Quasiaffine transforms of operators

by

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Abstract. We obtain a new sufficient condition (which may be useful elsewhere) that a compact perturbation of a normal operator be the quasiaffine transform of some normal operator. We also give some applications of this result.

Let \( H \) be a separable, infinite dimensional, complex Hilbert space, and let \( \mathcal{L}(H) \) denote the algebra of all bounded linear operators on \( H \). As usual, we will write \( K \) for the ideal of compact operators in \( \mathcal{L}(H) \). Recall from [8] that an \( X \in \mathcal{L}(H) \) is called a quasiaffinity if \( \ker X = \ker X^* = (0) \), and that if \( S, T \in \mathcal{L}(H) \) and there exists a quasiaffinity \( X \in \mathcal{L}(H) \) such that \( XS = TX \), then we say that \( S \) is a quasiaffine transform of \( T \) and we write \( S \prec T \). If both \( S \prec T \) and \( T \prec S \) then we say that \( S \) and \( T \) are quasisimilar, and we write \( S \sim T \). It is well known that quasisimilarity is an equivalence relation on \( \mathcal{L}(H) \) that preserves the existence of nontrivial hyperinvariant subspaces (cf. [5], [8]). One also knows that if \( S \) and \( T \) are normal and \( S \prec T \), then \( S \sim T \) and, in fact, \( S \) and \( T \) are unitarily equivalent. Below we also write \( \{T\}' \) for the commutant of an operator \( T \) in \( \mathcal{L}(H) \) and \( \sigma_p(T) \) for the point spectrum of \( T \).

The theory of quasiaffine transforms of operators is well developed and plays an important role in the study of operators on Hilbert space (cf., e.g., [2] and [8]). In particular, the following little-known but somewhat interesting result was obtained in [1].

Theorem 1 ([1, Th. 4.3]). Let \( T \) be an arbitrary operator in \( \mathcal{L}(H) \) and \( \varepsilon \) an arbitrary positive number. Then there exist a normal operator \( N \) and a compact operator \( K \) in \( \mathcal{L}(H) \) such that \( T \prec N + K \) and \( \|K\| < \varepsilon \).

Thus it is of interest to obtain sufficient conditions in order that an operator \( N + K \) as in Theorem 1 be a quasiaffine transform of a normal operator.

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operator $M$ (thus giving $T \prec N + K \prec M$), and we obtain one such condition below (Theorem 5).

Our result depends on an old construction that has been used by many authors (cf., e.g., [4], [6]). We first obtain a new Hilbert space $K_H$ from $\mathcal{H}$ as follows.

**Definition 2.** Let $K_1$ be the linear space of all (bounded) sequences $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $x_n \to 0$ weakly, and let LIM be a fixed Banach generalized limit on (the Banach space) $l^\infty$, with all of the properties of such limits (cf., e.g., [3, Ex. 14E]), which we use below without further explicit mention. Define a semi-inner product (and seminorm) on $K_1$ by

$$\langle \{x_n\}, \{y_n\} \rangle_{K_1} = \text{LIM} \langle x_n, y_n \rangle,$$

$$\|\{x_n\}\|_{K_1}^2 = \langle \{x_n\}, \{x_n\} \rangle_{K_1},$$

let $K_0$ be the linear manifold in $K_1$ consisting of all $\{x_n\}_{n \in \mathbb{N}}$ in $K_1$ such that $\langle \{x_n\}, \{x_n\} \rangle_{K_1} = 0$, and let $K = K_H$ be the (Hilbert space) completion of the quotient space $K_1/K_0$. We will denote some elements of $K_1$ by $[\{x_n\}]$, meaning the equivalence class of $K_1$ containing the sequence $\{x_n\}_{n \in \mathbb{N}}$ from $K_1$. It is easy to see that $K$ is nonseparable. (Note that if $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in $H$, then $[\{e_n\}]$ is a unit vector in $K$, and if $\pi : \mathbb{N} \to \mathbb{N}$ is any injective map with no fixed points (or only finitely many), then $[\{e_{\pi(n)}\}]$ is a unit vector in $K$ orthogonal to $[\{e_n\}]$.) Furthermore, we define a mapping $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ by setting, for every $S \in \mathcal{L}(H)$,

$$\Phi(S)[\{x_n\}] = [\{Sx_n\}], \quad [\{x_n\}] \in K.$$

A little cogitation, together with knowledge of the basic properties of generalized Banach limits and compact operators (cf., e.g., [7, Ch. 4]), convinces one of the truth of the following.

**Lemma 3.** The map $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ defined by (1) is a unital $C^*$-algebra homomorphism with $\ker \Phi \supset K$.

Next, let us consider the collection $\mathcal{C}$ of all (bounded) sequences $A = \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(H)$ such that $A_n \to 0$ in the weak operator topology (WOT) and $A_n^* A_n \to A_0^2 \neq 0$ (WOT), where $A_0 \geq 0$ (which implies, in particular, that for $y \in \mathcal{H}$, $\|A_n y\| \to \|A_0 y\|$). We can now state the following easy lemma, which follows, for instance, from [6, Lemma 1].

**Lemma 4.** For every $A = \{A_n\}_{n \in \mathbb{N}} \in \mathcal{C}$, there exists a nonzero bounded operator $X = X_A : \mathcal{H} \to K$ defined by

$$Xy = [\{A_n y\}], \quad y \in \mathcal{H},$$

and $\ker X \supset \{y \in \mathcal{H} : \|A_n y\| \to 0\}$. Moreover, if $T \in \mathcal{L}(H)$ and $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{C} \cap \{T\}'$, then

$$XT = \Phi(T)X.$$
Our first theorem, which uses J. Thomson’s deep result [9] on the existence of analytic bounded point evaluations, and which we believe to be new, is the following.

**Theorem 5.** Suppose that \( T = N + K \in \mathcal{L}(\mathcal{H}) \) with \( N \) normal and \( K \in \mathcal{K} \), and that the WOT on the unit ball of \( \{ T \}' \) is strictly weaker than the SOT there (equivalently, there exists a sequence \( \{ A_n \} \subset \{ T \}' \) such that \( A_n \to 0 \) (WOT) but \( A_n \rightharpoonup 0 \) (SOT)). Then either \( T \) has a nontrivial invariant subspace or there exists a normal operator \( M \in \mathcal{L}(\mathcal{H}) \) such that \( T \prec M \).

**Proof.** By dropping down to a subsequence (without changing the notation) we may suppose that \( A_n^* A_n \to A_0^2 \neq 0 \) (WOT) where \( A_0 \geq 0 \). Thus \( \{ A_n \} \in \mathcal{C} \), and by Lemma 4 this sequence generates a bounded nonzero operator \( X : \mathcal{H} \to \mathcal{K} \) satisfying

\[
Xy = \sum \{ A_n y \}, \quad \|Xy\|^2 = \|A_0 y\|^2, \quad y \in \mathcal{H},
\]

and also

\[
XT = \Phi(T)X = \Phi(N)X. \tag{4}
\]

It now follows immediately from (4) that if \( \ker X \neq (0) \) (i.e., \( A_0 \) is not a quasiaffinity), then \( \ker X \) is a nontrivial invariant subspace for \( T \), so, regarding \( X \) as a linear transformation from \( \mathcal{H} \) to \( \mathcal{R} = (\text{ran } X)^- \), we may suppose that \( X \) is a quasiaffinity and that \( \mathcal{R} \) is an invariant subspace for the normal operator \( \Phi(N) \). Thus (4) readily implies that \( T \prec \Phi(N)|_\mathcal{R} \), and it now suffices to show that \( \Phi(N)|_\mathcal{R} \) is normal. Suppose, to the contrary, that \( \Phi(N)|_\mathcal{R} \) is a nonnormal, subnormal operator. We may also suppose that every \( y \neq 0 \) in \( \mathcal{H} \) is cyclic for \( T \), and consequently \( \Phi(N)|_\mathcal{R} \) has a cyclic vector too. But then the pure subnormal part \( S \) of \( \Phi(N)|_\mathcal{R} \) has a cyclic vector, and by the deep theorem of J. Thomson [9], \( \sigma_p(S^*) \) is nonvoid. Thus also \( \sigma_p((\Phi(N)|_\mathcal{R})^*) \neq \emptyset \), and taking adjoints in (4), we get \( \sigma_p(T^*) \neq \emptyset \), which leads immediately to the existence of a nontrivial invariant subspace for \( T \).

This allows us to recover the following theorem, which, of course, dates from 1980 and thus was originally proved independently of Theorem 5.

**Theorem 6 ([6]).** Let \( T = N + K \in \mathcal{L}(\mathcal{H}) \) with \( N \) normal and \( K \in \mathcal{K} \), and suppose that on the unit balls of \( \{ T \}' \) and \( \{ T^* \}' \) the WOT is strictly weaker than the SOT. Then \( T \) has a nontrivial invariant subspace.

**Proof.** According to Theorem 5, either \( T \) has a nontrivial invariant subspace or there exist normal operators \( M_1 \) and \( M_2 \) such that \( T \prec M_1 \) and \( T^* \prec M_2 \). But then, as was noted above, \( M_1 \) and \( M_2 \) are unitarily equivalent, and consequently \( T \) is quasisimilar to \( M_1 \), from which the result follows.
Remark 7. To our knowledge, Lomonosov [6] was the first to realize the utility of the hypothesis that on the unit balls of \( \{T\}' \) and \( \{T^*\}' \) the WOT and SOT differ.

The following result, a special case of which is known (cf. [8, Chapter II, Prop. 5.3]), is an application of Theorem 5.

Theorem 8. Suppose \( T = N + K \in \mathcal{L}(\mathcal{H}) \) with \( N \) normal and \( K \in \mathcal{K} \), and \( \|T\| = r(T) = 1 \), where \( r(T) \) is the spectral radius of \( T \). If \( T \) does not belong to the class \( C_{00} \) (defined in [8]), then either \( T \) has a nontrivial invariant subspace or there exists a normal operator \( M \) satisfying \( T \prec M \) or \( M \prec T \).

Proof. As is well-known, if \( T \) has a unitary part, then \( T \) has a nontrivial hyperinvariant subspace ([8]). Thus we may suppose that \( T \) is completely nonunitary. Furthermore, since \( T \notin C_{00} \), if neither \( T \) nor \( T^* \) belongs to the class \( C_{10} \) (defined in [8]) then again \( T \) has a nontrivial hyperinvariant subspace. Thus, by taking adjoints if necessary, we may suppose that \( T \in C_{10} \). Since \( T^n \to 0 \) (WOT) via the \( H^\infty \)-functional calculus for completely nonunitary contractions, and \( T^n \nrightarrow 0 \) (SOT) by definition of the class \( C_{10} \), Theorem 5 is applicable.

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