,$$

where $l(x) = \frac{x-a}{b-a}f(b) + \frac{b-x}{b-a}f(a)$ and $f \in C^2[a,b]$. Using this inequality, strong regularity and the definition of ε_t we find, for $j \neq m-1, m, m+1$,

$$|\Lambda(s) - \varphi_{2,\varepsilon_t}(s-t)| \le C_\gamma \frac{\lambda_j^2}{(\operatorname{dist}(t, I_j) + \lambda_j)^3} \qquad \text{for } s \in I_j. \tag{3.41}$$

We need to estimate $|\Phi_t(s)| = |\Theta_t(s) - \varphi_{2,\varepsilon_t}(s-t)|$. It follows from (3.35) that for $s \in I_j$,

$$\begin{aligned}&|\Theta_t(s) - \varphi_{2,\varepsilon_t}(s-t)| \\ &\quad \le C_\gamma \frac{1}{2^{|m-j|_n}} \frac{1}{\lambda_m \vee \lambda_j} + \Big| \int_{\varepsilon_t \le |u| \le \pi} K_\sigma(t+u,s)\cot\frac{u}{2}\,du - \varphi_{2,\varepsilon_t}(s-t)\Big| \\ &\quad = C_\gamma \frac{1}{2^{|m-j|_n}} \frac{1}{\lambda_m \vee \lambda_j} + |P_\sigma(\varphi_{2,\varepsilon_t}(\cdot - t))(s) - \varphi_{2,\varepsilon_t}(s-t)|.\end{aligned}$$

Λ interpolates $\varphi_{2,\varepsilon_t}(\cdot - t)$ at the knots $\sigma' \subset \sigma$, so $\Lambda \in \mathcal{S}_\sigma$ and $P_\sigma \Lambda = \Lambda$. As $P_\sigma(\ldots)$ is linear on I_j, for $s \in I_j$ we have

$$\begin{aligned}|P_\sigma(\varphi_{2,\varepsilon_t}(\cdot - t))(s) - \varphi_{2,\varepsilon_t}(s-t)| \le |P_\sigma(\varphi_{2,\varepsilon_t}(\cdot - t) - \Lambda)(t_j)| \,\frac{t_{j+1} - s}{t_{j+1} - t_j}& \\ + |P_\sigma(\varphi_{2,\varepsilon_t}(\cdot - t) - \Lambda)(t_{j+1})| \,\frac{s - t_{j+1}}{t_{j+1} - t_j}& \\ + |\Lambda(s) - \varphi_{2,\varepsilon_t}(s-t)|.&\end{aligned}$$

Moreover, the definition of Λ and the bound for $\varphi_{2,\varepsilon_t}$ imply $|\Lambda - \varphi_{2,\varepsilon_t}(\cdot - t)| \le C_\gamma/\lambda_m$. Together with strong regularity this shows that (3.41) holds also for $j = m-1, m, m+1$. Therefore, by (3.41), estimates for the kernel from Theorem 3.1 and strong regularity we get

$$\begin{aligned}|P_\sigma(\varphi_{2,\varepsilon_t}(\cdot - t) - \Lambda)(t_j)| &\le \int_{\mathbb{T}} |K_\sigma(u,t_j)|\,|\Lambda(u) - \varphi_{2,\varepsilon_t}(u-t)|\,du \\ &= \sum_{l=1}^{n} \int_{I_l} |K_\sigma(u,t_j)|\,|\Lambda(u) - \varphi_{2,\varepsilon_t}(u-t)|\,du \\ &\le C_\gamma \sum_{l=1}^{n} \frac{1}{2^{|l-j|_n}}\,\frac{1}{\lambda_l \vee \lambda_j}\left(\frac{\lambda_l}{\operatorname{dist}(t,I_l) + \lambda_l}\right)^3.\end{aligned}$$

$|P_\sigma(\varphi_{2,\varepsilon}(\cdot - t) - \Lambda)(t_{j+1})|$ is estimated analogously. Summarizing, for $s \in \operatorname{supp} N_j$ with $j \ne m-1, m, m+1$ we have obtained

$$\begin{aligned}|\Phi_t(s)| = |\Theta_t(s) - \varphi_{2,\varepsilon_t}(s-t)| \le C_\gamma \bigg(\frac{1}{2^{|m-j|_n}}\,\frac{1}{\lambda_m \vee \lambda_j} + \frac{\lambda_j^2}{(\operatorname{dist}(t,I_j) + \lambda_j)^3}& \\ + \sum_{l=1}^{n} \frac{1}{2^{|l-j|_n}}\,\frac{1}{\lambda_l \vee \lambda_j}\left(\frac{\lambda_l}{\operatorname{dist}(t,I_l) + \lambda_l}\right)^3\bigg).&\end{aligned}$$

Consequently,

$$\alpha_j \le C_\gamma \bigg(\frac{1}{2^{|m-j|_n}} + \sum_{l=1}^{n} \frac{1}{2^{|l-j|_n}}\left(\frac{\lambda_l}{\operatorname{dist}(t,I_l) + \lambda_l}\right)^3\bigg),$$

and summing over $|j-m| > 1$, we get (3.40) (recall that $\alpha \le C_\gamma$). This completes the proof of (3.31).

We have already seen that (3.31) implies (3.32). Putting together (3.21), (3.22), (3.27) and (3.32) we get the inequality stated in Theorem 3.4. ∎

COMMENT. It is possible to show that $\|\Phi_t\|_1 \ge C_\gamma \mathcal{J}_\sigma^{(3)}(t)$. This implies that $\|\Phi_t\|_{\operatorname{Re} H^1} \sim_\gamma \mathcal{J}_\sigma^{(3)}(t)$, but to get a lower bound for $\widetilde{L}_\sigma(t)$ in terms of $\mathcal{J}_\sigma^{(3)}(t)$ we would need an estimate of the form $\|\widetilde{\Phi}_t\|_1 \ge C_\gamma \mathcal{J}_\sigma^{(3)}(t)$. We do not know how to achieve this. However, we are able to get a weaker estimate from below for $\widetilde{L}_\sigma(t)$, in terms of $\mathcal{J}_\sigma^{(4)}(t)$ instead of $\mathcal{J}_\sigma^{(3)}(t)$. This is the subject of Section 3.4.

3.4. Necessity of $\mathcal{J}_\sigma^{(4)}(t) < \infty$. Now, we consider necessary conditions for $\{\widetilde{f}_n, n \geq 1\}$ to be a basis in $C(\mathbb{T})$. Theorem 2.1 already shows that strong regularity is such a condition. We strengthen this result and get a condition similar to the sufficient condition from Theorem 3.4, but with $\mathcal{J}_\sigma^{(4)}(t)$. As previously, because of Theorem 2.1, we consider only the case when the sequence of partitions involved satisfies the strong regularity condition.

THEOREM 3.5. *Let $\gamma > 1$. Let $\sigma = \{s_i, 1 \leq i \leq n\}$ be a sequence of simple knots in $\mathbb{T}$ such that*

$$1/\gamma \leq \lambda_i/\lambda_{i+1} \leq \gamma \quad \textit{for all } 1 \leq i \leq n.$$

Then there is a constant C_γ, depending only on γ such that for $t \in \mathbb{T}$,

$$\mathcal{J}_\sigma^{(4)}(t) \leq C_\gamma \widetilde{L}_\sigma(t).$$

For the proof, we need several technical lemmas. We start with

LEMMA 3.6. *Let σ be a sequence of simple knots in $\mathbb{T}$ containing at least* 10 *points and satisfying the strong periodic regularity condition with parameter γ. Let $t \in \mathbb{T}$. Then there is $k_{\max} \in \mathbb{N}$ and a subsequence $\{j_k, 1 \leq k \leq k_{\max}\}$ such that*

$$2 \leq \lambda_{j_k}/\lambda_{j_{k-1}}, \tag{3.42}$$

$$\operatorname{dist}(t, I_{j_k}) \geq 2\operatorname{dist}(t, I_{j_{k-1}}), \tag{3.43}$$

$$\sum_{k=1}^{k_{\max}} \left(\frac{\lambda_{j_k}}{\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k}}\right)^3 \geq C_\gamma \mathcal{J}_\sigma^{(4)}(t), \tag{3.44}$$

$$\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k} \leq \gamma \operatorname{dist}(t, I_{j_k}). \tag{3.45}$$

Moreover, let $\Omega_1 = [t, t+\pi]$, $\Omega_2 = [t-\pi, t]$ be two periodic intervals on $\mathbb{T}$ with endpoints $t, t+\pi$. The subsequence $\{j_k, 1 \leq k \leq k_{\max}\}$ can be chosen to be either increasing with $I_{j_k} \subset \Omega_1$ for all $1 \leq k \leq k_{\max}$, or decreasing with $I_{j_k} \subset \Omega_2$ for all $1 \leq k \leq k_{\max}$.

Proof. Let m_1, m_2 be indices such that $t \in I_{m_1}$ and $t + \pi \in I_{m_2}$.

Consider the set $\mathbb{T} \setminus (I_{m_1-1} \cup I_{m_1} \cup I_{m_1+1} \cup I_{m_2})$; it consists of at most two periodic intervals $\Omega_1' \subset \Omega_1$ and $\Omega_2' \subset \Omega_2$. Let $\mathcal{C}_i = \{j : I_j \subset \Omega_i'\}$, $i = 1, 2$. Clearly, all intervals with indices in $\mathcal{C}_i$ are included in Ω_i. By strong regularity, the intervals with indices in both collections $\mathcal{C}_i$ satisfy (3.45), and moreover

$$\mathcal{J}_\sigma^{(4)}(t) \sim_\gamma \sum_{j \in \mathcal{C}_1 \cup \mathcal{C}_2} \left(\frac{\lambda_j}{\operatorname{dist}(t, I_j) + \lambda_j}\right)^4.$$

Let $\mathcal{C}$ be the one of $\mathcal{C}_1$, $\mathcal{C}_2$ for which $\sum_{j \in \mathcal{C}_i} \left(\frac{\lambda_j}{\operatorname{dist}(t, I_j) + \lambda_j}\right)^4$ is greater. Clearly,

then
$$\mathcal{J}_\sigma^{(4)}(t) \sim_\gamma \sum_{j\in\mathcal{C}} \left(\frac{\lambda_j}{\operatorname{dist}(t,I_j)+\lambda_j}\right)^4.$$

To simplify notation further, assume that $-\pi \le t < t+\pi < \pi$ and $\mathcal{C} = \mathcal{C}_1$, so $\mathcal{C} = \{j : m_1+2 \le j \le m_2-1\}$. First, we choose a subsequence satisfying (3.42) and (3.44). Let j_1 be the element of $\mathcal{C}$ with $\operatorname{dist}(t, I_{j_i})$ minimal, i.e. $j_1 = m_1+2$. Having chosen $j_1,\dots,j_{k-1}$, take j_k to be the first element of $\mathcal{C}$ such that $j_k > j_{k-1}$ and $\lambda_{j_k} \ge 2\lambda_{j_{k-1}}$. If this is not possible, stop the procedure; the last j_k chosen is $j_{k_{\max}}$. Thus, (3.42) is satisfied. Moreover, the definition of j_k implies that $\lambda_{j_k-1} < 2\lambda_{j_{k-1}}$, hence by strong regularity

$$\lambda_{j_k} \le \gamma\lambda_{j_k-1} \le 2\gamma\lambda_{j_{k-1}}. \tag{3.46}$$

Now, we check (3.44). If $j_k > j_{k-1}+1$, then the collection of indices $j_{k-1} \le j \le j_k-1$ is split into intervals $\zeta_l \le j \le \zeta_{l+1}-1$ so that $\sum_{j=\zeta_l}^{\zeta_{l+1}-1} \lambda_j \sim_\gamma \lambda_{j_{k-1}}$. We start with $\zeta_1 = j_{k-1}$. If $\sum_{j=j_{k-1}}^{j_k-1} \lambda_j < 2\lambda_{j_{k-1}}$, then it is enough to put $\zeta_2 = j_k$ and to stop the procedure—the desired lower estimate follows by strong regularity. If $\sum_{j=j_{k-1}}^{j_k-1} \lambda_j \ge 2\lambda_{j_{k-1}}$ and ζ_l, $l \ge 1$, is already defined with $\sum_{j=\zeta_l}^{j_k-1} \lambda_j \ge 2\lambda_{j_{k-1}}$, then ζ_{l+1} is chosen so that $2\lambda_{j_{k-1}} \le \sum_{j=\zeta_l}^{\zeta_{l+1}-1} \lambda_j \le 4\lambda_{j_{k-1}}$; this is possible, since by the definition of j_k, we have $\lambda_j \le 2\lambda_{j_{k-1}}$ for $j_{k-1} < j < j_k$. If $\sum_{j=\zeta_l}^{j_k-1} \lambda_j < 2\lambda_{j_{k-1}}$, then we change the definition of ζ_l by putting $\zeta_l = j_k$ and stop the procedure of choosing ζ_i's. Note that in this last case we have $2\lambda_{j_{k-1}} \le \sum_{j=\zeta_{l-1}}^{\zeta_l-1} \lambda_j \le 6\lambda_{j_{k-1}}$.

For the sequence ζ_l so chosen, note that
$$\operatorname{dist}(t, I_{\zeta_l}) \ge \operatorname{dist}(t, I_{j_{k-1}}) + C_\gamma(l-1)\lambda_{j_{k-1}}.$$
Therefore
$$\begin{aligned}\sum_{j=j_{k-1}}^{j_k-1}\left(\frac{\lambda_j}{\operatorname{dist}(t,I_j)+\lambda_j}\right)^4 &= \sum_{l=1}^{l_{\max}(k)}\sum_{j=\zeta_l}^{\zeta_{l+1}-1}\left(\frac{\lambda_j}{\operatorname{dist}(t,I_j)+\lambda_j}\right)^4\\ &\le \sum_{l=1}^{l_{\max}(k)} \frac{(\sum_{j=\zeta_l}^{\zeta_{l+1}-1}\lambda_j)^4}{\operatorname{dist}(t,I_{\zeta_l})^4}\\ &\le C_\gamma \sum_{l=1}^{l_{\max}(k)} \frac{\lambda_{j_{k-1}}^4}{(\operatorname{dist}(t,I_{j_{k-1}})+C_\gamma(l-1)\lambda_{j_{k-1}})^4}\\ &\le C_\gamma\left(\frac{\lambda_{j_{k-1}}}{\operatorname{dist}(t,I_{j_{k-1}})+\lambda_{j_{k-1}}}\right)^3.\end{aligned}$$
By a similar argument we find that
$$\sum_{j\in\mathcal{C},\, j\ge j_{k_{\max}}}\left(\frac{\lambda_j}{\operatorname{dist}(t,I_j)+\lambda_j}\right)^4 \le C_\gamma\left(\frac{\lambda_{j_{k_{\max}}}}{\operatorname{dist}(t,I_{j_{k_{\max}}})+\lambda_{j_{k_{\max}}}}\right)^3.$$

Thus, we have obtained a subsequence j_k satisfying (3.42) and (3.44). Now, we choose a further subsequence so that (3.42) and (3.44) are still satisfied and (3.43) holds as well. Again, this is done inductively. Take $j_{k_1} = j_1$. Then, with $j_{k_{l-1}}$ already defined, let j_{k_l} be the last index from the subsequence j_k such that $\operatorname{dist}(t, I_{j_{k_l}}) \le 2\operatorname{dist}(t, I_{j_{k_{l-1}}})$; however, in case $j_{k_l} = j_{k_{l-1}}$ we put $j_{k_l} = j_{k_{l-1}+1}$ to guarantee $j_{k_l} > j_{k_{l-1}}$. This choice of j_{k_l} guarantees that

$$\operatorname{dist}(t, I_{j_k}) \ge 2\operatorname{dist}(t, I_{j_{k_{l-1}}}) \quad \text{for } k > k_l. \tag{3.47}$$

This implies that both sequences $\{j_{k_{2l}}\}$ and $\{j_{k_{2l-1}}\}$ satisfy (3.43); clearly, they also satisfy (3.42). It remains to show that at least one of them also satisfies (3.44). For this, we need to estimate $\sum_{k=k_{l-1}+1}^{k_l} \big(\frac{\lambda_{j_k}}{\operatorname{dist}(t,I_{j_k})+\lambda_{j_k}}\big)^3$. Assume first that $k_l > k_{l-1}+1$. Then $\operatorname{dist}(t, I_{j_{k_l}}) \le 2\operatorname{dist}(t, I_{j_{k_{l-1}}})$, which implies, for $k_{l-1}+1 \le k \le k_l$,

$$\operatorname{dist}(t, I_{j_{k_l}}) + \lambda_{j_{k_l}} \sim_\gamma \operatorname{dist}(t, I_{j_{k_l}}) \le 2 \operatorname{dist}(t, I_{j_{k_{l-1}}}) \le 2(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k}).$$

Moreover, by (3.42),

$$\sum_{k=k_{l-1}+1}^{k_l} \lambda_{j_k}^3 \le \frac{8}{7}\lambda_{j_{k_l}}^3.$$

Therefore,

$$\sum_{k=k_{l-1}+1}^{k_l} \bigg(\frac{\lambda_{j_k}}{\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k}}\bigg)^3 \le C_\gamma \bigg(\frac{\lambda_{j_{k_l}}}{\operatorname{dist}(t, I_{j_{k_l}}) + \lambda_{j_{k_l}}}\bigg)^3.$$

Clearly, the same estimate holds when $k_l = k_{l-1}+1$. Altogether we get

$$\sum_{l=1}^{l_{\max}} \bigg(\frac{\lambda_{j_{k_l}}}{\operatorname{dist}(t, I_{j_k}) + \lambda_{j_{k_l}}}\bigg)^3 \ge C_\gamma \sum_{k=1}^{k_{\max}} \bigg(\frac{\lambda_{j_k}}{\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k}}\bigg)^3 \ge C_\gamma \mathcal{J}_\sigma^{(4)}(t).$$

This implies that at least one of the sequences $\{j_{2l}\}$, $\{j_{2l-1}\}$ satisfies (3.44) in addition to (3.42), (3.43). ∎

LEMMA 3.7. *Fix $\eta > 1$. Let $0 < a_1 < a_2 \le \eta a_1$. Let $\omega(x) = 1/x$. Then there is a constant α_η, depending only on η, such that the following holds: Let l be a function linear on $[a_1, a_2]$ and such that*

$$|\{x \in [a_1, a_2] : l(x) - \omega(x) > 0\}| < \alpha_\eta(a_2 - a_1). \tag{3.48}$$

Then, for $i = 1, 2$, there is $h_i > 0$ such that for

$$l_1(x) = l(x) + \zeta_1(x - a_1), \qquad l_2(x) = l(x) + \zeta_2(a_2 - x),$$

with $0 < \zeta_i < h_i$, we have

$$\int_{a_1}^{a_2} (\omega(x) - l(x))^2\,dx > \int_{a_1}^{a_2} (\omega(x) - l_i(x))^2\,dx.$$

REMARK. The constant α_η in Lemma 3.7 depends only on η, but h_1, h_2 may depend on a_1, a_2 and l.

Proof. For l_1, the statement is equivalent to

$$\zeta_1 \int_{a_1}^{a_2} (x-a_1)^2\,dx < 2\int_{a_1}^{a_2} (x-a_1)(\omega(x)-l(x))\,dx \qquad \text{for } 0<\zeta_1<h_1.$$

To prove the above inequality, it is enough to show that

$$\int_{a_1}^{a_2} (x-a_1)(\omega(x)-l(x))\,dx > 0, \tag{3.49}$$

and then take $h_1 = 2\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx / \int_{a_1}^{a_2}(x-a_1)^2\,dx$.

Since $x(\omega(x)-l(x))$ is a polynomial of degree 2, there are at most two points in $[a_1,a_2]$ for which $x(l(x)-\omega(x))=0$. Now, consider the following cases:

(i) $x(\omega(x)-l(x))$ either has no zeros in (a_1,a_2), or has one double zero in (a_1,a_2).
(ii) $x(\omega(x)-l(x))$ has one zero in (a_1,a_2), and $l(a_1)<\omega(a_1)$.
(iii) $x(\omega(x)-l(x))$ has one zero in (a_1,a_2), and $l(a_1)>\omega(a_1)$.
(iv) $x(\omega(x)-l(x))$ has two distinct zeros in (a_1,a_2).

CASE (i). Assumption (3.48) implies that in this case $\omega(x)-l(x)>0$ on (a_1,a_2) (except of the possible double zero of $x(\omega(x)-l(x))$). Consequently, (3.49) holds.

CASE (ii). We show that there is $\alpha'_\eta>0$ such that if l satisfies (ii) and (3.48) with α'_η, then (3.49) holds.

Let $u\in(a_1,a_2)$ be such that $l(u)=\omega(u)$. Note that in this case $l(a_2)>\omega(a_2)$. Moreover, $l(x)>\omega(x)$ for $x\in(u,a_2)$ and $l(x)<\omega(x)$ for $x\in(a_1,u)$. Thus, we can write $u=a_2-\alpha(a_2-a_1)$, and $0\le\alpha\le\alpha'_\eta$, because of (3.48). Further, for fixed u, we calculate the integral in (3.49) as a function of $l(a_2)$, and we find that the term depending on $l(a_2)$ is

$$l(a_2)(a_2-a_1)^2\left(\frac{1}{6}\,\frac{a_2-a_1}{a_2-u}-\frac{1}{2}\right) = l(a_2)(a_2-a_1)^2\left(\frac{1}{6\alpha}-\frac{1}{2}\right).$$

Assume $\alpha'_\eta<1/3$. Then for $\alpha\le\alpha'_\eta$ the integral in question is an increasing function of $l(a_2)$. Therefore, it is enough to consider $l(a_2)=\omega(a_2)$. Then $l(x)=\frac{a_2-x}{a_2u}+\frac{1}{a_2}$, and

$$\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx = (a_2-a_1)-a_1\log\frac{a_2}{a_1}-\frac{(a_2-a_1)^3}{6a_2u}-\frac{(a_2-a_1)^2}{2a_2}.$$

Write $a_2-a_1=\varepsilon a_1$. In this notation, $a_2=(1+\varepsilon)a_1$, $u=(1+\varepsilon-\varepsilon\alpha)a_1$. The assumption $a_1<a_2\le\eta a_1$ means that $0<\varepsilon\le\eta-1$. The integral in question is equal to $a_1 U(\alpha,\varepsilon)$, where

$$U(\alpha,\varepsilon) = \varepsilon - \log(1+\varepsilon) - \frac{\varepsilon^3}{6(1+\varepsilon)(1+\varepsilon-\alpha\varepsilon)} - \frac{\varepsilon^2}{2(1+\varepsilon)}.$$

Hence

$$\frac{\partial U}{\partial \varepsilon}(\alpha,\varepsilon) = \frac{\varepsilon^3(2-4\alpha+2\varepsilon-5\alpha\varepsilon+3\alpha^2\varepsilon)}{6(1+\varepsilon)^2(1+\varepsilon-\alpha\varepsilon)^2}.$$

It follows that there is $\alpha'_\eta > 0$ such that $\frac{\partial U}{\partial \varepsilon}(\alpha,\varepsilon) > 0$ for all $0 < \varepsilon \le \eta - 1$ and $0 \le \alpha \le \alpha'_\eta$. Since $U(\alpha,0) = 0$, this implies that $U(\alpha,\varepsilon) > 0$ for the same range of parameters. Consequently, for l as in (ii) and satisfying (3.48) with α'_η we have $\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx > 0$.

Case (iii). We show that there is $\alpha''_\eta > 0$ such that if l satisfies (iii) and (3.48) then both $\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx > 0$ and $\int_{a_1}^{a_2}(\omega(x)-l(x))\,dx > 0$.

Let $v \in [a_1,a_2]$ be such that $\omega(v) = l(v)$. Then we can write $v = a_1 + \alpha(a_2-a_1)$. Since $\omega(x)-l(x) < 0$ on (a_1,v) and $\omega(x)-l(x) > 0$ on (v,a_2), it follows from (3.48) that $\alpha \le \alpha''_\eta$.

First, for fixed v, we calculate the integral $\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx$ as a function of $l(a_1)$, and we find that the part depending on $l(a_1)$ is

$$l(a_1)(a_2-a_1)^2\left(\frac{1}{3}\,\frac{a_2-a_1}{v-a_1} - \frac{1}{2}\right) = l(a_1)(a_2-a_1)^2\left(\frac{1}{3\alpha} - \frac{1}{2}\right).$$

Without loss of generality we can assume $\alpha''_\eta < 2/3$. Then for $\alpha \le \alpha''_\eta$ the integral is an increasing function of $l(a_1)$, and it is enough to consider the case when $l(a_1) = \omega(a_1)$. Then $l(x) = \frac{a_1-x}{a_1 v} + \frac{1}{a_1}$, and

$$\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx = (a_2-a_1) - a_1\log\frac{a_2}{a_1} + \frac{(a_2-a_1)^3}{3a_1 v} - \frac{(a_2-a_1)^2}{2a_1}.$$

Write $a_2 - a_1 = \varepsilon a_1$; note again that $0 < \varepsilon \le \eta - 1$, $a_2 = (1+\varepsilon)a_1$, $v = (1+\alpha\varepsilon)a_1$, and the integral in question equals $a_1 V(\alpha,\varepsilon)$, where

$$V(\alpha,\varepsilon) = \varepsilon - \log(1+\varepsilon) + \frac{\varepsilon^3}{3(1+\varepsilon\alpha)} - \frac{\varepsilon^2}{2}.$$

Then

$$\frac{\partial V}{\partial \varepsilon}(\alpha,\varepsilon) = \frac{\varepsilon^3(3-4\alpha+2\alpha\varepsilon-3\alpha^2\varepsilon)}{3(1+\varepsilon)(1+\alpha\varepsilon)^2}.$$

This shows that there is $\alpha''_\eta > 0$ such that $\frac{\partial V}{\partial \varepsilon}(\alpha,\varepsilon) > 0$ for all $0 \le \alpha \le \alpha''_\eta$ and $0 < \varepsilon \le \eta - 1$. Since $V(\alpha,0) = 0$, this implies that $V(\alpha,\varepsilon) > 0$ for the same range of parameters. Consequently, for l as in (iii) and satisfying (3.48) with α''_η we have $\int_{a_1}^{a_2}(x-a_1)(\omega(x)-l(x))\,dx > 0$.

It remains to treat $\int_{a_1}^{a_2}(\omega(x)-l(x))\,dx$. We proceed analogously. First, for fixed v we consider the integral as a function of $l(a_1)$, and find that if $\alpha''_\eta < 1/2$ then the integral is an increasing function of $l(a_1)$ for each $0 \le \alpha \le \alpha''_\eta$. So again it is enough to consider the case $l(a_1) = \omega(a_1)$ and $l(x) = \frac{a_1 - x}{a_1 v} + \frac{1}{a_1}$. In this case

$$\int_{a_1}^{a_2} (\omega(x) - l(x))\,dx = \log\frac{a_2}{a_1} - \frac{a_2 - a_1}{a_1} + \frac{(a_2 - a_1)^2}{2a_1 v}.$$

Put $a_2 - a_1 = \varepsilon a_1$; again $0 < \varepsilon \le \eta - 1$, and the integral equals $V_1(\alpha, \varepsilon) = \log(1+\varepsilon) - \varepsilon + \frac{\varepsilon^2}{2(1+\alpha\varepsilon)}$. Then

$$\frac{\partial V_1}{\partial \varepsilon}(\alpha, \varepsilon) = \frac{\varepsilon^2(2 - 3\alpha + \alpha\varepsilon - 2\alpha^2\varepsilon)}{2(1+\varepsilon)(1+\alpha\varepsilon)^2}.$$

Again, there is $\alpha''_\eta > 0$ such that $\frac{\partial V_1}{\partial \varepsilon}(\alpha, \varepsilon) > 0$ for all $0 \le \alpha \le \alpha''_\eta$ and $0 < \varepsilon \le \eta - 1$. Since $V_1(\alpha, 0) = 0$, this implies that $V_1(\alpha, \varepsilon) > 0$ for the same range of parameters, and so $\int_{a_1}^{a_2} (x - a_1)(\omega(x) - l(x))\,dx > 0$ for l as in (iii) and satisfying (3.48) with α''_η.

Case (iv). Let $b_1, b_2 \in [a_1, a_2]$, $b_1 < b_2$, be zeros of $\omega(x) - l(x)$. Clearly, $\omega(x) - l(x) > 0$ on $[a_1, b_1)$ and $(b_2, a_2]$, and $\omega(x) - l(x) < 0$ on (b_1, b_2). Consequently, $\int_{a_1}^{b_1} (x - a_1)(\omega(x) - l(x))\,dx > 0$ and $\int_{b_2}^{a_2} (x - a_1)(\omega(x) - l(x))\,dx > 0$. Moreover, $\max(b_2 - a_1, a_2 - b_1) \ge (a_2 - a_1)/2$. It should be clear that $a_1 < b_1 \le \eta a_1$ and $b_2 < a_2 \le \eta b_2$. Now, fix $\alpha_\eta = \min(\alpha'_\eta/2, \alpha''_\eta/2)$, where $\alpha'_\eta, \alpha''_\eta$ are taken from (ii), (iii), respectively. It follows that either l satisfies condition (ii) on $[a_1, b_2]$ and (3.48) with α'_η, or it satisfies condition (iii) on $[b_1, a_2]$ and (3.48) with α''_η. In the first case, condition (ii) yields $\int_{a_1}^{b_2} (x - a_1)(\omega(x) - l(x))\,dx > 0$. In the second case, (iii) gives $\int_{b_1}^{a_2} (x - b_1)(\omega(x) - l(x))\,dx > 0$ and $\int_{b_1}^{a_2} (\omega(x) - l(x))\,dx > 0$, which implies

$$\begin{aligned}\int_{b_1}^{a_2} (x - a_1)(\omega(x) - l(x))\,dx &= \int_{b_1}^{a_2} (x - b_1)(\omega(x) - l(x))\,dx \\ &\quad + (b_1 - a_1)\int_{b_1}^{a_2} (\omega(x) - l(x))\,dx > 0.\end{aligned}$$

This completes the proof of $\int_{a_1}^{a_2} (x - a_1)(\omega(x) - l(x))\,dx > 0$ in case (iv).

The case of l_2 is treated analogously. ∎

Lemma 3.8. *Fix $\eta > 1$. Let $0 < a_1 < a_2 \le \eta a_1$. Let $\omega(x) = 1/x$. Then there is a constant α_η, depending only on η, such that the following holds: Let l be a function linear on $[a_1, a_2]$ and such that*

$$|\{x \in [a_1, a_2] : l(x) - \omega(x) < 0\}| < \alpha_\eta (a_2 - a_1). \tag{3.50}$$

Then, for $i = 1, 2$, there is $h_i > 0$ such that for

$$l_1(x) = l(x) - \zeta_1(x - a_1), \qquad l_2(x) = l(x) - \zeta_2(a_2 - x),$$

with $0 < \zeta_i < h_i$, *we have*

$$\int_{a_1}^{a_2} (\omega(x) - l(x))^2\,dx > \int_{a_1}^{a_2} (\omega(x) - l_i(x))^2\,dx.$$

REMARK. The constant α_η in Lemma 3.8 depends only on η, but h_1, h_2 may depend on a_1, a_2 and l.

Proof. The statement for l_2 is equivalent to

$$\zeta_2 \int_{a_1}^{a_2} (a_2 - x)^2\,dx < 2\int_{a_1}^{a_2} (a_2 - x)(l(x) - \omega(x))\,dx \quad \text{for } 0 < \zeta_2 < h_2.$$

To prove the above inequality, it is enough to show that

$$\int_{a_1}^{a_2} (a_2 - x)(l(x) - \omega(x))\,dx > 0, \tag{3.51}$$

and then take $h_2 = 2\int_{a_1}^{a_2} (a_2 - x)(l(x) - \omega(x))\,dx / \int_{a_1}^{a_2} (a_2 - x)^2\,dx$.

As $x(l(x) - \omega(x))$ is a polynomial of degree 2, there are at most two points in $[a_1, a_2]$ for which $x(l(x) - \omega(x)) = 0$. Note that (3.50) implies that $x(l(x) - \omega(x))$ cannot have a double zero in (a_1, a_2). Now, consider the following cases:

(i) $x(l(x) - \omega(x))$ has no zeros in (a_1, a_2).
(ii) $x(l(x) - \omega(x))$ has one zero in (a_1, a_2), and $l(a_1) < \omega(a_1)$.
(iii) $x(l(x) - \omega(x))$ has one zero in (a_1, a_2), and $l(a_1) > \omega(a_1)$.
(iv) $x(l(x) - \omega(x))$ has two distinct zeros in (a_1, a_2).

CASE (i). Assumption (3.50) implies that in this case $l(x) - \omega(x) > 0$ on (a_1, a_2). Consequently, (3.51) holds.

CASE (ii). We show that in this case there is $\alpha'_\eta > 0$ such that if l satisfies (3.50) with α'_η, then both $\int_{a_1}^{a_2} (a_2 - x)(l(x) - \omega(x))\,dx > 0$ and $\int_{a_1}^{a_2} (l(x) - \omega(x))\,dx > 0$.

Let $u \in (a_1, a_2)$ be such that $l(u) = \omega(u)$. Note that in this case $l(a_2) > \omega(a_2)$. Moreover, $l(x) > \omega(x)$ for $x \in (u, a_2)$ and $l(x) < \omega(x)$ for $x \in (a_1, u)$. Thus, we can write $u = a_1 + \alpha(a_2 - a_1)$, and $0 \le \alpha \le \alpha'_\eta$, because of (3.50).

For fixed u, we first calculate the integral $\int_{a_1}^{a_2} (a_2 - x)(l(x) - \omega(x))\,dx$ as a function of $l(a_2)$, and we find that the term depending on $l(a_2)$ is equal to

$$l(a_2)(a_2 - a_1)^2\left(\frac{1}{2} - \frac{a_2 - a_1}{3(a_2 - u)}\right) = l(a_2)(a_2 - a_1)^2\left(\frac{1}{2} - \frac{1}{3(1 - \alpha)}\right).$$

Assume $\alpha'_\eta < 1/3$. Then for $\alpha \le \alpha'_\eta$ the integral is an increasing function of $l(a_2)$. Therefore, it is enough to consider $l(a_2) = \omega(a_2)$. Then $l(x) =$

$\frac{a_2-x}{a_2u}+\frac{1}{a_2}$, and so

$$\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx=\frac{(a_2-a_1)^3}{3a_2u}+\frac{(a_2-a_1)^2}{2a_2}+(a_2-a_1)-a_2\log\frac{a_2}{a_1}.$$

Put $a_2-a_1=\varepsilon a_2$. In this notation, $a_1=(1-\varepsilon)a_2$ and $u=(1-\varepsilon+\alpha\varepsilon)a_2$. The condition $a_1<a_2\le\eta a_1$ is equivalent to $0<\varepsilon\le 1-1/\eta$. The integral in question equals $a_2U(\alpha,\varepsilon)$, where

$$U(\alpha,\varepsilon)=\frac{\varepsilon^3}{3(1-\varepsilon+\alpha\varepsilon)}+\frac{\varepsilon^2}{2}+\varepsilon-\log\frac{1}{1-\varepsilon}.$$

Hence

$$\frac{\partial U}{\partial\varepsilon}(\alpha,\varepsilon)=\frac{\varepsilon^3(1-\varepsilon-4\alpha+4\varepsilon\alpha-3\alpha^2\varepsilon)}{3(1-\varepsilon)(1-\varepsilon+\alpha\varepsilon)^2}.$$

By this formula there is $\alpha'_\eta>0$ such that $\frac{\partial U}{\partial\varepsilon}(\alpha,\varepsilon)>0$ for all $0<\varepsilon\le 1-1/\eta$ and $0\le\alpha\le\alpha'_\eta$. Since $U(\alpha,0)=0$, this implies that $U(\alpha,\varepsilon)>0$ for the same range of parameters. Consequently, for l as in (ii) and satisfying (3.50) with α'_η we have $\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx>0$.

For $\int_{a_1}^{a_2}(l(x)-\omega(x))\,dx$, the argument is similar. First, for fixed u we consider the integral as a function of $l(a_2)$, and find that if $\alpha'_\eta<1/2$ then the integral is an increasing function of $l(a_2)$ for each $0\le\alpha\le\alpha'_\eta$. So again it is enough to consider the case $l(a_2)=\omega(a_2)$ and $l(x)=\frac{a_2-x}{a_2u}+\frac{1}{a_2}$, and then

$$\int_{a_1}^{a_2}(l(x)-\omega(x))\,dx=\frac{(a_2-a_1)^2}{2a_2u}+\frac{a_2-a_1}{a_2}-\log\frac{a_2}{a_1}.$$

Put $a_2-a_1=\varepsilon a_2$; again $0<\varepsilon\le 1-1/\eta$, and the integral equals $U_1(\alpha,\varepsilon)=\frac{\varepsilon^2}{2(1-\varepsilon+\alpha\varepsilon)}+\varepsilon-\log\frac{1}{1-\varepsilon}$. Hence

$$\frac{\partial U_1}{\partial\varepsilon}(\alpha,\varepsilon)=\frac{\varepsilon^2(1-\varepsilon-3\alpha+3\alpha\varepsilon-2\alpha^2\varepsilon)}{2(1-\varepsilon)(1-\varepsilon+\alpha\varepsilon)^2}.$$

Again, there is $\alpha'_\eta>0$ such that $\frac{\partial U_1}{\partial\varepsilon}(\alpha,\varepsilon)>0$ for all $0\le\alpha\le\alpha'_\eta$ and $0<\varepsilon\le 1-1/\eta$. Since $U_1(\alpha,0)=0$, this implies that $U_1(\alpha,\varepsilon)>0$ for the same range of parameters, and so $\int_{a_1}^{a_2}(l(x)-\omega(x))\,dx>0$ for l as in (ii) and satisfying (3.50) with α'_η.

Case (iii). We show that there is $\alpha''_\eta>0$ such that if l satisfies (iii) and (3.50) then $\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx>0$.

Let $v\in[a_1,a_2]$ be such that $\omega(v)=l(v)$. Write $v=a_2-\alpha(a_2-a_1)$. Since $l(x)-\omega(x)>0$ on (a_1,v) and $l(x)-\omega(x)<0$ on (v,a_2), it follows by (3.50) that $\alpha\le\alpha''_\eta$. Now, for fixed v, the part of $\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx$

depending on $l(a_1)$ is

$$l(a_1)(a_2-a_1)^2\left(\frac{1}{2}-\frac{1}{6}\,\frac{a_2-a_1}{v-a_1}\right)=l(a_1)(a_2-a_1)^2\left(\frac{1}{2}-\frac{1}{6(1-\alpha)}\right).$$

We can assume $\alpha''_\eta < 2/3$. Then for $\alpha \le \alpha''_\eta$ the integral in question is an increasing function of $l(a_1)$, and it is enough to consider the case when $l(a_1)=\omega(a_1)$. Then $l(x)=\frac{a_1-x}{a_1 v}+\frac{1}{a_1}$, and so

$$\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx=\frac{-(a_2-a_1)^3}{6a_1 v}+\frac{(a_2-a_1)^2}{2a_1}+(a_2-a_1)-a_2\log\frac{a_2}{a_1}.$$

Set $a_2-a_1=\varepsilon a_2$; then $0<\varepsilon\le 1-1/\eta$, $a_1=(1-\varepsilon)a_2$, $v=(1-\alpha\varepsilon)a_2$, and the integral equals $a_2 V(\alpha,\varepsilon)$, where

$$V(\alpha,\varepsilon)=\frac{-\varepsilon^3}{6(1-\varepsilon)(1-\alpha\varepsilon)}+\frac{\varepsilon^2}{2(1-\varepsilon)}+\varepsilon-\log\frac{1}{1-\varepsilon}.$$

Then

$$\frac{\partial V}{\partial\varepsilon}(\alpha,\varepsilon)=\frac{\varepsilon^3(2-4\alpha-\alpha\varepsilon+3\alpha^2\varepsilon)}{6(1-\varepsilon)^2(1-\alpha\varepsilon)^2}.$$

Again there is $\alpha''_\eta>0$ such that $\frac{\partial V}{\partial\varepsilon}(\alpha,\varepsilon)>0$ for all $0\le\alpha\le\alpha''_\eta$ and $0<\varepsilon\le 1-1/\eta$. Since $V(\alpha,0)=0$, this implies that $V(\alpha,\varepsilon)>0$ for the same range of parameters. Consequently, for l as in (iii) and satisfying (3.50) with α''_η we have $\int_{a_1}^{a_2}(a_2-x)(\omega(x)-l(x))\,dx>0$.

Case (iv). Let $b_1,b_2\in[a_1,a_2]$, $b_1<b_2$, be zeros of $\omega(x)-l(x)$. Set $b=(a_1+a_2)/2$. Clearly, $l(x)-\omega(x)<0$ on $[a_1,b_1)$ and $(b_2,a_2]$, and $l(x)-\omega(x)>0$ on (b_1,b_2). Note that $b_1<b<b_2$: if $b\le b_1$, then $l(x)-\omega(x)<0$ on $[a_1,b)$; as $\alpha_\eta<1/2$, this contradicts (3.50). By an analogous argument we exclude the possibility of $b_2\le b$. Note that $a_1<b\le\eta a_1$ and $b<a_2\le\eta b$. Take $\alpha_\eta=\min(\alpha'_\eta/2,\alpha''_\eta/2)$, where $\alpha'_\eta,\alpha''_\eta$ are taken from (ii), (iii), respectively. It follows that l satisfies condition (ii) on $[a_1,b]$ and (3.50) with α'_η, and also satisfies condition (iii) on $[b,a_2]$ and (3.50) with α''_η.

Applying (ii) to l on $[a_1,b]$ we get $\int_{a_1}^{b}(b-x)(l(x)-\omega(x))\,dx>0$ and $\int_{a_1}^{b}(l(x)-\omega(x))\,dx>0$. It follows that

$$\int_{a_1}^{b}(a_2-x)(l(x)-\omega(x))\,dx=\int_{a_1}^{b}(b-x)(l(x)-\omega(x))\,dx+(a_2-b)\int_{a_1}^{b}(l(x)-\omega(x))\,dx>0.$$

Moreover, applying (iii) to l on $[b,a_2]$ we find $\int_{b}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx>0$. This completes the proof of $\int_{a_1}^{a_2}(a_2-x)(l(x)-\omega(x))\,dx>0$ in case (iv).

The case of l_1 is treated analogously. ∎

To formulate Proposition 3.9 we need to introduce some notation. Let $\psi(u) = 2/u$ for $|u| \le \pi$. For $\varepsilon > 0$, let $\psi_{1,\varepsilon} = \psi(u)$ for $0 \le |u| \le \varepsilon$ and $\psi_{1,\varepsilon} = 0$ for $\varepsilon < |u| \le \pi$, $\psi_{2,\varepsilon} = 0$ for $0 \le |u| \le \varepsilon$ and $\psi_{2,\varepsilon} = \psi(u)$ for $\varepsilon < |u| \le \pi$, i.e. $\psi = \psi_{1,\varepsilon} + \psi_{2,\varepsilon}$. The functions $\psi, \psi_{1,\varepsilon}, \psi_{2,\varepsilon}$ are regarded as defined on $\mathbb{R}$ and 2π-periodic. Next, for a given sequence of points σ and $t \in \mathbb{T}$, let ε_t be given by formula (3.28) and $g_t = \psi_{2,\varepsilon_t}(\cdot - t)$.

PROPOSITION 3.9. *Let σ be a sequence of simple knots in $\mathbb{T}$ containing at least 10 points and satisfying the strong regularity condition with parameter γ. For $t \in \mathbb{T}$, let g_t be as defined above. Then there are constants B_γ, D_γ, depending only on γ, such that for each $t \in \mathbb{T}$ there is $F_t \in C(\mathbb{T})$ such that*

$$\|F_t\|_\infty = 1, \quad \|\widetilde{F}_t\|_\infty \le B_\gamma, \quad \Big|\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u))F_t(u)\,du\Big| \ge D_\gamma \mathcal{J}_\sigma^{(4)}(t).$$

Proof. We keep the notation from Lemma 3.6. Let $\{j_k, 1 \le k \le k_{\max}\}$ be the subsequence obtained in Lemma 3.6. To fix ideas, assume that $I_{j_k} \subset \Omega_1$. In this case, $\{j_k\}$ is increasing, and we can assume that $j_k + 1 < j_{k+1}$. For fixed j_k, consider $\Delta_{j_k} = I_{j_k} \cup I_{j_k+1}$. Note that the intervals Δ_{j_k} are parwise disjoint. We show that there are intervals $\Delta^-_{j_k}, \Delta^+_{j_k} \subset \Delta_{j_k}$ such that

$$\text{(3.52)} \quad P_\sigma g_t(u) - g_t(u) \ge C_\gamma \frac{\lambda_{j_k}^2}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3} \qquad \text{on } \Delta^+_{j_k},\ |\Delta^+_{j_k}| \sim_\gamma \lambda_{j_k},$$

$$\text{(3.53)} \quad P_\sigma g_t(u) - g_t(u) \le -C_\gamma \frac{\lambda_{j_k}^2}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3} \qquad \text{on } \Delta^-_{j_k},\ |\Delta^-_{j_k}| \sim_\gamma \lambda_{j_k}.$$

Let us show (3.52). Condition (3.45) and strong regularity imply

$$\begin{aligned}\operatorname{dist}(t, I_{j_k+1}) + \lambda_{j_k+1} &= \operatorname{dist}(t, I_{j_k}) + \lambda_{j_k} + \lambda_{j_k+1}\\ &\le (\gamma+1)(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})\\ &\le \gamma(\gamma+1)\operatorname{dist}(t, I_{j_k}) \le \gamma(\gamma+1)\operatorname{dist}(t, I_{j_k+1}).\end{aligned}$$

Hence both intervals $I_{j_k} - t$, $I_{j_k+1} - t$ satisfy the assumptions of Lemma 3.7 with common parameter $\eta_\gamma = \gamma(\gamma+1)$. $P_\sigma g_t(s-t)$ is linear on both $I_{j_k} - t$, $I_{j_k+1} - t$. If we had both

$$\text{(3.54)} \qquad |\{s \in I_{j_k} : P_\sigma g_t(s) - g_t(s) > 0\}| \le \alpha_{\eta_\gamma} \lambda_{j_k},$$

$$\text{(3.55)} \qquad |\{s \in I_{j_k+1} : P_\sigma g_t(s) - g_t(s) > 0\}| \le \alpha_{\eta_\gamma} \lambda_{j_k+1},$$

then by Lemma 3.7 there would be $\zeta > 0$ such that

$$\int_{I_{j_k}} (P_\sigma g_t(s) - g_t(s))^2\,ds > \int_{I_{j_k}} \Big(P_\sigma g_t(s) + \zeta\,\frac{s - t_{j_k}}{\lambda_{j_k}} - g_t(s)\Big)^2 ds,$$

$$\int_{I_{j_k+1}} (P_\sigma g_t(s) - g_t(s))^2\,ds > \int_{I_{j_k+1}} \Big(P_\sigma g_t(s) + \zeta\,\frac{t_{j_k+2} - s}{\lambda_{j_k+1}} - g_t(s)\Big)^2 ds.$$

But this implies that

$$\int_{\mathbb{T}} (P_\sigma g_t(s) - g_t(s))^2\, ds > \int_{\mathbb{T}} (P_\sigma g_t(s) + \zeta N_{j_k+1}(s) - g_t(s))^2\, ds.$$

Since $g_t \in L^2(\mathbb{T})$, this contradicts the definition of $P_\sigma g_t(s)$. So at least one of (3.54), (3.55) does not hold. Assume that (3.54) does not hold; the other case is analogous. As $(s-t)(P_\sigma g_t(s) - g_t(s))$ is a polynomial in s of degree 2 on I_{j_k}, it follows that there is an interval $I'_{j_k} \subset I_{j_k}$ such that $|I'_{j_k}| \sim_\gamma \lambda_{j_k}$ and

$$P_\sigma g_t(s) - g_t(s) > 0 \quad \text{for } s \in I'_{j_k}.$$

Further, we have

$$\operatorname{dist}(t, I'_{j_k}) + |I'_{j_k}| \le \operatorname{dist}(t, I_{j_k}) + |I_{j_k}| \le \eta_\gamma \operatorname{dist}(t, I_{j_k}) \le \eta_\gamma \operatorname{dist}(t, I'_{j_k}).$$

Let l_t be a linear function interpolating g_t at the endpoints of I'_{j_k}. Since $\|l_t - g_t\|_{L^\infty(I'_{j_k})} \le 2 \inf_l \|l - g_t\|_{L^\infty(I'_{j_k})}$, where the infimum is taken over all functions l linear on I'_{j_k}, we obtain

$$\max_{u \in I'_{j_k}} |P_\sigma g_t(u) - g_t(u)| \ge \frac{1}{2} \max_{u \in I'_{j_k}} |l_t(u) - g_t(u)| \ge C_\gamma \frac{\lambda_{j_k}^2}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3}.$$

Using again the argument with a quadratic polynomial, we find an interval $\Delta^+_{j_k} \subset I'_{j_k}$ with $|\Delta^+_{j_k}| \sim_\gamma |I'_{j_k}|$ such that

$$P_\sigma g_t(s) - g_t(s) \ge \frac{1}{2} \max_{u \in I'_{j_k}} |P_\sigma g_t(u) - g_t(u)| \quad \text{for } s \in \Delta^+_{j_k}.$$

Summarizing, we get (3.52).

Inequality (3.53) is proved analogously, with the use of Lemma 3.8.

Let $\Delta^+_{j_k}, \Delta^-_{j_k}$ satisfy (3.52), (3.53), respectively. Let $h^-_{j_k}, h^+_{j_k}$ be piecewise linear functions with supports $\Delta^-_{j_k}, \Delta^+_{j_k}$, respectively, with values 0 at the endpoints of the supports and $-1, 1$, respectively, at their midpoints. Observe that

$$\int_{\Delta^\pm_{j_k}} |g_t(u) h^\pm_{j_k}(u)|\, du \le C_\gamma \frac{|\Delta^\pm_{j_k}|}{\operatorname{dist}(t, \Delta^\pm_{j_k})} \le C_\gamma. \tag{3.56}$$

Since $\operatorname{supp} h^+_{j_k} = \Delta^+_{j_k}$, we have

$$\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) h^+_{j_k}(u)\, du = \int_{\Delta^+_{j_k}} (P_\sigma g_t(u) - g_t(u)) h^+_{j_k}(u)\, du.$$

As $h^+_{j_k} \ge 0$, it follows by (3.52) that $(P_\sigma g_t(u) - g_t(u)) h^+_{j_k}(u) \ge 0$. Combining the above with the estimates (3.52) we find that

$$\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) h_{j_k}^+(u)\,du \ge C_\gamma \frac{\lambda_{j_k}^3}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3}. \tag{3.57}$$

Since $\operatorname{supp} h_{j_k}^- = \Delta_{j_k}^-$ and $h_{j_k}^- \le 0$, an analogous argument combined with (3.53) gives $(P_\sigma g_t(u) - g_t(u)) h_{j_k}^-(u) \ge 0$ and

$$\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) h_{j_k}^-(u)\,du \ge C_\gamma \frac{\lambda_{j_k}^3}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3}. \tag{3.58}$$

Let h_{j_k} be one of $h_{j_k}^-, h_{j_k}^+$, where the choice of signs $\pm$ is done so that

$$\Big|\sum_{k=1}^{m} \int_{\Delta_{j_k}} g_t(u) h_{j_k}(u)\,du\Big| \le 2C_\gamma \quad \text{for each } m \ge 1, \tag{3.59}$$

where $\Delta_{j_k} = \operatorname{supp} h_{j_k}$. This choice is possible because of (3.56). Inequality (3.59) implies in particular

$$\Big|\sum_{k=m_1}^{m_2} \int_{\Delta_{j_k}} g_t(u) h_{j_k}(u)\,du\Big| \le 4C_\gamma \quad \text{for each } m_2 \ge m_1 \ge 1. \tag{3.60}$$

Now, put

$$F_t = \sum_{k=1}^{k_{\max}} h_{j_k}. \tag{3.61}$$

Observe that $F_t \in C(\mathbb{T})$ and $\|F_t\|_\infty = 1$. Moreover, by (3.57), (3.58) (note the positive signs)

$$\begin{aligned}\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) F_t(u)\,du &= \sum_{k=1}^{k_{\max}} \int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) h_{j_k}(u)\,du \\ &\ge C_\gamma \sum_{k=1}^{k_{\max}} \frac{\lambda_{j_k}^3}{(\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k})^3}.\end{aligned}$$

This inequality and (3.44) give

$$\Big|\int_{\mathbb{T}} (P_\sigma g_t(u) - g_t(u)) F_t(u)\,du\Big| \ge D_\gamma \mathcal{J}_\sigma^{(4)}(t).$$

It remains to show $\|\widetilde{F}_t\|_\infty \le B_\gamma$. To simplify the calculations, let

$$f^*(x) = \frac{-1}{2\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon \le |u| \le \pi} f(x+u)\psi(u)\,du.$$

Note that for $\varphi(u) = \cot(u/2)$ we have $|\varphi(u) - \psi(u)| \le C|u|$ for $|u| \le \pi$, and so $\|\widetilde{f} - f^*\|_\infty \le C\|f\|_\infty$. Therefore, it is enough to show $\|F^*\|_\infty \le B_\gamma$.

Fix $x \in \mathbb{T}$, and consider $\Gamma_1 = [x, x+\pi]$ and $\Gamma_2 = [x-\pi, x]$, two periodic intervals with endpoints $x, x+\pi$. We have $F_t^*(x) = \sum_{k=1}^{k_{\max}} h_{j_k}^*$.

There are two simple estimates for $h_{j_k}^*(x)$. Each h_{j_k} is a Lipschitz function with Lipschitz constant $\sim_\gamma \lambda_{j_k}^{-1}$. Define

$$A_{j_k}(x) = \{u : |u| \le \pi,\ h_{j_k}(x \pm u) \ne 0\},$$

and observe that $|A_{j_k}(x)| \le 2|\Delta_{j_k}| \le C_\gamma \lambda_{j_k}$. Therefore

$$\begin{aligned} |h_{j_k}^*(x)| &= \left| \frac{1}{2\pi} \int_0^\pi (h_{j_k}(x-u) - h_{j_k}(x+u))\psi(u)\, du \right| \\ &\le C_\gamma \int_{A_{j_k}(x)} \left| \frac{u}{\lambda_{j_k}} \cdot \frac{1}{u} \right| du \le C_\gamma. \end{aligned} \tag{3.62}$$

If $x \notin \Delta_{j_k}$ and $u \in A_{j_k}(x)$ then $|u| \ge \operatorname{dist}(x, I_{j_k})$. Therefore for $x \notin \Delta_{j_k}$,

$$\begin{aligned} |h_{j_k}^*(x)| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} \psi(u) h_{j_k}(x-u)\, du \right| \\ &\le \frac{1}{2\pi} \int_{A_{j_k}(x)} \frac{2}{|u|}\, du \le C_\gamma \frac{\lambda_{j_k}}{\operatorname{dist}(x, I_{j_k})}. \end{aligned} \tag{3.63}$$

Take ξ such that $x \in I_\xi$. Now, the sequence of indices $\{j_k, 1 \le k \le k_{\max}\}$ is split as follows:

$$\begin{aligned} M_0 &= \{k : |j_k - \xi| \le 2 \text{ or } x+\pi \in I_{j_k}\}, \\ M_1 &= \{k : I_{j_k} \subset \Gamma_1\}, \\ M_2 &= \{k : I_{j_k} \subset \Gamma_2\}. \end{aligned}$$

Note that if $k \notin M_0$ then $I_{j_k} \subset \Gamma_1$ or $I_{j_k} \subset \Gamma_2$, so $\{j_k, 1 \le k \le k_{\max}\} = M_0 \cup M_1 \cup M_2$. Let

$$U_i(x) = \sum_{k \in M_i} h_{j_k}^*(x), \qquad i = 0, 1, 2.$$

Since $\# M_0 \le 8$, it follows by (3.62) that

$$|U_0(x)| \le C_\gamma. \tag{3.64}$$

To estimate $U_1(x)$ and $U_2(x)$, consider first the case when $x \in \Omega_1$. Let us begin with $U_2(x)$. Let

$$p = \max\{k : k \in M_2\}.$$

Then, by the definition of M_0, $I_{j_{p+1}} \subset \Omega_1 \cap \Gamma_2$. Therefore, because of strong regularity we have, for $k \in M_2 = \{1, \ldots, p\}$,

$$\operatorname{dist}(x, I_{j_k}) \ge \operatorname{dist}(x, I_{j_p}) \ge \lambda_{j_{p+1}} \ge \lambda_{j_p}/\gamma.$$

Moreover, it follows from (3.42) that $\sum_{k=1}^{p} \lambda_{j_k} \le 2\lambda_{j_p}$. Using these two facts and (3.63) we obtain

$$(3.65) \qquad |U_2(x)| \le C_\gamma \sum_{k=1}^{p} \frac{\lambda_{j_k}}{\operatorname{dist}(x, I_{j_k})} \le C_\gamma \frac{\sum_{k=1}^{p} \lambda_{j_k}}{\lambda_{j_p}} \le C_\gamma.$$

It remains to estimate $U_1(x)$. Let $r = \min\{k : k \in M_1\}$; note that $M_1 = \{r, r+1, \dots, k_{\max}\}$. It follows from (3.43) that for $k > r$,

$$\operatorname{dist}(t, I_{j_k}) \ge 2\operatorname{dist}(t, I_{j_{k-1}}) \ge 2\operatorname{dist}(t, I_{j_r}),$$

which implies

$$(3.66) \qquad \operatorname{dist}(t, I_{j_k}) - \operatorname{dist}(t, I_{j_r}) \ge \operatorname{dist}(t, I_{j_k})/2 \quad \text{for } k > r.$$

For $k > r$ we write

$$h^*_{j_k}(x) = \int_{\mathbb{T}} \psi(u-x) h_{j_k}(u)\, du = \int_{\Delta_{j_k}} \frac{\psi(u-x)}{g_t(u)} g_t(u) h_{j_k}(u)\, du.$$

On Δ_{j_k} both $\psi(u-x)/g_t(u)$ and $g_t(u)h_{j_k}(u)$ are of constant sign, hence by the mean value theorem there is $u_k \in \Delta_{j_k}$ such that

$$h^*_{j_k}(x) = \frac{\psi(u_k - x)}{g_t(u_k)} \int_{\mathbb{T}} g_t(u) h_{j_k}(u)\, du.$$

Because $u_k \in \Delta_{j_k} \subset \Omega_1 \cap \Gamma_1$ for $k \in M_1$, we have $\psi(u_k - x) > 0$ and $g_t(u_k) > 0$, and moreover $u_{k+1} > u_k$. Therefore, $z_k = \psi(u_k - x)/g_t(u_k) = (u_k - t)/(u_k - x)$ for $k \in M_1$ is a decreasing sequence of positive numbers. Moreover, using (3.45) and (3.66) we obtain, for $k > r$,

$$|z_k| = \frac{\operatorname{dist}(t, u_k)}{\operatorname{dist}(x, u_k)} \le \frac{\operatorname{dist}(t, I_{j_k}) + \lambda_{j_k}}{\operatorname{dist}(t, I_{j_k}) - \operatorname{dist}(t, I_{j_r})} \le \frac{\gamma \operatorname{dist}(t, I_{j_k})}{\frac{1}{2}\operatorname{dist}(t, I_{j_k})} = 2\gamma.$$

Combining boundedness and monotonicity of $\{z_k\}$ with (3.60) we find

$$\Big| \sum_{k=r+1}^{k_{\max}} z_k \int_{\mathbb{T}} g_t(u) h_{j_k}(u)\, du \Big| \le C_\gamma.$$

This inequality and (3.62) imply

$$|U_1(x)| \le |h^*_{j_r}(x)| + \Big| \sum_{k=r+1}^{k_{\max}} z_k \int_{\mathbb{T}} g_t(u) h_{j_k}(u)\, du \Big| \le C_\gamma.$$

Since $F^*_t(x) = U_0(x) + U_1(x) + U_2(x)$, we obtain $|F^*_t(x)| \le B_\gamma$ for $x \in \Omega_1$.

When $x \in \Omega_2$, estimating $U_1(x)$ is analogous to the argument for $U_2(x)$ in case $x \in \Omega_1$, and estimating $U_2(x)$ is analogous to the argument for $U_1(x)$ in case $x \in \Omega_1$ (but the signs and type of monotonicity of z_k's may be different).

This completes the proof of Proposition 3.9. ∎

Proof of Theorem 3.5. First, suppose that σ contains at least 10 points. Let $t \in \mathbb{T}$. We keep the notation from the proof of Theorem 3.4 and introduced before Proposition 3.9. In particular, m is such that $t \in I_m$ and ε_t is defined by (3.28), and n is the number of points in σ.

By (3.22) and (3.27) we have

$$\int_{\mathbb{T}} \left| \frac{1}{2\pi} + V_2(t,s) + V_3(t,s) \right| ds \le C. \tag{3.67}$$

Now, we estimate $\|V_1(t,\cdot)\|_1$ from below. Recall that $V_1(t,\cdot) = \frac{-1}{2\pi}\widetilde{\Phi}_t$, where Φ_t is given by (3.29). Let Ψ_t be defined by a formula analogous to (3.29), but with $\psi(u) = 2/u$ replacing $\varphi(u) = \cot(u/2)$, and let F_t be as in Proposition 3.9. Then

$$\begin{aligned}\|\widetilde{\Psi}_t\|_1 &= \sup_{\|F\|_\infty \le 1} \left| \int_{\mathbb{T}} \widetilde{\Psi}_t(s) F(s)\, ds \right| = \sup_{\|F\|_\infty \le 1} \left| \int_{\mathbb{T}} \Psi_t(s) \widetilde{F}(s)\, ds \right| \\ &\ge \frac{1}{B_\gamma} \left| \int_{\mathbb{T}} \Psi_t(s) F_t(s)\, ds \right|.\end{aligned}$$

Observe that $\Psi_t(s) = q_t(s) + P_\sigma g_t(s) - g_t(s)$, where

$$\begin{aligned} q_t(s) &= \int_0^\pi (K_\sigma(t+u,s) - K_\sigma(t-u,s))\psi_{1,\varepsilon_t}(u)\, du \\ &= \int_0^{\varepsilon_t} (K_\sigma(t+u,s) - K_\sigma(t-u,s)) \frac{2}{u}\, du. \end{aligned}$$

Calculations analogous to those leading to (3.35) give

$$|q_t(s)| \le C_\gamma \frac{1}{2^{|m-k|_n}} \frac{1}{\lambda_m \vee \lambda_k} \qquad \text{for } s \in I_k.$$

Since $\|F_t\|_\infty = 1$, we find

$$\left| \int_{\mathbb{T}} q_t(s) F_t(s)\, ds \right| \le \sum_{k=1}^n \int_{I_k} |q_t(s)|\, ds \le C_\gamma \sum_{k=1}^n \frac{1}{2^{|m-k|_n}} \le C_\gamma.$$

By Proposition 3.9,

$$\left| \int_{\mathbb{T}} (P_\sigma g_t(s) - g_t(s)) F_t(s)\, ds \right| \ge D_\gamma \mathcal{J}_\sigma^{(4)}(t).$$

Combining these estimates we find

$$\|\widetilde{\Psi}_t\|_1 \ge \frac{D_\gamma \mathcal{J}_\sigma^{(4)}(t) - C_\gamma}{B_\gamma}.$$

Note that $|\varphi(u) - \psi(u)| \le C|u|$ for $|u| \le \pi$. This observation and estimate (3.2) imply that $\|\Phi_t - \Psi_t\|_\infty \le C$. Now the boundedness of the conjugate operator on $L^2(\mathbb{T})$ yields

$$\|\widetilde{\Phi}_t - \widetilde{\Psi}_t\|_1 \le (2\pi)^{1/2}\|\widetilde{\Phi}_t - \widetilde{\Psi}_t\|_2 \le (2\pi)^{1/2}\|\Phi_t - \Psi_t\|_2 \le 2\pi\|\Phi_t - \Psi_t\|_\infty \le C.$$

Thus we get

$$\|\widetilde{\Phi}_t\|_1 \ge \frac{D_\gamma}{B_\gamma}\,\mathcal{J}_\sigma^{(4)}(t) - C_\gamma.$$

Combining the above inequality with (3.21) and (3.67) we find

$$\widetilde{L}_\sigma(t) \ge C_\gamma \mathcal{J}_\sigma^{(4)}(t) - C'_\gamma.$$

On the other hand, $\int_{\mathbb{T}} \widetilde{K}_\sigma(t,s)\,ds = 1$, which implies that

$$\widetilde{L}_\sigma(t) \ge 1.$$

Combination of the last two inequalities completes the proof of Theorem 3.5 in the case when σ contains at least 10 points.

Finally, if σ contains fewer than 10 points, then $\mathcal{J}_\sigma^{(4)}(t) \sim_\gamma 1$. Also in this case we have $\widetilde{L}_\sigma(t) \ge 1$, which gives the desired estimate. ∎

4. EXAMPLES AND FINAL REMARKS

For completeness, let us discuss some examples of sequences $\mathcal{T}$ with $\sup_{n,t} \mathcal{J}_n^{(3)}(t) < \infty$ or $\sup_{n,t} \mathcal{J}_n^{(4)}(t) = \infty$.

Example 1 (Sequences with given bounds for the mesh ratio). Let $\{\gamma_k, k \ge 1\}$ be a sequence of positive numbers. Define $\varrho_k = 1 + \sum_{j=1}^{k-1} 1/\gamma_j$. Let $\mathcal{T}$ be a sequence of points such that the partitions $\mathcal{T}_n, n \ge 2$, satisfy

$$\text{(4.1)}\qquad \frac{1}{\gamma_{|k_1-k_2|_n}}\,\lambda_{n,k_1} \le \lambda_{n,k_2} \le \gamma_{|k_1-k_2|_n}\lambda_{n,k_1} \text{ for all } n \ge 2,\ 1 \le k_1, k_2 \le n.$$

Recall that $|k_1 - k_2|_n = \min(|k_1 - k_2|, n - |k_1 - k_2|)$. Then $\mathcal{T}$ satisfies the strong periodic regularity condition with parameter γ_1. Observe that

$$\operatorname{dist}(I_{n,k_1}, I_{n,k_2}) + \lambda_{n,k_1} \ge \lambda_{n,k_1}\Big(1 + \sum_{j=1}^{|k_1-k_2|_n-1} \frac{1}{\gamma_j}\Big) = \lambda_{n,k_1}\varrho_{|k_1-k_2|_n-1}.$$

If $\sum_{k=1}^\infty 1/\varrho_k^3 < \infty$, then the last inequality implies $\sup_{n,t} \mathcal{J}_n^{(3)}(t) < \infty$. Therefore, by Theorem 2.2, the conjugate to the corresponding general periodic Franklin system is a basis in $C(\mathbb{T})$.

The condition $\sum_{k=1}^\infty 1/\varrho_k^3 < \infty$ is satisfied e.g. when $\gamma_k = M$, where M is a fixed constant. Condition (4.1) then means that the partitions $\mathcal{T}_n, n \ge 1$, generated by $\mathcal{T}$ have uniformly bounded global mesh ratio. Another example is $\gamma_k = k^\varepsilon$ with $0 < \varepsilon < 2/3$.

4.1. More on the strong regularity condition. Theorem 2.1 states that the strong regularity condition is necessary for the conjugate Franklin system to be a basis in $C(\mathbb{T})$. The following example shows that it is not sufficient. However, in the class of quasi-dyadic partitions it is a sufficient condition.

EXAMPLE 2 (Sequence satisfying the strong periodic regularity condition with $\sup_{n,t} \mathcal{J}_n^{(4)}(t) = \infty$). Set $\tau_0 = 0$, $\tau_1 = 1$, $\tau_2 = 1/2$, and $\tau_{2s-1} = 1/2 - 1/2^s$, $\tau_{2s} = 1/2 + 1/2^s$ for $s \geq 2$. This sequence satisfies the strong periodic regularity condition with parameter $\gamma = 2$. For $\mu \geq 2$, let $\mathcal{G}_\mu = \{\tau_s, 0 \leq s \leq 2\mu\}$. Then

$$\mathcal{J}_{\mathcal{G}_\mu}^{(k)}\left(\frac{1}{2}\right) = 2 + \frac{2(\mu - 2)}{2^k}.$$

Further, we can pass from $\mathcal{G}_\mu$ to a uniform partition with step $1/2^\mu$ by dividing each time one of the longest intervals in half.

A dense sequence satisfying the strong regularity condition with $\gamma = 2$ and $\sup_{n,t} \mathcal{J}_n^{(4)}(t) = \infty$ can be constructed as follows:

(i) Fix a sequence $\mu_i \in \mathbb{N}$ with $\limsup_{i\to\infty} \mu_i = \infty$.
(ii) Take $\mathcal{G}_{\mu_1}$ and rescale it to $[-\pi, \pi]$, and then pass to a uniform partition with step $2\pi/2^{\mu_1}$ by halving longest intervals.
(iii) Fix one of the intervals in the uniform partition of $[-\pi, \pi]$; call it Δ_1. Take a partition $\mathcal{G}_{\mu_2,\Delta_1}$ of Δ_1 obtained by rescaling $\mathcal{G}_{\mu_2}$ from $[0, 1]$ to Δ_1. Then pass to a uniform partition of $[-\pi, \pi]$ with step $2\pi/2^{\nu_2}$, $\nu_2 = \mu_1 + \mu_2$, by halving each time one of the longest intervals.
(iv) For $i \geq 2$, repeat step (iii): choose Δ_i, rescale $\mathcal{G}_{\mu_{i+1}}$ to Δ_i, and then pass to a uniform partition of $[-\pi, \pi]$ with step $2\pi/2^{\nu_{i+1}}$, $\nu_{i+1} = \nu_i + \mu_{i+1}$.

Quasi-dyadic partitions. By a *quasi-dyadic sequence* we mean a sequence of points $\mathcal{T} = \{t_j, j \geq 1\}$ with the following structure: $\mathcal{T}_0 = \{t_1, t_2\} \subset \mathbb{T}$; these two points define two arcs in $\mathbb{T}$; call them $I_{0,1}, I_{0,2}$. Then take $\mathcal{T}_1 = \{t_3, t_4\} \subset \mathbb{T}$ so that $t_3 \in I_{0,1}$ and $t_4 \in I_{0,2}$. Assume $\mathcal{T}_0$ and $\mathcal{T}_s = \{t_l, 2^s + 1 \leq l \leq 2^{s+1}\}$, $1 \leq s \leq \mu$, have been defined. The collection of points $\{t_l, 1 \leq l \leq 2^{\mu+1}\}$ defines $2^{\mu+1}$ disjoint arcs in $\mathbb{T}$; call them $I_{\mu,l}$, $1 \leq l \leq 2^{\mu+1}$. Now, take $\mathcal{T}_{\mu+1} = \{t_l, 2^{\mu+1} + 1 \leq l \leq 2^{\mu+2}\}$ such that $t_l \in I_{\mu,l}$. Finally, set $\mathcal{T} = \bigcup_{j\geq 0} \mathcal{T}_j$.

PROPOSITION 4.1. *Let $\mathcal{T}$ be a quasi-dyadic sequence satisfying the strong periodic regularity condition. Then*

$$\sup_{n,t} \mathcal{J}_n^{(3)}(t) < \infty.$$

Proof. To simplify notation, consider $n = 2^{\mu+1}$, i.e. the case when all dyadic levels $\mathcal{T}_0, \ldots, \mathcal{T}_\mu$ are present. For given $t \in \mathbb{T}$, let Δ_s be the unique interval from $I_{s,1}, \ldots, I_{s,2^{s+1}}$ such that $t \in \Delta_s$. Let Δ_s^-, Δ_s^+ be the left and right neighbours of Δ_s, and $\widetilde{\Delta}_s = \Delta_s^- \cup \Delta_s \cup \Delta_s^+$. By strong regularity,

$$|\Delta_s^-| \sim_\gamma |\Delta_s| \sim_\gamma |\Delta_s^+| \sim_\gamma |\widetilde{\Delta}_s|.$$

Observe that $\widetilde{\Delta}_s \subset \widetilde{\Delta}_{s-1}$, and each $\widetilde{\Delta}_s$ is a union of some arcs $I_{\mu,l}$.

Let

$$A_s = \{l : I_{\mu,l} \subset \widetilde{\Delta}_s \text{ and } I_{\mu,l} \not\subset \widetilde{\Delta}_{s+1}\}.$$

Note that for $l \in A_s$, $s \geq \mu - 1$ we have $\operatorname{dist}(t, I_{\mu,l}) \sim_\gamma |\Delta_s|$. Moreover, strong regularity implies that $|I_{\mu,l}| \leq (\gamma/(\gamma+1))^{\mu-s}|\Delta_s|$ when $l \in A_s$. Now,

$$\begin{aligned}\sum_{l\in A_s}\left(\frac{|I_{\mu,l}|}{|I_{\mu,l}| + \operatorname{dist}(t, I_{\mu,l})}\right)^3 &\leq C_\gamma \frac{\max_{l\in A_s}|I_{\mu,l}|}{|\Delta_s|^3}\sum_{l\in A_s}|I_{\mu,l}|^2 \\ &\leq C_\gamma\left(\frac{\gamma}{\gamma+1}\right)^{\mu-s}|\Delta_s|^{-2}\Big(\sum_{l\in A_s}|I_{\mu,l}|\Big)^2 \\ &\leq C_\gamma\left(\frac{\gamma}{\gamma+1}\right)^{\mu-s}.\end{aligned}$$

This implies

$$\mathcal{J}_n^{(3)}(t) = \sum_{s\leq\mu}\sum_{l\in A_s}\left(\frac{|I_{\mu,l}|}{|I_{\mu,l}| + \operatorname{dist}(t, I_{\mu,l})}\right)^3 \leq C_\gamma \sum_{s\leq\mu}\left(\frac{\gamma}{\gamma+1}\right)^{\mu-s} \leq C_\gamma. \ \blacksquare$$

Combining Theorems 2.1 and 2.2 with Proposition 4.1 we get

COROLLARY 4.2. *Let $\mathcal{T}$ be a quasi-dyadic sequence of points with the corresponding general periodic Franklin system $\{f_n, n \geq 1\}$. Then the conjugate system $\{\widetilde{f}_n, n \geq 1\}$ is a basis in $C(\mathbb{T})$ if and only if $\mathcal{T}$ satisfies the strong periodic regularity condition.*

Acknowledgments. G. G. Gevorkyan was supported by the Armenian National Science and Education Fund, Grant No. 05-PS-math-87-65. Part of this work was done while G. G. Gevorkyan was visiting the Institute of Mathematics of the Polish Academy of Sciences in Sopot in April 2005. G. G. Gevorkyan's stay in Sopot was partially supported by the Foundation for Polish Science.

A. Kamont was supported by KBN grant 1 P03A 038 27. Part of this work was done while A. Kamont was visiting the Institute of Mathematics of the National Academy of Sciences of Armenia in October 2005.

References

[1] S. V. Bochkarev, *Existence of a basis in the space of functions analytic in the disc and some properties of the Franklin system*, Mat. Sb. 95 (1974), 3–18 (in Russian).

[2] —, *Some inequalities for Franklin series*, Anal. Math. 1 (1975), 249–257.

[3] S. V. Bochkarev, *Conjugate Franklin system—a basis in the space of continuous functions*, Dokl. Akad. Nauk SSSR 285 (1985), 521–526 (in Russian).
[4] —, *Bases in function spaces and the Franklin system*, Trudy Mat. Inst. Steklova 190 (1989), 22–39 (in Russian).
[5] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), 141–157.
[6] —, *Properties of the orthonormal Franklin system II*, ibid. 27 (1966), 289–323.
[7] Z. Ciesielski and A. Kamont, *Projections onto piecewise linear functions*, Funct. Approx. Comment. Math. 25 (1997), 129–143.
[8] Ph. Franklin, *A set of continuous orthogonal functions*, Math. Ann. 100 (1928), 522–528.
[9] G. G. Gevorkyan and A. Kamont, *On general Franklin systems*, Dissertationes Math. (Rozprawy Mat.) 374 (1998).
[10] —, —, *Unconditionality of general Franklin system in* $L^p[0,1]$, $1 < p < \infty$, Studia Math. 164 (2004), 161–201.
[11] —, —, *General Franklin systems as bases in* $H^1[0,1]$, ibid. 167 (2005), 259–292.
[12] —, —, *Some properties of the conjugate general Franklin systems*, Dokl. Nats. Akad. Nauk Armen. 106 (2006), no. 2, 108–113 (in Russian).
[13] K. A. Keryan, *The unconditional basis property of a general Franklin periodic system in* $L_p[0,1]$, $1 < p < \infty$, Izv. Nats. Akad. Nauk Armenii Mat. 40 (2005), no. 1, 18–60 (in Russian); English transl.: J. Contemp. Math. Anal. 40 (2005), no. 1, 15–55.
[14] K. A. Keryan and M. P. Pogosyan, *A general Franklin periodic system as a basis in* $H^1[0,1]$, Izv. Nats. Akad. Nauk Armenii Mat. 40 (2005), no. 1, 61–84 (in Russian); English transl.: J. Contemp. Math. Anal. 40 (2005), no. 1, 56–79.
[15] P. Sjölin and J. O. Strömberg, *Basis properties of Hardy spaces*, Ark. Mat. 21 (1983), 111–125.
[16] P. Wojtaszczyk, *The Franklin system is an unconditional basis in* H^1, ibid. 20 (1982), 293–300.
[17] A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge Univ. Press, Cambridge, 1959.

Department of Mathematics
Yerevan State University
Alex Manoukian St. 1
375049 Yerevan, Armenia
E-mail: ggg@ysu.am

Institute of Mathematics
Polish Academy of Sciences
Abrahama 18
81-825 Sopot, Poland
E-mail: A.Kamont@impan.gda.pl

Received March 27, 2007
Revised version December 4, 2008 (6129)