STUDIA MATHEMATICA 183 (1) (2007)

Volume thresholds for Gaussian and spherical random polytopes and their duals

by

PETER PIVOVAROV (Edmonton)

Abstract. Let g be a Gaussian random vector in $\mathbb{R}^n$. Let $N = N(n)$ be a positive integer and let K_N be the convex hull of N independent copies of g. Fix $R > 0$ and consider the ratio of volumes $V_N := \mathbb{E}\,\mathrm{vol}(K_N \cap RB_2^n)/\mathrm{vol}(RB_2^n)$. For a large range of $R = R(n)$, we establish a sharp threshold for N, above which $V_N \to 1$ as $n \to \infty$, and below which $V_N \to 0$ as $n \to \infty$. We also consider the case when K_N is generated by independent random vectors distributed uniformly on the Euclidean sphere. In this case, similar threshold results are proved for both $R \in (0,1)$ and $R = 1$. Lastly, we prove complementary results for polytopes generated by random facets.

1. Introduction. A remarkable result due to M. E. Dyer, Z. Füredi, and C. McDiarmid gives a threshold for the expected volume of random polytopes generated by vertices of the cube $[-1,1]^n$. Specifically, let μ be the uniform probability measure on $\{-1,1\}$ and let $Z = (z_1,\ldots,z_n)$ be a random vector whose coordinates are independent and identically distributed according to μ. Consider $N = N(n)$ independent random vectors $Z_1,\ldots,Z_N$, each with the same distribution as Z, and form their convex hull $C_N = \mathrm{conv}\{Z_1,\ldots,Z_N\}$. In [2], a threshold value for N is established at which C_N captures significant volume in the following sense: for each $\varepsilon > 0$, we have

$$(1)\qquad \frac{\mathbb{E}\,\mathrm{vol}_n(C_N)}{\mathrm{vol}_n([-1,1]^n)} \xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{if } N \le (\nu-\varepsilon)^n, \\ 1 & \text{if } N \ge (\nu+\varepsilon)^n, \end{cases}$$

where $\nu = 2/\sqrt{e}$. The corresponding result for the case when μ is uniform on $[-1,1]$ is also proved. Their method has since been significantly generalized; namely, D. Gatzouras and A. Giannopoulos, in [3], obtain analogous results for a large class of compactly supported probability measures μ on $\mathbb{R}$.

2000 *Mathematics Subject Classification*: Primary 52A22; Secondary 52B11, 52A38.
Key words and phrases: high-dimensional convex bodies, random polytopes.
This article is part of the author's Ph.D. thesis, currently being written under the supervision of N. Tomczak-Jaegermann at the University of Alberta. The author holds a Natural Sciences and Engineering Research Council of Canada Postgraduate Scholarship.

We consider similar problems for Gaussian random polytopes and polytopes generated by random points on the Euclidean sphere. In the Gaussian case, let $\gamma_1, \ldots, \gamma_n$ be independent $\mathrm{N}(0,1)$ random variables and let $g = (\gamma_1, \ldots, \gamma_n)$. Consider $N = N(n)$ independent copies of g, say $g_1, \ldots, g_N$, and set $K_N := \operatorname{conv}\{g_1, \ldots, g_N\}$. The Gaussian measure is not compactly supported and so the following question arises: what does it mean for K_N to capture significant volume? Let B_2^n denote the Euclidean ball and let R be an arbitrary positive constant, possibly dependent on the dimension n. We investigate the quantity

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)}.$$

For a large range of $R = R(n)$, we establish sharp thresholds, analogous to that of (1).

For the spherical setting, let u be a random vector distributed uniformly on the Euclidean sphere S^{n-1}. Consider $N = N(n)$ independent copies of u, say $u_1, \ldots, u_N$, and set $L_N := \operatorname{conv}\{u_1, \ldots, u_N\}$. This case presents a different model of randomness as the coordinates of u are not independent. We study the quantity

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N \cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)}$$

for the case when R is any fixed value in $(0,1)$ and the case of the entire ball, i.e., $R = 1$. Sharp thresholds for N are obtained in both cases.

We follow the same approach as that of Dyer, Füredi and McDiarmid. The tools developed in [2] have a simple realization in our setting; this simplicity nicely illustrates the geometry behind the method. The lack of independence of coordinates in the spherical case presents no difficulty as the argument depends more on geometric considerations than on probabilistic techniques such as the theory of large deviations, as in [2] and [3]. Also noteworthy is the threshold for N in the spherical case: it is super-exponential in the dimension n. The results in [2] are exponential in n and the authors of [3] considered only measures for which there is an exponential threshold in n.

Finally, as a natural complement to the above results, we prove corresponding theorems for Gaussian and spherical polytopes generated by random facets, i.e., for the polytopes

$$K'_N := \{x \in \mathbb{R}^n : \langle g_i, x\rangle \leq 1 \text{ for each } i = 1, \ldots, N\},$$
$$L'_N := \{x \in \mathbb{R}^n : \langle u_i, x\rangle \leq 1 \text{ for each } i = 1, \ldots, N\},$$

where the g_i's and u_i's are as above. In this case, the arguments do not invoke duality and use only elementary properties of the random vectors involved.

NOTATION. We shall denote the canonical Euclidean norm on $\mathbb{R}^n$ by $\|\cdot\|_2$, and B_2^n will denote the Euclidean ball. Lebesgue measure on $\mathbb{R}^n$ will be denoted by $\mathrm{vol}_n(\cdot)$; the unit sphere in $\mathbb{R}^n$ by S^{n-1}; and the surface area of S^{n-1} by $\mathrm{vol}_{n-1}(\partial B_2^n)$.

2. Volume thresholds for polytopes generated by random vertices

2.1. *Results in the Gaussian case.* Let $\gamma_1, \dots, \gamma_n$ be independent Gaussian $\mathrm{N}(0,1)$ random variables. Denote the standard unit vector basis in $\mathbb{R}^n$ by $e_1, \dots, e_n$. Consider the random vector $g = \sum_{i=1}^n \gamma_i e_i$; then g satisfies $\mathbb{E}\|g\|_2 \approx \sqrt{n}$. Let $N = N(n) > n$ be an integer and consider N independent random vectors $g_1, \dots, g_N$, each with the same distribution as g. Form their convex hull

$$K_N := \mathrm{conv}\{g_1, \dots, g_N\}.$$

We shall use the following standard notation:

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2}\, dx \qquad (a \in \mathbb{R}). \tag{2}$$

Note that for $a > -1$ we have the approximation (see [6])

$$\frac{2}{a + (a^2+4)^{1/2}} \le \sqrt{2\pi}\exp(a^2/2)(1 - \Phi(a)) \le \frac{4}{3a + (a^2+8)^{1/2}}. \tag{3}$$

Using the method of [2], we establish the following theorem for the expected volume of K_N lying inside RB_2^n, where $R > 0$.

THEOREM 2.1. *Let $R > 0$.*

(a) *If $0 < t < R$ then for all n we have*

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)} \le (t/R)^n + N(1 - \Phi(t)). \tag{4}$$

(b) *For all $n > 2e$ we have*

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)} \ge 1 - 2\exp(n \ln N - N(1 - \Phi(R))). \tag{5}$$

For suitable values of $R = R_n$, the theorem implies that, as $n \to \infty$, a threshold occurs around $N \approx (1 - \Phi(R_n))^{-1}$. As a precise illustration of this, we shall establish the following family of results for $R_n = cn^\kappa$, where $c > 0$ and $\kappa > 0$ are arbitrary absolute constants. Since $\mathbb{E}\|g\|_2 \approx \sqrt{n}$, the case $\kappa = 1/2$ is of particular interest.

COROLLARY 2.2. *Let $\kappa > 0$, $c > 0$ and let $0 < \varepsilon < c$. Then, as $n \to \infty$,*

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap cn^\kappa B_2^n)}{\mathrm{vol}_n(cn^\kappa B_2^n)} \to \begin{cases} 0 & \text{if } N \le (1 - \Phi((c-\varepsilon)n^\kappa))^{-1}, \\ 1 & \text{if } N \ge (1 - \Phi((c+\varepsilon)n^\kappa))^{-1}. \end{cases} \tag{6}$$

The proofs of the above statements are in Section 2.4. Compare Corollary 2.2 with Theorem 3.4 in Section 3.1.

2.2. *Results in the spherical case.* Let μ denote Haar measure on S^{n-1} and let u be a random vector distributed according to μ, i.e., $\mathbb{P}(u \in A) = \mu(A)$ for measurable sets $A \subset S^{n-1}$. Consider $N = N(n) > n$ independent random vectors $u_1, \ldots, u_N$, each with the same distribution as u, and set

$$L_N := \operatorname{conv}\{u_1, \ldots, u_N\}.$$

We shall prove the following theorems.

THEOREM 2.3. *Let* $0 < R < 1$ *and let* $0 < \varepsilon < 1$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}\operatorname{vol}_n(L_N \cap RB_2^n)}{\operatorname{vol}_n(RB_2^n)} \to \begin{cases} 0 & \text{if } N \le \exp((1-\varepsilon)n\ln(1/\sqrt{1-R^2})), \\ 1 & \text{if } N \ge \exp((1+\varepsilon)n\ln(1/\sqrt{1-R^2})). \end{cases} \tag{7}$$

For the case of the entire ball, i.e. $R = 1$, the threshold is super-exponential in n; this is in contrast with the results of [2] and [3], which are exponential in n.

THEOREM 2.4. *Let* $0 < \varepsilon < 1$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}\operatorname{vol}_n(L_N)}{\operatorname{vol}_n(B_2^n)} \to \begin{cases} 0 & \text{if } N \le \exp((1-\varepsilon)n\ln\sqrt{n}), \\ 1 & \text{if } N \ge \exp((1+\varepsilon)n\ln\sqrt{n}). \end{cases} \tag{8}$$

The proofs of the above theorems are in Section 2.5. The rate of convergence in each theorem is explained in the remark following its proof.

The complementary results for polytopes generated by random facets are Theorems 3.8 and 3.9 in Section 3.2. See also the comments preceding Theorem 3.9.

2.3. *Related results and further directions.* Theorems 2.3 and 2.4 are similar to a result of J. S. Müller, in [4], about approximation of the Euclidean ball by random polytopes (see [5] and the references cited therein for related results and a discussion of similar questions of approximation). In our notation, Müller's result is an asymptotic formula for the difference $\operatorname{vol}_n(B_2^n) - \mathbb{E}\operatorname{vol}_n(L_N)$. The asymptotic treatment in [4], however, is for the case when n is fixed and $N \to \infty$. A major extension of Müller's result was done by Schütt and Werner in [5]. Namely, let K be a convex body whose boundary satisfies certain regularity conditions. Let C_N be the convex hull of N points chosen randomly from the boundary of K. The authors of [5] derive an asymptotic formula for $\operatorname{vol}_n(K) - \mathbb{E}\operatorname{vol}_n(C_N)$, where, as in [4], n is fixed and $N \to \infty$.

Further study of volume thresholds, involving more general random models, has been suggested by A. Giannopoulos. One might consider the case when the polytope C_N is generated by independent random vectors distributed uniformly in a convex body K and examine the quantity

$$\frac{\mathbb{E}\,\mathrm{vol}_n(C_N \cap rK)}{\mathrm{vol}_n(rK)}$$

for both $r \in (0,1)$ and $r = 1$. Similar problems may also be considered in the framework of [5].

Another direction is the case of polytopes C_N generated by independent random vectors distributed according to an arbitrary measure. Since the analysis in [3] is carried out for measures with compact support $[-R,R]^n$, one might examine the quantity

$$\frac{\mathbb{E}\,\mathrm{vol}_n(C_N \cap [-R,R]^n)}{\mathrm{vol}_n([-R,R]^n)}$$

for various values of R.

2.4. *Proofs in the Gaussian case.* We shall use the notation defined in Section 2.1. Before proving Theorem 2.1, we will present some tools that were used in [2]. For $x \in \mathbb{R}^n$, set

$$q(x) := \inf\{\mathbb{P}(g \in H) : H \text{ is a halfspace containing } x\}.$$

Throughout this paper we assume that all halfspaces are closed.

Claim 2.5. *$\mathbb{P}(x \in K_N) \le Nq(x)$.*

Proof. Let H be a halfspace containing x. If none of $g_1,\dots,g_n$ belongs to H then K_N lies in $\mathbb{R}^n \setminus H$ and hence $x \notin K_N$. Consequently,

$$\{x \in K_N\} \subset \bigcup_{i=1}^{N} \{g_i \in H\}.$$

Since H was an arbitrary halfspace containing x, the result follows. ∎

In the Gaussian case we can actually calculate $q(x)$. For a closed set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, let $d(x,A) := \inf\{\|x-a\|_2 : a \in A\}$.

Lemma 2.6.

(a) *Let H be a halfspace with $d := d(0,H) > 0$. Then $\mathbb{P}(g \in H) = 1 - \Phi(d)$.*
(b) *For $x \in \mathbb{R}^n$, we have $q(x) = 1 - \Phi(\|x\|_2)$.*

Proof. (a) The density of g with respect to Lebesgue measure is given by

$$f(x) := (1/\sqrt{2\pi})^n \exp(-\|x\|_2^2/2).$$

By rotational invariance we may assume that $H := \{x \in \mathbb{R}^n : x_1 \ge d\}$.

Consequently,

$$\mathbb{P}(g \in H) = \int_H f(x)\,dx = \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x_n^2/2}\,dx_n\right)^{n-1}\left(\frac{1}{\sqrt{2\pi}}\int_d^\infty e^{-x_1^2/2}\,dx_1\right)$$
$$= 1 - \Phi(d).$$

(b) If $x = 0$ then $q(0) = 1/2 = 1 - \Phi(0)$. Suppose that $x \neq 0$. Let $H(x)$ be the halfspace bounded by the tangent hyperplane to $\|x\|_2 B_2^n$ at x and which does not contain 0. Then by part (a) we have

$$1 - \Phi(\|x\|_2) = \mathbb{P}(g \in H(x)) \geq q(x).$$

Conversely, let H be any halfspace containing x. Set $d = d(0, H)$. If $d = 0$ then $\mathbb{P}(g \in H) \geq 1/2 \geq 1 - \Phi(\|x\|_2)$. If $d > 0$ then

$$\mathbb{P}(g \in H) = 1 - \Phi(d) \geq 1 - \Phi(\|x\|_2)$$

since $d \leq \|x\|_2$. It follows that $q(x) \geq 1 - \Phi(\|x\|_2)$. ∎

An important consequence of Lemma 2.6 is the following simple observation:

$$RB_2^n = \{x \in \mathbb{R}^n : q(x) \geq 1 - \Phi(R)\} \tag{9}$$

for any $R > 0$. Equality (9) allows us to use an argument from [2] to establish the next lemma; for the reader's convenience, we have included the proof.

Lemma 2.7. *Let $R > 0$. The inclusion $RB_2^n \subset K_N$ holds with probability greater than $1 - 2\binom{N}{n}(\Phi(R))^{N-n}$.*

Proof. For any $J \subset \{1, \ldots, N\}$ with $|J| = n$, the set $\{g_j\}_{j\in J}$ is linearly (hence affinely) independent almost surely. In particular, the affine hull of $\{g_j\}_{j\in J}$ is a hyperplane almost surely. Let us now define the event E_J: one of the two halfspaces H determined by $\{g_j\}_{j\in J}$ contains K_N and $\mathbb{P}(g \notin H) \geq 1 - \Phi(R)$.

Suppose $x \in RB_2^n \setminus K_N$. Then there exists $J \subset \{1, \ldots, N\}$ with $|J| = n$ such that one of the two halfspaces H determined by $\{g_j\}_{j\in J}$ contains K_N but excludes x. But then x belongs to the complementary halfspace $\widetilde{H}$, so

$$\mathbb{P}(g \notin H) \geq q(x) \geq 1 - \Phi(R)$$

since $\|x\|_2 \leq R$. It follows that

$$\{RB_2^n \not\subset K_N\} \subset \bigcup_{\substack{J\subset\{1,\ldots,n\}\\ |J|=n}} E_J.$$

Thus if we set $D = \{1, \ldots, n\}$ we have

$$\mathbb{P}(RB_2^n \not\subset K_N) \leq \binom{N}{n}\mathbb{P}(E_D).$$

Now let us estimate $\mathbb{P}(E_D)$ by conditioning on $g_1, \dots, g_n$. Let H and $\widetilde{H}$ denote the two halfspaces generated by $g_1, \dots, g_n$. If $\mathbb{P}(g \notin H) \geq 1 - \Phi(R)$ then

$$\mathbb{P}(g_j \in H : j = n+1, \dots, N) \leq (\Phi(R))^{N-n}$$

and similarly for $\widetilde{H}$. It follows that

$$\mathbb{P}(E_D \,|\, g_1, \dots, g_n) \leq 2(\Phi(R))^{N-n}.$$

Now since

$$\mathbb{P}(E_D) = \mathbb{E}(1_{E_D}) = \mathbb{E}(\mathbb{E}(1_{E_D} \,|\, g_1, \dots, g_n)) = \mathbb{E}(\mathbb{P}(E_D \,|\, g_1, \dots, g_n)),$$

we obtain $\mathbb{P}(E_D) \leq 2(\Phi(R))^{N-n}$ and hence

$$\mathbb{P}(RB_2^n \not\subset K_N) \leq 2\binom{N}{n}(\Phi(R))^{N-n}. \ \blacksquare$$

LEMMA 2.8. *Let B be a bounded, measurable subset of $\mathbb{R}^n$. Then*

$$\mathrm{vol}_n(B)\mathbb{P}(B \subset K_N) \leq \mathbb{E}\,\mathrm{vol}_n(K_N \cap B) \leq N\,\mathrm{vol}_n(B) \sup_{x \in B}(1 - \Phi(\|x\|_2)).$$

Proof. Note that

$$\mathbb{E}\,\mathrm{vol}_n(K_N \cap B) = \mathbb{E}\int_B 1_{\{x \in K_N\}}\,dx = \int_B \mathbb{P}(x \in K_N)\,dx.$$

The upper bound follows from Claim 2.5 and Lemma 2.6, and the lower bound is trivial. $\blacksquare$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. (a) Set $B = RB_2^n \setminus tB_2^n$ and write

$$K_N \cap RB_2^n = (K_N \cap tB_2^n) \cup (K_N \cap B).$$

Clearly, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap tB_2^n)}{\mathrm{vol}_n(RB_2^n)} \leq (t/R)^n.$$

Referring to Lemma 2.8, we obtain

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap B)}{\mathrm{vol}_n(RB_2^n)} \leq N(1 - \Phi(t)),$$

which gives us (4).

(b) By Lemma 2.8, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K_N \cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)} \geq \mathbb{P}(RB_2^n \subset K_N).$$

Applying Lemma 2.7 gives us

$$\begin{aligned}
(10)\qquad \mathbb{P}(RB_2^n \not\subset K_N) &\leq 2\binom{N}{n}(\Phi(R))^{N-n} \\
&\leq 2(eN/n)^n \exp((N-n)\ln \Phi(R)) \\
&= 2\exp(n\ln(eN/n) + (N-n)\ln \Phi(R)).
\end{aligned}$$

Note that $1/2 < \Phi(R) < 1$ and hence for $n > 2e$ the latter expression is less than

$$(11)\qquad 2\exp(n\ln N + N\ln \Phi(R)) \leq 2\exp(n\ln N - N(1-\Phi(R))),$$

where we have used the estimate $\ln x \leq x - 1$. This proves (5). ∎

Proof of Corollary 2.2. Set $R_n := cn^\kappa$. Assume first that we have $N \leq (1-\Phi((c-\varepsilon)n^\kappa))^{-1}$. Set $t_n = (c-\varepsilon/2)n^\kappa$. Then

$$(12)\qquad \lim_{n\to\infty}(t_n/R_n)^n = 0$$

and, using (3), we see that

$$(13)\qquad N(1-\Phi(t_n)) \leq 2\exp((c-\varepsilon)^2 n^{2\kappa}/2 - (c-\varepsilon/2)^2 n^{2\kappa}/2) \to 0$$

as $n \to \infty$, giving us half of (6).

Assume now that $N \geq (1-\Phi((c+\varepsilon)n^\kappa))^{-1}$. For convenience of notation set $r_n := (c+\varepsilon)n^\kappa$. Without loss of generality we may assume that $N = \lceil (1-\Phi(r_n))^{-1}\rceil$, where $\lceil x\rceil$ denotes the smallest integer larger than x. Appealing to (3) yields

$$\begin{aligned}
n\ln N &\leq n\ln((\sqrt{2\pi}/2)(r_n + (r_n^2+4)^{1/2})\exp(r_n^2/2)) \\
&\leq n\ln(\sqrt{2\pi}(r_n+1)) + nr_n^2/2 \leq nr_n^2
\end{aligned}$$

provided that

$$(14)\qquad \ln(\sqrt{2\pi}\,(r_n+1)) \leq r_n^2/2.$$

Another application of (3) gives

$$\begin{aligned}
N(1-\Phi(R_n)) &\geq \frac{1}{2}\,\frac{3r_n + (r_n^2+8)^{1/2}}{R_n + (R_n^2+4)^{1/2}}\exp((r_n^2 - R_n^2)/2) \\
&\geq (1/2)\exp((r_n^2 - R_n^2)/2).
\end{aligned}$$

Thus if n satisfies (14) and $n > 2e$, we have

$$\begin{aligned}
(15)\qquad &\frac{\mathbb{E}[\mathrm{vol}_n(K_N \cap cn^\kappa B_2^n)]}{\mathrm{vol}_n(cn^\kappa B_2^n)} \\
&\geq 1 - 2\exp(nr_n^2 - (1/2)\exp((r_n^2 - R_n^2)/2)) \\
&= 1 - 2\exp((c+\varepsilon)^2 n^{2\kappa+1} - (1/2)\exp(((c+\varepsilon)^2 n^{2\kappa} - c^2 n^{2\kappa})/2)) \to 1
\end{aligned}$$

as $n \to \infty$, which yields the other half of (6). ∎

REMARK 2.9. The bounds on N in Corollary 2.2 may be improved in the following sense: let $(\varepsilon_n)_{n\geq 1} \subset (0, c)$ with $\varepsilon_n \to 0$ such that (12)–(15) are satisfied when ε is replaced by ε_n; then the corollary is true with ε replaced by ε_n. For instance, we could take $\varepsilon_n = \min(c/2, n^{-\gamma})$ for any fixed $\gamma \in (0, \min(1, 2\kappa))$.

2.5. *Proofs in the spherical case.* We will use the notation defined in Section 2.2. For $x \in B_2^n$, set

$$q(x) := \inf\{\mathbb{P}(u \in H) : H \text{ is a halfspace containing } x\}.$$

For $v \in S^{n-1}$ and $0 \leq R \leq 1$ set

$$C(R, v) := \{x \in S^{n-1} : \langle x, v\rangle \geq R\}. \tag{16}$$

Since we are interested in surface area, we will omit the reference to v and write $C(R) := C(R, v)$. Upper and lower estimates for $\mu(C(R))$ are included in the Appendix.

The proofs of the following statements are similar to the Gaussian case.

CLAIM 2.10. *$\mathbb{P}(x \in L_N) \leq Nq(x)$.*

LEMMA 2.11.

(a) *Let H be a halfspace with $d := d(0, H)$. If $d \in (0, 1]$ then $\mathbb{P}(u \in H) = \mu(C(d))$.*
(b) *For $x \in B_2^n$, we have $q(x) = \mu(C(\|x\|_2))$.*

LEMMA 2.12. *Let $R \in (0, 1)$. For each $n > 2e$, the inclusion $RB_2^n \subset L_N$ holds with probability greater than $1 - 2\exp(n \ln N - N\mu(C(R)))$.*

Proof. The proof is analogous to that of Lemma 2.7 and the estimates starting with (10) and ending with (11). ∎

LEMMA 2.13. *Let B be a measurable subset of B_2^n. Then*

$$\mathrm{vol}_n(B)\mathbb{P}(B \subset L_N) \leq \mathbb{E}\,\mathrm{vol}_n(L_N \cap B) \leq N\,\mathrm{vol}_n(B) \sup_{x\in B} \mu(C(\|x\|_2)).$$

Proof. Argue as in the proof of Lemma 2.8 and apply Claim 2.10 and Lemma 2.11. ∎

We now have all the tools for proving Theorem 2.3.

Proof of Theorem 2.3. It is more convenient to prove the theorem with the bounds on N being

$$N \leq \exp((1-\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})), \tag{17}$$

$$N \geq \exp((1+\varepsilon)(n-1)\ln(1/\sqrt{1-R^2})). \tag{18}$$

Assume first that (17) holds. Let $t := \sqrt{1-(1-R^2)^{1-\varepsilon/2}}$ so that $0 < t < R$. Set $B := RB_2^n \setminus tB_2^n$ and write

$$L_N \cap RB_2^n = (L_N \cap tB_2^n) \cup (L_N \cap B).$$

Since

$$\lim_{n\to\infty}\frac{\mathrm{vol}_n(tB_2^n)}{\mathrm{vol}_n(RB_2^n)}=\lim_{n\to\infty}(t/R)^n=0, \tag{19}$$

we need only show that $\lim_{n\to\infty}\mathbb{E}\,\mathrm{vol}_n(L_N\cap B)/\mathrm{vol}_n(RB_2^n)=0$.

By Lemma 4.2 of the Appendix and the fact that $(1-\varepsilon/2)\ln(\sqrt{1-R^2})=\ln(\sqrt{1-t^2})$, we have

$$\begin{aligned}\mu(C(t)) &\le 3\exp((n-1)\ln(\sqrt{1-t^2}))\\ &=3\exp(-(1-\varepsilon/2)(n-1)\ln(1/\sqrt{1-R^2}))\end{aligned} \tag{20}$$

for all $n\ge 3$.

Thus by Lemma 2.13 and (20), we have

$$\begin{aligned}\frac{\mathbb{E}\,\mathrm{vol}_n(L_N\cap B)}{\mathrm{vol}_n(RB_2^n)} &\le N\mu(C(t))\\ &\le 3\exp(-(\varepsilon/2)(n-1)\ln(1/\sqrt{1-R^2}))\to 0\end{aligned} \tag{21}$$

as $n\to\infty$, giving us half of (7).

Assume now that (18) holds. Then by Lemma 2.13 we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N\cap RB_2^n)}{\mathrm{vol}_n(RB_2^n)}\ge \mathbb{P}(RB_2^n\subset L_N).$$

For convenience of notation, set $r=\sqrt{1-R^2}$. Without loss of generality, we may assume that $N=\lceil \exp((1+\varepsilon)(n-1)\ln(1/r))\rceil$. Lemma 4.1 of the Appendix implies that

$$\begin{aligned}N\mu(C(R)) &\ge \exp((1+\varepsilon)(n-1)\ln(1/r)-(n-1)\ln(1/r)-\ln(6\sqrt{n}))\\ &\ge \exp((\varepsilon/2)(n-1)\ln(1/r))\end{aligned}$$

for all n satisfying

$$\ln(6\sqrt{n})\le(\varepsilon/2)(n-1)\ln(1/r). \tag{22}$$

Thus if n satisfies (22) and $n>2e$, Lemma 2.12 gives us

$$\mathbb{P}(RB_2^n\not\subset L_N)\le 2\exp(2n^2\ln(1/r)-\exp((\varepsilon/2)(n-1)\ln(1/r)))\to 0 \tag{23}$$

as $n\to\infty$, which completes the proof of (7). ∎

Remark 2.14. The rate of convergence in Theorem 2.3 can be obtained from lines (19), (21), and (23).

Remark 2.15. In Theorem 2.3, we may replace ε by ε_n where $(\varepsilon_n)_{n\ge1}\subset(0,1)$ with $\varepsilon_n\to 0$ provided that (ε_n) satisfies (19), (21), (22) and (23). One may verify that $\varepsilon_n=n^{-\gamma}$, for any fixed $\gamma\in(0,1)$, serves this purpose.

Proof of Theorem 2.4. It is more convenient to prove the theorem with the bounds on N being

$$N \leq \exp((1-\varepsilon)(n-1)\ln\sqrt{n}), \tag{24}$$
$$N \geq \exp((1+\varepsilon)(n-1)\ln\sqrt{n}). \tag{25}$$

We shall use the following elementary fact:

$$\lim_{n\to\infty}(1-n^{-\beta})^{n/2} = \begin{cases} 0 & \text{if } 0<\beta<1, \\ 1 & \text{if } \beta>1. \end{cases} \tag{26}$$

Assume first that (24) holds. Let $\beta := 1-\varepsilon/2$ and set $R_n := \sqrt{1-n^{-\beta}}$. Let $B := B_2^n \setminus R_n B_2^n$ and write

$$L_N = (L_N \cap R_n B_2^n) \cup (L_N \cap B).$$

By (26), we have

$$\lim_{n\to\infty} \frac{\mathrm{vol}_n(R_n B_2^n)}{\mathrm{vol}_n(B_2^n)} = \lim_{n\to\infty} R_n^n = 0, \tag{27}$$

and thus we need only show that $\lim_{n\to\infty} \mathbb{E}\,\mathrm{vol}_n(L_N\cap B)/\mathrm{vol}_n(B_2^n) = 0$.

By Lemma 4.2 of the Appendix and the fact that $\sqrt{1-R_n^2} = n^{-\beta/2}$, we obtain

$$\begin{aligned} \mu(C(R_n)) &\leq 3\exp((n-1)\ln(\sqrt{1-R_n^2})) \\ &= 3\exp(-(1-\varepsilon/2)(n-1)\ln\sqrt{n}) \end{aligned} \tag{28}$$

for all $n \geq 3$. Thus by Lemma 2.13 and (28), we have

$$\begin{aligned} \frac{\mathbb{E}\,\mathrm{vol}_n(L_N\cap B)}{\mathrm{vol}_n(B_2^n)} &\leq N\mu(C(R_n)) \\ &\leq 3\exp(-(\varepsilon/2)(n-1)\ln\sqrt{n}) \to 0 \end{aligned} \tag{29}$$

as $n\to\infty$, giving us half of (8).

Let us now assume that (25) holds. Let $\gamma = 1+\varepsilon/2$ and set $r_n := \sqrt{1-n^{-\gamma}}$. Applying Lemma 2.13, we get

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N)}{\mathrm{vol}_n(B_2^n)} \geq \frac{\mathbb{E}\,\mathrm{vol}_n(L_N\cap r_n B_2^n)}{\mathrm{vol}_n(B_2^n)} \geq r_n^n \cdot \mathbb{P}(r_n B_2^n \subset L_N).$$

Using (26), we have

$$\lim_{n\to\infty} r_n^n = 1, \tag{30}$$

and thus we need only prove that $\mathbb{P}(r_n B_2^n \subset L_N) \to 1$ as $n\to\infty$. Without loss of generality, we may assume that $N = \lceil \exp((1+\varepsilon)(n-1)\ln\sqrt{n})\rceil$. Using Lemma 4.1 of the Appendix and the fact that $\sqrt{1-r_n^2} = n^{-\gamma/2}$, we get

$$\begin{aligned} N\mu(C(r_n)) &\geq \exp((1+\varepsilon)(n-1)\ln\sqrt{n} + (n-1)\ln\sqrt{1-r_n^2} - \ln(6\sqrt{n})) \\ &= \exp((1+\varepsilon)(n-1)\ln\sqrt{n} - (1+\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n})) \\ &\geq \exp((\varepsilon/4)(n-1)\ln\sqrt{n}) \end{aligned}$$

for all n satisfying

$$\ln(6\sqrt{n}) \leq (\varepsilon/4)(n-1)\ln\sqrt{n}. \tag{31}$$

Thus if n satisfies (31) and $n > 2e$, Lemma 2.12 yields

$$\mathbb{P}(r_n B_2^n \not\subset L_N) \leq 2\exp(2n^2\ln\sqrt{n} - \exp((\varepsilon/4)(n-1)\ln\sqrt{n})) \to 0 \tag{32}$$

as $n \to \infty$, which completes the proof of (8). ∎

Remark 2.16. The rate of convergence in Theorem 2.4 can be obtained from (27), (29), and (32).

Remark 2.17. In Theorem 2.4, we may replace ε by ε_n where $(\varepsilon_n)_{n\geq 1} \subset (0,1)$ with $\varepsilon_n \to 0$ provided that (ε_n) satisfies (27) and (29)–(32). One can check that $\varepsilon_n = 1/\ln(\ln n)$ works.

3. Volume thresholds for polytopes generated by random facets

3.1. *Gaussian case.* In this section we consider Gaussian polytopes generated by random facets and prove a complement of Corollary 2.2. We shall use the notation defined in Section 2.1. Consider the polytope

$$K'_N := \{x \in \mathbb{R}^n : \langle g_i, x\rangle \leq 1 \text{ for each } i = 1,\dots,N\}.$$

Before stating the main result, we will prove two lemmas.

Lemma 3.1. *For each $x \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\mathbb{P}(x \in K'_N) = (\Phi(1/\|x\|_2))^N.$$

Proof. By independence and rotational invariance of the g_i's, we have

$$\begin{aligned} \mathbb{P}(x \in K'_N) &= \mathbb{P}\{\langle x, g_i\rangle \leq 1 \text{ for each } i = 1,\dots,N\} = (\mathbb{P}\{\langle x, g_1\rangle \leq 1\})^N \\ &= (\mathbb{P}\{\gamma_1 \leq 1/\|x\|_2\})^N \qquad (\gamma_1 \sim \mathrm{N}(0,1)) \\ &= (\Phi(1/\|x\|_2))^N. \ ∎ \end{aligned}$$

Lemma 3.2. *Let $0 < t < R$ and set $B = RB_2^n \setminus tB_2^n$. Then for each n we have*

$$\mathrm{vol}_n(B)(\Phi(1/R))^N \leq \mathbb{E}\,\mathrm{vol}_n(K'_N \cap B) \leq \mathrm{vol}_n(B)(\Phi(1/t))^N. \tag{33}$$

Proof. Argue as in the proof of Lemma 2.8 and apply Lemma 3.1. ∎

Remark 3.3. Let $a > 0$. The identity $(\Phi(a))^N = \exp(N\ln\Phi(a))$ and the estimate

$$x - 1 - (x-1)^2 \leq \ln x \leq x - 1, \qquad x \in [1/2, 1], \tag{34}$$

imply that

$$(\Phi(a))^N \geq \exp(-N(1-\Phi(a)) - N(1-\Phi(a))^2), \tag{35}$$

$$(\Phi(a))^N \leq \exp(-N(1-\Phi(a))). \tag{36}$$

These estimates will be used in conjunction with Lemma 3.2.

THEOREM 3.4. *Let $\kappa > 0$, $c > 0$ and $0 < \varepsilon < c$. Then, as $n \to \infty$,*

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap (cn^\kappa)^{-1}B_2^n)}{\mathrm{vol}_n((cn^\kappa)^{-1}B_2^n)} \to \begin{cases} 1 & \text{if } n < N \leq (1-\Phi((c-\varepsilon)n^\kappa))^{-1}, \\ 0 & \text{if } N \geq (1-\Phi((c+\varepsilon)n^\kappa))^{-1}. \end{cases} \tag{37}$$

Proof. Let $R_n := (cn^\kappa)^{-1}$, $t_n := ((c+\varepsilon/2)n^\kappa)^{-1}$ and $B := R_nB_2^n \setminus t_nB_2^n$, and write

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap R_nB_2^n)}{\mathrm{vol}_n(R_nB_2^n)} = \frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap t_nB_2^n)}{\mathrm{vol}_n(R_nB_2^n)} + \frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap B)}{\mathrm{vol}_n(R_nB_2^n)}.$$

In particular, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap B)}{\mathrm{vol}_n(R_nB_2^n)} \leq \frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap R_nB_2^n)}{\mathrm{vol}_n(R_nB_2^n)} \leq (t_n/R_n)^n + \frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap B)}{\mathrm{vol}_n(R_nB_2^n)}. \tag{38}$$

Assume first that $n < N \leq (1-\Phi((c-\varepsilon)n^\kappa))^{-1}$. Without loss of generality, we shall assume that $N = \lfloor (1-\Phi((c-\varepsilon)n^\kappa))^{-1} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

The bound from (3) gives us

$$N(1-\Phi(1/R_n)) \leq 2\exp((c-\varepsilon)^2n^{2\kappa}/2 - c^2n^{2\kappa}/2) \to 0 \tag{39}$$

as $n \to \infty$, and hence

$$N(1-\Phi(1/R_n))^2 \to 0 \quad \text{as } n \to \infty. \tag{40}$$

By (38) and Lemma 3.2, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K'_N \cap R_nB_2^n)}{\mathrm{vol}_n(R_nB_2^n)} \geq (1-(t_n/R_n)^n)\Phi(1/R_n)^N.$$

The latter term tends to 1 as $n \to \infty$ by (35), (39) and (40). This proves half of (37).

Assume now that $N \geq (1-\Phi((c+\varepsilon)n^\kappa))^{-1}$. For convenience of notation set $r_n := (c+\varepsilon)n^\kappa$. Without loss of generality we may assume that $N = \lceil (1-\Phi(r_n))^{-1} \rceil$.

Note that

$$(41)\quad N(1-\Phi(1/t_n)) \geq \frac{1}{2}\,\frac{3r_n+(r_n^2+8)^{1/2}}{(1/t_n)+((1/t_n)^2+4)^{1/2}}\exp((r_n^2-(1/t_n)^2)/2)$$
$$\geq (1/2)\exp((r_n^2-(1/t_n)^2)/2)$$
$$= (1/2)\exp(((c+\varepsilon)^2-(c+\varepsilon/2)^2)n^{2\kappa}/2)\to\infty$$

as $n\to\infty$. By (38) and Lemma 3.2, we get

$$\frac{\mathbb{E}\,\mathrm{vol}_n(K'_N\cap R_nB_2^n)}{\mathrm{vol}_n(R_nB_2^n)} \leq (t_n/R_n)^n+(1-(t_n/R_n)^n)\Phi(1/t_n)^N.$$

The latter term tends to 0 as $n\to\infty$ by (36) and (41). This yields the other half of (37). ∎

3.2. *Spherical case.* In this section we prove counterparts to Theorems 2.3 and 2.4 for polytopes generated by random facets. We shall use the notation defined in Section 2.2. Set

$$L'_N := \{x\in\mathbb{R}^n : \langle u_i,x\rangle\leq 1 \text{ for each } i=1,\ldots,N\}.$$

Before stating the main results we will prove several lemmas. Recall that the notation for a spherical cap $C(R)$ was introduced in (16).

Lemma 3.5. *For each $x\in\mathbb{R}^n\setminus B_2^n$ we have*

$$(42)\qquad \mathbb{P}(x\in L'_N) = (1-\mu(C(1/\|x\|_2)))^N.$$

Proof. Let $x\in\mathbb{R}^n\setminus B_2^n$. Observe first that

$$(43)\qquad \{\theta\in S^{n-1}:\langle\theta,x\rangle\leq 1\} = \{\theta\in S^{n-1}:\langle\theta,x/\|x\|_2\rangle\leq 1/\|x\|_2\}$$
$$= S^{n-1}\setminus \operatorname{int} C(1/\|x\|_2, x/\|x\|_2),$$

where $\operatorname{int} A$ denotes the interior of A.

By independence of the u_i's, we have

$$\mathbb{P}(x\in L'_N) = \mathbb{P}\{\langle u_i,x\rangle\leq 1 \text{ for each } i=1,\ldots,N\} = (\mathbb{P}\{\langle u_1,x\rangle\leq 1\})^N$$
$$= (1-\mu(C(1/\|x\|_2)))^N.\ ∎$$

Lemma 3.6. *Let $1\leq t<s$ and set $B:=sB_2^n\setminus tB_2^n$. Then for each n,*

$$(44)\qquad \mathrm{vol}_n(B)(1-\mu(C(1/s)))^N \leq \mathbb{E}\,\mathrm{vol}_n(L'_N\cap B)$$
$$\leq \mathrm{vol}_n(B)(1-\mu(C(1/t)))^N.$$

Proof. Argue as in the proof of Lemma 2.8 and apply Lemma 3.5. ∎

Remark 3.7. Let $0\leq a\leq 1$. The identity $(1-\mu(C(a)))^N=\exp(N\ln(1-\mu(C(a))))$ and (34) imply that

$$(45)\qquad (1-\mu(C(a)))^N \geq \exp(-N\mu(C(a))-N\mu(C(a))^2),$$

$$(46)\qquad (1-\mu(C(a)))^N \leq \exp(-N\mu(C(a))).$$

These estimates will be used in conjunction with Lemma 3.6.

THEOREM 3.8. *Let* $0 < R < 1$ *and let* $0 < \varepsilon < 1$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap R^{-1}B_2^n)}{\mathrm{vol}_n(R^{-1}B_2^n)} \to \begin{cases} 1 & \text{if } n < N \le \exp((1-\varepsilon)n\ln(1/\sqrt{1-R^2})), \\ 0 & \text{if } N \ge \exp((1+\varepsilon)n\ln(1/\sqrt{1-R^2})). \end{cases} \tag{47}$$

Proof. As in the proof of Theorem 2.3, it is more convenient to use the bounds on N in (17) and (18). Set $t := 1/\sqrt{1-(1-R^2)^{1+\varepsilon/2}}$ so that

$$\ln\sqrt{1-(1/t)^2} = (1+\varepsilon/2)\ln\sqrt{1-R^2}. \tag{48}$$

Let $s = 1/R$ so that $1 < t < s$. Let $B = sB_2^n \setminus tB_2^n$ and write

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap sB_2^n)}{\mathrm{vol}_n(sB_2^n)} = \frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap tB_2^n)}{\mathrm{vol}_n(sB_2^n)} + \frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap B)}{\mathrm{vol}_n(sB_2^n)}.$$

In particular, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap B)}{\mathrm{vol}_n(sB_2^n)} \le \frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap sB_2^n)}{\mathrm{vol}_n(sB_2^n)} \le (t/s)^n + \frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap B)}{\mathrm{vol}_n(sB_2^n)}. \tag{49}$$

Assume first that (17) holds. Without loss of generality, we shall assume that $N = \lfloor \exp((1-\varepsilon)(n-1)\ln(1/\sqrt{1-R^2}))\rfloor$.

Lemma 4.2 of the Appendix gives us

$$\mu(C(1/s)) = \mu(C(R)) \le 3\exp(-(n-1)\ln(1/\sqrt{1-R^2})).$$

Thus

$$N\mu(C(1/s)) \le 3\exp(-\varepsilon(n-1)\ln(1/\sqrt{1-R^2})) \to 0 \tag{50}$$

as $n \to \infty$, and also

$$N\mu(C(1/s))^2 \to 0 \quad \text{as } n \to \infty. \tag{51}$$

By (49) and Lemma 3.6, we obtain

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap sB_2^n)}{\mathrm{vol}_n(sB_2^n)} \ge (1-(t/s)^n)(1-\mu(C(1/s)))^N.$$

The latter term tends to 1 as $n \to \infty$ by (45), (50) and (51). This proves half of (47).

Assume now that (18) holds. Without loss of generality we may assume that $N = \lceil \exp((1+\varepsilon)(n-1)\ln(1/\sqrt{1-R^2}))\rceil$.

By Lemma 4.1 of the Appendix and our choice (48) of t, we have

$$\begin{aligned} \mu(C(1/t)) &\ge \exp((n-1)\ln\sqrt{1-(1/t)^2} - \ln(6\sqrt{n})) \\ &= \exp(-(1+\varepsilon/2)(n-1)\ln(1/\sqrt{1-R^2}) - \ln(6\sqrt{n})) \end{aligned}$$

and hence

$$N\mu(C(1/t)) \geq \exp((\varepsilon/2)(n-1)\ln(1/\sqrt{1-R^2}) - \ln(6\sqrt{n})) \geq \exp((\varepsilon/4)(n-1)\ln(1/\sqrt{1-R^2})) \tag{52}$$

provided that

$$\ln(6\sqrt{n}) \leq (\varepsilon/4)(n-1)\ln(1/\sqrt{1-R^2}). \tag{53}$$

By (49) and Lemma 3.6, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap sB_2^n)}{\mathrm{vol}_n(sB_2^n)} \leq (t/s)^n + (1-(t/s)^n)(1-\mu(C(1/t)))^N.$$

The latter term tends to 0 as $n \to \infty$ by (46) and (52). This yields the other half of (47). ∎

Next we turn our attention to similar threshold results for the entire body L'_N. Since $L'_N \supset B_2^n$, it is natural to consider the quantity

$$\frac{\mathrm{vol}_n(B_2^n)}{\mathbb{E}\,\mathrm{vol}_n(L'_N)}.$$

In fact, $\mathbb{E}\,\mathrm{vol}_n(L'_N) = \infty$. To see this, let $1 = t < s$, set $B = sB_2^n \setminus B_2^n$ and apply Lemma 3.6:

$$\mathbb{E}\,\mathrm{vol}_n(L'_N \cap B) \geq (1/2)^N \mathrm{vol}_n(sB_2^n \setminus B_2^n).$$

Thus if n is fixed, $\mathbb{E}\,\mathrm{vol}_n(L'_N \cap B) \to \infty$ as $s \to \infty$. Nevertheless, we can still prove the following threshold result.

Theorem 3.9. *Let* $0 < \varepsilon < 1$.

(a) *There exists a sequence* $(t_n)_{n=1}^\infty = (t_n(\varepsilon))_{n=1}^\infty$ *with* $t_n > 1$ *and* $\lim_{n\to\infty} t_n = 1$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}_n(B_2^n)}{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap t_nB_2^n)} = 0 \quad \text{if } n < N \leq \exp((1-\varepsilon)n\ln\sqrt{n}). \tag{54}$$

(b) *There exists a sequence* $(R_n)_{n=1}^\infty = (R_n(\varepsilon))_{n=1}^\infty$ *with* $R_n > 1$ *and* $\lim_{n\to\infty} R_n = \infty$ *such that*

$$\lim_{n\to\infty} \frac{\mathrm{vol}_n(B_2^n)}{\mathbb{E}\,\mathrm{vol}_n(L'_N \cap R_nB_2^n)} = 1 \quad \text{if } N \geq \exp((1+\varepsilon)n\ln\sqrt{n}). \tag{55}$$

Proof. As in the proof of Theorem 2.4, it is more convenient to use the bounds for N in (24) and (25).

Assume first that (24) holds. Without loss of generality we shall assume that $N = \lfloor \exp((1-\varepsilon)(n-1)\ln\sqrt{n}) \rfloor$.

Let $(t_n)_{n=2}^\infty \subset (1,\infty)$ be any sequence satisfying the following conditions:

(i) $\lim_{n\to\infty} t_n = 1$,
(ii) $\lim_{n\to\infty} t_n^n = \infty$,
(iii) $\lim_{n\to\infty} N\mu(C(1/t_n)) = 0$.

For instance, let $t_n := 1/\sqrt{1-n^{-(1-\varepsilon/2)}}$. Then $t_n > 1$, $t_n \to 1$ as $n \to \infty$ and t_n satisfies condition (ii) by (26) (in the proof of Theorem 2.4).

To see that condition (iii) is satisfied, apply Lemma 4.2 of the Appendix and the fact that $\ln\sqrt{1-(1/t_n)^2} = -(1-\varepsilon/2)\ln\sqrt{n}$ to get

$$\begin{aligned}\mu(C(1/t_n)) &\le 3\exp((n-1)\ln\sqrt{1-(1/t_n)^2})\\ &= 3\exp(-(1-\varepsilon/2)(n-1)\ln\sqrt{n})\end{aligned}$$

and thus

$$N\mu(C(1/t_n)) \le 3\exp(-(\varepsilon/2)(n-1)\ln\sqrt{n}) \to 0 \tag{56}$$

as $n \to \infty$, and hence also

$$N\mu(C(1/t_n))^2 \longrightarrow 0 \quad \text{as } n \to \infty. \tag{57}$$

Set $B = t_nB_2^n \setminus B_2^n$. Since

$$\begin{aligned}\frac{\mathbb{E}\,\mathrm{vol}_n(L_N' \cap t_nB_2^n)}{\mathrm{vol}_n(B_2^n)} &= \frac{\mathrm{vol}_n(B_2^n) + \mathbb{E}\,\mathrm{vol}_n(L_N' \cap B)}{\mathrm{vol}_n(B_2^n)}\\ &= 1 + \frac{\mathbb{E}\,\mathrm{vol}_n(L_N' \cap B)}{\mathrm{vol}_n(B_2^n)},\end{aligned} \tag{58}$$

it suffices to prove that

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N' \cap B)}{\mathrm{vol}_n(B_2^n)} \to \infty \quad \text{as } n \to \infty. \tag{59}$$

By Lemma 3.6, we have

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N' \cap B)}{\mathrm{vol}_n(B_2^n)} \ge (t_n^n - 1)(1-\mu(C(1/t_n)))^N.$$

The latter term tends to ∞ as $n \to \infty$ by our choice of (t_n) and by (45), (56), and (57).

Let us now assume that (25) holds. Without loss of generality, we shall assume that $N = \lceil \exp((1+\varepsilon)(n-1)\ln\sqrt{n}) \rceil$.

Before defining conditions for choosing the sequence R_n, we introduce an auxiliary sequence. Let $(r_n)_{n=2}^{\infty} \subset (1,\infty)$ be any sequence such that

(a) $\lim_{n\to\infty} r_n = 1$,
(b) $\lim_{n\to\infty} r_n^n = 1$,
(c) $\lim_{n\to\infty} N\mu(C(1/r_n)) = \infty$.

For instance, let $r_n := 1/\sqrt{1-n^{-(1+\varepsilon/2)}}$. Then $r_n > 1$, $r_n \to 1$ as $n \to \infty$ and, by (26), condition (b) also holds. By Lemma 4.1 of the Appendix and the fact that $\ln\sqrt{1-(1/r_n)^2} = -(1+\varepsilon/2)\ln\sqrt{n}$, condition (c) is satisfied since

$$\mu(C(1/r_n)) \geq \exp((n-1)\ln\sqrt{1-(1/r_n)^2} - \ln(6\sqrt{n}))$$
$$= \exp(-(1+\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n}))$$

and hence

$$N\mu(C(1/r_n)) \geq \exp((\varepsilon/2)(n-1)\ln\sqrt{n} - \ln(6\sqrt{n})) \geq \exp((\varepsilon/4)(n-1)\ln\sqrt{n}) \tag{60}$$

provided that

$$\ln(6\sqrt{n}) \leq (\varepsilon/4)(n-1)\ln\sqrt{n}. \tag{61}$$

Now let $(R_n)_{n=2}^{\infty} \subset (1,\infty)$ be any sequence such that

(A) $R_n > r_n$ for each n,
(B) $\lim_{n\to\infty} R_n = \infty$,
(C) $\lim_{n\to\infty} R_n^n(1-\mu(C(1/r_n)))^N = 0$.

For instance, choose (R_n) such that

$$n\ln R_n \leq (1/2)\exp((\varepsilon/4)(n-1)\ln\sqrt{n}).$$

In this case, if n satisfies (61), then (46) and (60) imply that

$$R_n^n(1-\mu(C(1/r_n)))^N \leq \exp(n\ln R_n - \exp((\varepsilon/4)(n-1)\ln\sqrt{n})) \to 0 \tag{62}$$

as $n\to\infty$. Set $B = R_nB_2^n \setminus B_2^n$. Since

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N'\cap R_nB_2^n)}{\mathrm{vol}_n(B_2^n)} = \frac{\mathrm{vol}_n(B_2^n) + \mathbb{E}\,\mathrm{vol}_n(L_N'\cap B)}{\mathrm{vol}_n(B_2^n)} = 1 + \frac{\mathbb{E}\,\mathrm{vol}_n(L_N'\cap B)}{\mathrm{vol}_n(B_2^n)}, \tag{63}$$

it suffices to prove that

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N'\cap B)}{\mathrm{vol}_n(B_2^n)} \to 0 \quad \text{as } n\to\infty. \tag{64}$$

Writing $B = (R_nB_2^n \setminus r_nB_2^n) \cup (r_nB_2^n \setminus B_2^n)$ and applying Lemma 3.6 twice gives

$$\frac{\mathbb{E}\,\mathrm{vol}_n(L_N'\cap B)}{\mathrm{vol}_n(B_2^n)} \leq (r_n^n - 1) + R_n^n(1-\mu(C(1/r_n)))^N. \tag{65}$$

The right-hand side of the latter inequality tends to 0 by our choice of (r_n) and (R_n). ∎

Concluding remark. The author recently obtained analogous results with volume replaced by general log-concave probability measures. The details will be given in a forthcoming paper.

4. Appendix: Area of caps on the sphere. Estimates for the area of spherical caps are well-known. Such estimates, however, are not commonly stated in the form that best serves our purpose. Therefore we have included this Appendix for the reader's convenience.

For $v \in S^{n-1}$ and $0 < R < 1$, consider the cap

$$C(R, v) := \{x \in S^{n-1} : \langle x, v\rangle \geq R\}.$$

Since we are interested in surface area, we will omit the reference to v and write $C(R) := C(R, v)$. Let α be the angle of the cap, i.e., $\cos\alpha = R$. Fix $0 < t < \alpha$. Let H be a hyperplane at distance $\cos t$ from the origin. Then $B_2^n \cap H$ is an $(n-1)$-dimensional Euclidean ball of radius $\sin t$. Thus if we let μ denote Haar measure on S^{n-1} then

$$\mu(C(R)) = \frac{\int_0^\alpha \mathrm{vol}_{n-2}(\partial(\sin t B_2^{n-1}))\,dt}{\int_0^\pi \mathrm{vol}_{n-2}(\partial(\sin t B_2^{n-1}))\,dt} = \frac{\int_0^\alpha \sin^{n-2} t\,dt}{\int_0^\pi \sin^{n-2} t\,dt}.$$

Let $I_n := \int_0^{\pi/2} \sin^n t\,dt$. Integrating by parts gives $I_n = ((n-1)/n)I_{n-2}$. The latter recurrence and Stirling's formula may be used to verify that $\sqrt{n}\, I_n \to \sqrt{\pi/2}$; in fact, for $n \geq 3$ we have

$$\frac{1}{2}\sqrt{\frac{2\pi}{n}} \leq \int_0^\pi \sin^{n-2} t\,dt \leq 2\sqrt{\frac{2\pi}{n}}. \tag{66}$$

LEMMA 4.1. *Let $R \in (0,1)$. Then for each $n \geq 3$, we have*

$$\mu(C(R)) \geq \frac{(1-R^2)^{(n-1)/2}}{6\sqrt{n}}. \tag{67}$$

Proof. Observe that

$$\int_0^\alpha \sin^{n-2} t\,dt \geq \int_0^\alpha \sin^{n-2} t \cos t\,dt = \frac{\sin^{n-1}\alpha}{n-1}.$$

Applying (66) and noting that $\sin\alpha = \sqrt{1-R^2}$ yields the result. ∎

LEMMA 4.2. *Let $R \in (0,1)$. Then for each $n \geq 3$, we have*

$$\mu(C(R)) \leq 3(1-R^2)^{(n-1)/2}. \tag{68}$$

Proof. Assume first that $1/\sqrt{2} < R < 1$. Using the inequality

$$1 - \cos t \leq 2\sin^2 t \cos t \qquad (t \in [0, \pi/4]), \tag{69}$$

and recalling that $R = \cos\alpha$, we have

$$\begin{aligned}\int_0^\alpha \sin^{n-2} t\,dt &= \int_0^\alpha \sin^{n-2} t\cos t\,dt + \int_0^\alpha \sin^{n-2} t(1-\cos t)\,dt\\ &\le \int_0^\alpha \sin^{n-2} t\cos t\,dt + 2\int_0^\alpha \sin^{n} t\cos t\,dt\\ &= \frac{\sin^{n-1}\alpha}{n-1} + \frac{2\sin^{n+1}\alpha}{n+1} \le \frac{3\sin^{n-1}\alpha}{n-1}.\end{aligned}$$

Applying again (66) and noting that $\sin\alpha = \sqrt{1-R^2}$ gives the result.

Finally, for $0 < R \le 1/\sqrt{2}$, one may argue, for example, as in the proof of [1, Lemma 2.2]. ∎

Acknowledgements. Research for this article started while I was visiting the Institut Henri Poincaré in Paris, France, for the Phenomena in High Dimensions trimester. I am grateful to the organizers of the trimester for the opportunity to take part as well as for financial support. I also thank the institute staff for their hospitality. The topic of the present article stemmed from discussions between N. Tomczak-Jaegermann and A. Giannopoulos. I am indebted to the former for guidance through each stage of the project, from its inception through to the final draft. Lastly, I benefitted greatly from discussions with A. Giannopoulos, H. Koenig, and P. Mankiewicz.

References

[1] K. Ball, *An Elementary Introduction to Modern Convex Geometry*, Flavors of Geometry, Math. Sci. Res. Inst. Publ. 31, Cambridge Univ. Press, Cambridge, 1997.

[2] M. E. Dyer, Z. Füredi, and C. McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms 3 (1992), 91–106.

[3] D. Gatzouras and A. Giannopoulos, *Threshold for the volume spanned by random points with independent coordinates*, preprint, July 2006.

[4] J. S. Müller, *Approximation of the ball by random polytopes*, J. Approx. Theory 63 (1990), 198–209.

[5] C. Schütt and E. Werner, *Polytopes with vertices chosen randomly from the boundary of a convex body*, in: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1807, Springer, Berlin, 2003, 241–422.

[6] S. J. Szarek and E. Werner, *A nonsymmetric correlation inequality for Gaussian measure*, J. Multivariate Anal. 68 (1999), 193–211.

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta, Canada T6G 2G1
E-mail: ppivovarov@math.ualberta.ca

Received January 16, 2007
Revised version July 10, 2007 (6086)