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Homomorphisms of commutative Banach algebras and extensions to multiplier algebras with applications to Fourier algebras

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Abstract. Let A and B be semisimple commutative Banach algebras with bounded approximate identities. We investigate the problem of extending a homomorphism $\varphi : A \to B$ to a homomorphism of the multiplier algebras $M(A)$ and $M(B)$ of A and B, respectively. Various sufficient conditions in terms of B (or B and φ) are given that allow the construction of such extensions. We exhibit a number of classes of Banach algebras to which these criteria apply. In addition, we prove a polar decomposition for homomorphisms from A into A with closed range. Our results are applied to Fourier algebras of locally compact groups.

Introduction. Let A and B be semisimple commutative Banach algebras and suppose that A has a bounded approximate identity $(e_\alpha)_\alpha$. We study homomorphisms φ from A to B from various aspects. Let I_φ be the largest ideal of B for which $(\varphi(e_\alpha))_\alpha$ serves as an approximate identity, and let Z_φ denote the zero set of I_φ in the Gelfand spectrum $\Delta(B)$ of B. In Section 1 we find criteria for Z_φ to be open in $\Delta(B)$ and I_φ to be complemented by a certain ideal J_φ (Theorems 1.4 and 1.5). The results are applied to the related question of when a homomorphism $\varphi : A \to B$ extends to a homomorphism, $\phi : M(A) \to M(B)$, between the multiplier algebras $M(A)$ and $M(B)$. This extension problem is the main objective of the paper.

When I_φ is complemented as above, we give in Theorem 2.1 (which is a basic result of the paper) an explicit construction of an extension $\phi : M(A) \to M(B)$. Moreover, if in addition B is a BSE-algebra (named after Bochner–Schoenberg–Eberlein), then all homomorphisms from $M(A)$

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to $M(B)$ extending φ can be described (Theorem 2.7). BSE-algebras have been investigated by several authors (see, for instance, [27] and [28]). In Section 2 we also add two more examples of BSE-algebras, namely L^1-algebras of compact commutative hypergroups and ideals $k(E)$ of Tauberian BSE-algebras A, where E is a set of synthesis in $\Delta(A)$.

Section 3, which is the main part of the paper, is devoted to establishing conditions on A and B which guarantee that every homomorphism from A into B extends to a homomorphism from $M(A)$ to $M(B)$. We exhibit several situations for this to happen. The first, and most important one, is when B is the dual of some Banach space and multiplication of B is separately continuous in the w^*-topology (Theorem 3.1). Theorem 3.1 applies in several situations, such as homomorphisms into Fourier and Fourier–Stieltjes algebras of locally compact groups and L^1-algebras of commutative hypergroups. The second case is when B is weakly sequentially complete and A has a sequential bounded approximate identity and, in addition, the equality $A \cdot A^* = A^*$ is satisfied (Theorem 3.8). This latter equality holds, for instance, when A is Arens regular. Other sufficient conditions include: A is Arens regular and regular and B is weakly sequentially complete (Theorem 3.11(i)); B is Arens regular and weakly sequentially complete (Theorem 3.11(ii)). The proofs of all these results are not constructive. However, in Theorem 3.9, we exploit a situation which is slightly more special than the one of Theorem 3.1, but has the advantage of admitting a constructive proof of the existence of an extending homomorphism. In view of the diversity of these sufficient conditions for extendability of a homomorphism it appears very unlikely that necessary and sufficient conditions can be found.

Finally, in Section 4 we establish a polar decomposition of homomorphisms $\varphi : A \to A$ with closed range (Theorem 4.2), similar to the polar decomposition of bounded linear operators in Hilbert spaces.

Of course, homomorphisms between Banach algebras have been studied for a long time and by numerous authors from various different aspects (see [5], [7], [11], [15] and [16], to mention just a few, and the extensive list of references in [6]).

Preliminaries. Let A be a commutative Banach algebra and $\Delta(A)$ the set of all homomorphisms from A onto $\mathbb{C}$. Then $\Delta(A) \subseteq A^*$ and the Gelfand topology on $\Delta(A)$ is the restriction to $\Delta(A)$ of the w^*-topology on A^*. The algebra A is called *regular* if, given any closed subset E of $\Delta(A)$ and $\gamma_0 \in \Delta(A) \setminus E$, there exists $a \in A$ such that $\widehat{a}(\gamma) = \gamma(a) = 0$ for all $\gamma \in E$ and $\widehat{a}(\gamma_0) \neq 0$. For $\gamma \in \Delta(A)$, let $\ker\gamma$ denote the kernel of γ, and recall that $\gamma \mapsto \ker\gamma$ sets up a bijection between $\Delta(A)$ and the set of all maximal modular ideals of A. To $E \subseteq \Delta(A)$ and $M \subseteq A$, respectively, associate $k(E) = \bigcap_{\gamma\in E} \ker\gamma$, the *kernel* of E, and $h(M) = \{\gamma \in \Delta(A) : M \subseteq \ker\gamma\}$, the *hull* of M. There is a unique topology on $\Delta(A)$ such that $\overline{E} = h(k(E))$ for every subset E of $\Delta(A)$. This so-called *hull-kernel* (*hk*) *topology* is weaker than the Gelfand topology and the two topologies agree if and only if A is regular.

On the second dual A^{**} of A there exist two natural multiplications extending that of A, known as the first and second Arens products. In this paper, A^{**} will always be equipped with the *first Arens product*, which is defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f \cdot a$ and $m \cdot f$ of A^* and mn of A^{**} are defined by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \quad \langle m \cdot f, b\rangle = \langle m, f \cdot b\rangle, \quad \langle mn, f\rangle = \langle m, n \cdot f\rangle,$$

respectively. With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**}. In general, A^{**} is not commutative, but for $a \in A$ and $m \in A^{**}$, $ma = am$. Moreover, this multiplication is not separately continuous with respect to the w^*-topology on A^{**}. But, for fixed $n \in A^{**}$, the mapping $m \mapsto mn$ is w^*-continuous. Also, for all $m, n \in A^{**}$ and $\gamma \in \Delta(A)$, we have $\langle mn, \gamma\rangle = \langle m, \gamma\rangle\langle n, \gamma\rangle$.

By a *bounded approximate identity* of A we mean a bounded net $(e_\alpha)_\alpha$ in A such that $\|ae_\alpha - a\| \to 0$ for every $a \in A$. We shall frequently use the fact that if $(e_\alpha)_\alpha$ is such a bounded approximate identity in A, then each w^*-cluster point u of this net in A^{**} is a right unit for A^{**}, that is, $mu = m$ for all $m \in A^{**}$. Regarding all these facts, we refer the reader to [4, Section 28] and [6, Chapter 2].

A bounded linear operator $T : A \to A$ is said to be a *multiplier* of A if $T(ab) = aT(b)$ for all $a, b \in A$. The space $M(A)$ of multipliers is a commutative semisimple closed unital subalgebra of the algebra of all bounded linear operators on A. Each $a \in A$ defines an element $L_a \in M(A)$ by $L_a(b) = ab$, $b \in A$. Since A is semisimple and has a bounded approximate identity, the map $a \mapsto L_a$ is a topological isomorphism of A onto the ideal $\{L_a : a \in A\}$ of $M(A)$ (see [17]). Moreover, each $\gamma \in \Delta(A)$ extends uniquely to an element of $\Delta(M(A))$. In this manner, A will always be considered as a closed ideal of $M(A)$, and $\Delta(A)$ as a subset of $\Delta(M(A))$.

Several applications of our results and examples will concern the Fourier and the Fourier–Stieltjes algebras, $A(G)$ and $B(G)$, of a locally compact group G. These algebras have been introduced and extensively studied by Eymard in his fundamental paper [9]. Recall that $B(G)$ is the set of all coefficient functions $x \mapsto u_{\pi,\xi,\eta}(x) = \langle \pi(x)\xi, \eta\rangle$, where π is a unitary representation of G in a Hilbert space and ξ and η are elements of that space. With pointwise operations and the norm $\|u\| = \inf \|\xi\| \cdot \|\eta\|$, the infimum being taken over all representations $u = u_{\pi,\xi,\eta}$ of u, $B(G)$ is a commutative Banach algebra. The Fourier algebra $A(G)$ is the closure of the subalgebra of compactly supported functions in $B(G)$. When G is abelian, $B(G)$ is isometrically isomorphic to the measure algebra $M(\widehat{G})$ of the dual group $\widehat{G}$ of G and this identification maps $A(G)$ onto $L^1(\widehat{G})$. The space $A(G)$ is a closed ideal of $B(G)$, and $\Delta(A(G))$ can be canonically identified with G. In fact, the mapping $x \mapsto \gamma_x$, where $\gamma_x(u) = u(x)$ for $u \in A(G)$, is a homeomorphism from G onto $\Delta(A(G))$. The algebra $A(G)$ has a bounded approximate identity if and only if the group G is amenable [19]. Moreover, in this case, the mapping $w \mapsto T_w$, where $T_w(u) = uw$ for $u \in A(G)$, is an isometric isomorphism between $B(G)$ and the multiplier algebra $M(A(G))$ of $A(G)$ (see [8]). The monographs [13] and [22] are very good accounts of the theory of amenable groups.

1. Ideals and zero sets associated to a homomorphism. Let A and B be two semisimple commutative Banach algebras, and throughout the entire paper suppose that A has a bounded approximate identity, $(e_\alpha)_\alpha$ say. Let $\varphi : A \to B$ be a homomorphism. Note that φ is continuous since A and B are semisimple. Let $\varphi^* : B^* \to A^*$ and $\varphi^{**} : A^{**} \to B^{**}$ denote the adjoints of φ and of φ^*, respectively. We associate to φ two ideals defined by

$$I_\varphi = \{b \in B : b\varphi(e_\alpha) \to b\} \quad \text{and} \quad J_\varphi = \{b \in B : b\varphi(e_\alpha) \to 0\},$$

and two subsets of $\Delta(B)$ defined by

$$Z_\varphi = \{\gamma \in \Delta(B) : \varphi^*(\gamma) = 0\} \quad \text{and} \quad E_\varphi = \{\gamma \in \Delta(B) : \varphi^*(\gamma) \neq 0\}.$$

Obviously, I_φ is the largest closed ideal of B for which $(\varphi(e_\alpha))_\alpha$ serves as an approximate identity.

We study the ideals I_φ and J_φ, in particular the question of when $B = I_\varphi \oplus J_\varphi$, and (topological) properties of the sets Z_φ and E_φ. These results will turn out to be of relevance when addressing the extension problem for homomorphisms.

LEMMA 1.1.

(i) *The definition of the ideal I_φ does not depend on the choice of the bounded approximate identity of A.*

(ii) *The set Z_φ equals the hull of I_φ and Z_φ is hull-kernel closed in $\Delta(B)$.*

Proof. (i) Let $(d_\beta)_\beta$ be another bounded approximate identity for A, and let $I = \{b \in B : b\varphi(d_\beta) \to b\}$. If $b \in I$, then $b\varphi(d_\beta) \to b$ and $b\varphi(d_\beta) \in I_\varphi$ since $\varphi(A) \subseteq I_\varphi$. Since I_φ is closed, $I \subseteq I_\varphi$. Interchanging the roles of $(e_\alpha)_\alpha$ and $(d_\beta)_\beta$ shows the reverse inclusion.

(ii) Let $\gamma \in Z_\varphi$ and $b \in I_\varphi$. Then

$$\gamma(b) = \lim_\alpha \gamma(b\varphi(e_\alpha)) = \gamma(b) \lim_\alpha \gamma(\varphi(e_\alpha)) = \gamma(b) \lim_\alpha \varphi^*(\gamma)(e_\alpha) = 0.$$

Hence $Z_\varphi \subseteq h(I_\varphi)$. Conversely, if $\gamma \in h(I_\varphi)$, then $\gamma(\varphi(A)) \subseteq \gamma(I_\varphi) = \{0\}$, whence $\varphi^*(\gamma) = 0$ and hence $\gamma \in Z_\varphi$. So $Z_\varphi = h(I_\varphi)$, which is hk-closed. ■

LEMMA 1.2.

(i) *The sum* $I_\varphi + J_\varphi$ *is closed in* B *and* $J_\varphi = k(E_\varphi)$. *In particular, the definition of the ideal* J_φ *does not depend on the choice of the bounded approximate identity in* A.

(ii) *The set* $\varphi^*(E_\varphi)$ *is closed in* $\Delta(A)$ *and* $\ker\varphi = k(\varphi^*(E_\varphi))$.

Proof. (i) It is clear from the definition of I_φ and J_φ that $I_\varphi \cap J_\varphi = \{0\}$. Since I_φ has a bounded approximate identity, it follows from a theorem of Rudin [23, Theorem 4.2] that the sum $I_\varphi + J_\varphi$ is closed in B (see also [16, Lemma 3.2]).

To prove that $k(E_\varphi) \subseteq J_\varphi$, let b be an arbitrary element of $k(E_\varphi)$. Then $\gamma(b\varphi(e_\alpha)) = 0$ for every α and every $\gamma \in \Delta(B)$ since $\gamma(b) = 0$ for $\gamma \in E_\varphi$, whereas $\gamma(\varphi(e_\alpha)) = 0$ for $\gamma \in Z_\varphi$. Thus, since B is semisimple, $b\varphi(e_\alpha) = 0$ for all α, whence $b \in J_\varphi$. For the reverse inclusion it suffices to show that $E_\varphi \subseteq h(J_\varphi)$, since then $J_\varphi \subseteq k(h(J_\varphi)) \subseteq k(E_\varphi)$.

To this end, consider an element γ of E_φ and let $b \in J_\varphi$. Since $\varphi^*(\gamma) \neq 0$, $\gamma(\varphi(a)) \neq 0$ for some $a \in A$ and

$$\gamma(\varphi(a)) = \lim_\alpha \gamma(\varphi(ae_\alpha)) = \gamma(\varphi(a)) \lim_\alpha \gamma(\varphi(e_\alpha)).$$

This implies that $\gamma(\varphi(e_\alpha)) \to 1$. On the other hand, $b\varphi(e_\alpha) \to 0$ and hence $\gamma(b)\gamma(\varphi(e_\alpha)) \to 0$. It follows that $\gamma(b) = 0$. Thus $E_\varphi \subseteq h(J_\varphi)$.

(ii) Let $(\gamma_\alpha)_\alpha$ be a net in E_φ such that $(\varphi^*(\gamma_\alpha))_\alpha$ converges pointwise on A to some $\delta \in \Delta(A)$. Let γ be a w^*-cluster point of $(\gamma_\alpha)_\alpha$ in B^*. Then either $\gamma = 0$ or $\gamma \in \Delta(B)$. Since φ^* is w^*-w^*-continuous, it follows that $\delta = \varphi^*(\gamma)$. Thus $\gamma \notin Z_\varphi \cup \{0\}$, and hence $\gamma \in E_\varphi$ and $\delta \in \varphi^*(E_\varphi)$.

Now, since B is semisimple, an element $a \in A$ belongs to $\ker\varphi$ if and only if $\langle \varphi(a), \gamma\rangle = \langle a, \varphi^*(\gamma)\rangle = 0$ for all $\gamma \in E_\varphi$. It follows that $\ker\varphi = k(\varphi^*(E_\varphi))$. ∎

LEMMA 1.3.

(i) Z_φ *is open in the hull-kernel topology* (*equivalently*, *open in the Gelfand topology*) *of* $\Delta(B)$ *if and only if* $h(I_\varphi + J_\varphi) = \emptyset$.

(ii) *If* $A/\ker\varphi$ *is unital*, *then* Z_φ *is open in* $\Delta(B)$.

Proof. (i) Since $\Delta(B)$ is the disjoint union of Z_φ and E_φ, Z_φ is hk-open in $\Delta(B)$ if and only if $Z_\varphi \cap \overline{E_\varphi}^{\mathrm{hk}} = \emptyset$. Since $Z_\varphi = h(I_\varphi)$ (Lemma 1.1) and $J_\varphi = k(E_\varphi)$ (Lemma 1.2), this in turn is equivalent to

$$\emptyset = h(I_\varphi) \cap h(k(E_\varphi)) = h(I_\varphi + k(E_\varphi)) = h(I_\varphi + J_\varphi).$$

Moreover, if Z_φ is open in the Gelfand topology, then $Z_\varphi = \Delta(k(E_\varphi))$ and hence $E_\varphi = h(k(E_\varphi))$. Thus Z_φ is hk-open.

(ii) Let $e \in A$ be such that $e + \ker\varphi$ is the identity of $A/\ker\varphi$. Since $\varphi(e)$ is an idempotent in B, the decomposition $b = b\varphi(e) + [b - b\varphi(e)]$, $b \in B$, shows that $B = I_\varphi \oplus J_\varphi$. The statement now follows from (i). ∎

In what follows we let $E(A)$ denote the set of all w^*-cluster points of bounded approximate identities of A.

THEOREM 1.4. *Let $\varphi : A \to B$ be a homomorphism. Then the following conditions are equivalent*:

(i) *$B\varphi^{**}(e) \subseteq B$ for some $e \in E(A)$.*
(ii) *$B = I_\varphi \oplus J_\varphi$.*
(iii) *$B\varphi^{**}(f) \subseteq B$ for every $f \in E(A)$.*

Proof. (i)⇒(ii). The element $\varphi^{**}(e)$ is an idempotent in B^{**}, and therefore the map $b \mapsto b\varphi^{**}(e)$ is a projection in B by hypothesis. Then

$$I_\varphi = \{b \in B : b\varphi(e_\alpha) \to b\} = \{b \in B : b\varphi^{**}(e) = b\},$$
$$J_\varphi = \{b \in B : b\varphi(e_\alpha) \to 0\} = \{b \in B : b\varphi^{**}(e) = 0\}.$$

Thus $B\varphi^{**}(e) \subseteq I_\varphi$ and $B(1 - \varphi^{**}(e)) \subseteq J_\varphi$, and hence $B = I_\varphi \oplus J_\varphi$.

(ii)⇒(iii). Let $f \in E(A)$ and let $(f_\beta)_\beta$ be a bounded approximate identity for A such that $f = w^*\text{-}\lim_\beta f_\beta$. Note that, by Lemmas 1.1 and 1.2, the definition of the ideals I_φ and J_φ does not depend on the choice of the bounded approximate identity. Thus, for $b \in I_\varphi$, we have $b\varphi(f_\beta) \to b$ and hence $b\varphi^{**}(f) = b$. On the other hand, for $b \in J_\varphi$, $b\varphi(f_\beta) \to 0$ and hence $b\varphi^{**}(f) = 0$. Consequently, $B\varphi^{**}(f) \subseteq B$.

The implication (iii)⇒(i) is trivial. ∎

It follows from Theorem 1.4 that the condition that $B\varphi^{**}(e) \subseteq B$ does not depend on the choice of $e \in E(A)$.

THEOREM 1.5. *Let $\varphi : A \to B$ be a homomorphism, and let $I = I_\varphi \oplus J_\varphi$. Then the following two conditions are equivalent*:

(i) *Z_φ is open in $\Delta(B)$.*
(ii) *There exists a subset Z of $\Delta(M(I))$ which is open and closed in the hk-topology of $\Delta(M(I))$ and satisfies $Z \cap \Delta(B) = Z_\varphi$.*

Proof. Suppose that (i) holds and notice first that $\Delta(B) = \Delta(I)$ since $h(I) = \emptyset$ by Lemma 1.3. Let $P : I \to I$ be the projection with $P(I) = I_\varphi$ and $\ker P = J_\varphi$. By the closed graph theorem, P is continuous, and since both the range and the kernel of P are ideals, P is a multiplier of I. Let $K = M(I)P$ and

$$Z = \{\gamma \in \Delta(M(I)) : \widehat{P}(\gamma) = 0\}.$$

Then, since P is an idempotent, K is a closed ideal of $M(I)$ and, by definition of K and Z, $h(K) = Z$. Moreover, since $M(I)$ is semisimple,

$$k(Z) = \{T \in M(I) : \gamma(T - TP) = 0 \text{ for all } \gamma \in \Delta(M(I))\} = K.$$

So $Z = h(k(Z))$ is hk-closed in $\Delta(M(I))$. Similarly, with $L = M(I)(\mathrm{id} - P)$, where id denotes the identity operator on I, we have $h(L) = \Delta(M(I)) \setminus Z$ and $L = k(\Delta(M(I)) \setminus Z)$, so that $\Delta(M(I)) \setminus Z$ is also hk-closed in $\Delta(M(I))$. Finally, since $\Delta(B) = \Delta(I)$,

$$Z \cap \Delta(B) = \{\gamma \in \Delta(I) : \langle \gamma, P(b) \rangle = 0 \text{ for all } b \in I\} = h(I_\varphi) = Z_\varphi.$$

This shows (ii).

(ii)$\Rightarrow$(i). Let $Z_\varphi = Z \cap \Delta(B)$ for some subset Z of $\Delta(M(I))$ which is open and closed in the hk-topology of $\Delta(M(I))$. Then

$$\begin{aligned} h(I_\varphi + J_\varphi) &= h(I_\varphi) \cap h(J_\varphi) = Z_\varphi \cap h(k(E_\varphi)) \\ &= (Z \cap \Delta(B)) \cap h(k((\Delta(M(I)) \setminus Z) \cap \Delta(B))) \\ &\subseteq Z \cap \Delta(B) \cap (\Delta(M(I)) \setminus Z) = \emptyset, \end{aligned}$$

where the hulls and hk-closures are taken in $\Delta(B)$. It follows that the hull of $I_\varphi + J_\varphi$ in $\Delta(B)$ is empty, and hence Z_φ is open in $\Delta(B)$ (Lemma 1.3). ∎

REMARK 1.6. In (ii) of Theorem 1.5 the condition that Z is open and closed in the hull-kernel topology can be replaced by the condition that Z is open and closed in the Gelfand topology. Indeed, in this case, by the Shilov idempotent theorem (see [6, Theorem 2.4.33]), there exists $T \in M(B)$ such that $\widehat{T} = 1_Z$, the characteristic function of Z. Since

$$\begin{aligned} Z &= \{\gamma \in \Delta(M(B)) : \gamma(I - T) = 0\} \\ &= \Delta(M(B)) \setminus \{\gamma \in \Delta(M(B)) : \gamma(T) = 0\}, \end{aligned}$$

it follows that Z is open and closed in the hull-kernel topology.

For a semisimple Banach algebra B, let $\mathcal{Z}(B)$ denote the collection of all subsets of $\Delta(B)$ of the form Z_φ, where φ is a homomorphism from a semisimple commutative Banach algebra A with bounded approximate identity into B.

PROPOSITION 1.7. *The set $\mathcal{Z}(B)$ is closed under forming finite unions and intersections.*

Proof. Let A_1 and A_2 be two semisimple commutative Banach algebras with bounded approximate identities and let $\varphi_1 : A_1 \to B$ and $\varphi_2 : A_2 \to B$ be homomorphisms.

We first show that $Z_{\varphi_1} \cup Z_{\varphi_2} \in \mathcal{Z}(B)$. Let $\varphi_1 \widehat{\otimes} \varphi_2$ be the unique homomorphism from the projective tensor product $A_1 \widehat{\otimes} A_2$ into B satisfying

$$\varphi_1 \widehat{\otimes} \varphi_2(a_1 \otimes a_2) = \varphi_1(a_1)\varphi_2(a_2), \qquad a_j \in A_j,\ j = 1, 2.$$

It is clear that $Z_{\varphi_1 \widehat{\otimes} \varphi_2} = Z_{\varphi_1} \cup Z_{\varphi_2}$. However, $A_1 \widehat{\otimes} A_2$ need not be semisimple unless one of A_1 and A_2 has the approximation property (see [21] and [4, p. 236]). Therefore, let R denote the radical of $A_1 \widehat{\otimes} A_2$. Then R is contained in the kernel of $\varphi_1 \widehat{\otimes} \varphi_2$. Indeed, if $a \in A_1 \widehat{\otimes} A_2$ is such that $\varphi_1 \widehat{\otimes} \varphi_2(a) \neq 0$, then $\gamma \circ (\varphi_1 \widehat{\otimes} \varphi_2)(a) \neq 0$ for some $\gamma \in \Delta(B)$ since B is semisimple, and hence $a \notin R$. Thus $\varphi_1 \widehat{\otimes} \varphi_2$ induces a homomorphism φ from the semisimple algebra $(A_1 \widehat{\otimes} A_2)/R$ to B with $Z_\varphi = Z_{\varphi_1} \cup Z_{\varphi_2}$.

To see that $Z_{\varphi_1} \cap Z_{\varphi_2} \in \mathcal{Z}(B)$, define $\varphi : A_1 \oplus A_1 \to B \oplus B \to B$ by $\varphi = \sigma \circ (\varphi_1 \oplus \varphi_2)$, where $\sigma(b_1, b_2) = b_1 + b_2$, $b_1 \in B_1$, $b_2 \in B_2$. Then, for $\gamma \in \Delta(B)$ and $a_1 \in A_1$, $a_2 \in A_2$,

$$\varphi^*(\gamma)(a_1, a_2) = \gamma(\sigma(\varphi_1(a_1), \varphi_2(a_2))) = \gamma(\varphi_1(a_1)) + \gamma(\varphi_2(a_2)).$$

Thus $\varphi^*(\gamma) = 0$ if and only if $\varphi_1^*(\gamma) = 0$ and $\varphi_2^*(\gamma) = 0$. ■

EXAMPLE 1.8. Let G be an amenable locally compact group. Then $\mathcal{Z}(A(G)) = \mathcal{R}_c(G)$, the collection of closed sets in the coset ring of G (see [10] for the definition of the coset ring). Clearly, if $E \in \mathcal{R}_c(G)$ then the ideal $k(E)$ is semisimple and has a bounded approximate identity [10, Lemma 2.2], and if φ is the embedding of $k(E)$ into $A(G)$ then $Z_\varphi = E$. Conversely, let φ be a homomorphism from a semisimple commutative Banach algebra A with bounded approximate identity into $A(G)$. Then the ideal I_φ has a bounded approximate identity. By Theorem 2.3 of [10], $I_\varphi = k(E)$ for some $E \in \mathcal{R}_c(G)$. Since $A(G)$ is regular, it follows that $Z_\varphi = h(I_\varphi) = h(k(E)) = E \in \mathcal{R}_c(G)$.

In passing, we recall that if A is regular and semisimple then given any closed subset E of $\Delta(A)$, there exists a smallest ideal $j(E)$ of A with hull equal to E. More precisely, $j(E)$ consists of all $a \in A$ such that $\widehat{a}$ has compact support and vanishes on a neighbourhood of E. The set E is called a *set of synthesis* if $\overline{j(E)} = k(E)$, and A is said to be *Tauberian* if $\emptyset$ is a set of synthesis.

PROPOSITION 1.9. *Suppose that A and B are regular and that A is Tauberian. Let $\varphi : A \to B$ be a homomorphism, and let I denote the closed ideal of B generated by $\varphi(A)$. Then I has a bounded approximate identity and $I = \overline{j(Z_\varphi)}$. In particular, Z_φ is a set of synthesis if and only if $I = k(Z_\varphi)$.*

Proof. Since A is regular and Tauberian, the ideal

$$j(\emptyset) = \{a \in A : \operatorname{supp} \widehat{a} \text{ is compact}\}$$

is dense in A. Thus we can assume that the bounded approximate identity $(e_\alpha)_\alpha$ of A is contained in $j(\emptyset)$. Then, using regularity again, for every α there exists $a_\alpha \in A$ such that $\widehat{a}_\alpha = 1$ on $\operatorname{supp} \widehat{e}_\alpha$. Since A is semisimple, $a_\alpha e_\alpha = e_\alpha$. This implies that, for every $\gamma \in \Delta(B)$, $\widehat{\varphi(a_\alpha)}(\gamma) = 1$ whenever $\widehat{\varphi(e_\alpha)}(\gamma) \neq 0$, and hence $\widehat{\varphi(a_\alpha)} = 1$ on $\operatorname{supp} \widehat{\varphi(e_\alpha)}$. Since $\widehat{\varphi(a_\alpha)} \in C_0(\Delta(B))$ and $\widehat{\varphi(a_\alpha)}$ vanishes on Z_φ, it follows that $\operatorname{supp} \widehat{\varphi(e_\alpha)}$ is compact and disjoint from Z_φ. Thus we have seen that $\varphi(e_\alpha) \in j(Z_\varphi)$ for every α, and hence $I \subseteq \overline{j(Z_\varphi)}$. Now, since $(e_\alpha)_\alpha$ is an approximate identity for A and hence I

is generated by the set of all $\varphi(e_\alpha)$,

$$h(I) = \{\gamma \in \Delta(B) : \gamma(\varphi(e_\alpha)) = 0 \text{ for all } \alpha\} = Z_\varphi.$$

Finally, B being regular, $\overline{j(Z_\varphi)}$ is the smallest closed ideal of B with hull equal to Z_φ. It follows that $I = \overline{j(Z_\varphi)}$, and hence Z_φ is a set of synthesis if and only if $I = k(Z_\varphi)$.

It remains to observe that I has a bounded approximate identity. Given $x \in I$ and $\varepsilon > 0$, there exist $b_1, \ldots, b_n \in B$ and $a_1, \ldots, a_n \in A$ such that $\|x - \sum_{j=1}^n b_j\varphi(a_j)\| \le \varepsilon$. Since A has a bounded approximate identity, there exists $e \in A$ such that

$$\|\varphi(a_j) - \varphi(a_je)\| \le \varepsilon\Big(\sum_{j=1}^n \|b_j\|\Big)^{-1}$$

for $j = 1, \ldots, n$. Then

$$\begin{aligned}\|x - x\varphi(e)\| &\le \Big\|x - \sum_{j=1}^n b_j\varphi(a_j)\Big\| + \Big\|\sum_{j=1}^n b_j(\varphi(a_j) - \varphi(a_je))\Big\| \\ &\le \varepsilon + \sum_{j=1}^n \|b_j\| \cdot \|\varphi(a_j) - \varphi(a_je)\| \le 2\varepsilon.\end{aligned}$$

This finishes the proof. ∎

We conclude this section with an observation that will be used in the next section. If $\varphi : A \to B$ is a homomorphism and $(e_\alpha)_\alpha$ is a bounded approximate identity of A, then the net of functions $\widehat{\varphi(e_\alpha)}$ converges to 1 uniformly on compact subsets of E_φ. To see this, let $\gamma_0 \in E_\varphi$ be given, choose $a \in A$ such that $|\gamma_0(a)| > 1$ and let $V = \{\gamma \in E_\varphi : |\gamma(a)| > 1\}$. Then V is an open neighbourhood of γ_0 in $\Delta(B)$ and, for all $\gamma \in V$ and all α,

$$\begin{aligned}|\widehat{\varphi(e_\alpha)}(\gamma) - 1| &\le |\widehat{\varphi(a)}(\gamma)[\widehat{\varphi(e_\alpha)}(\gamma) - 1]| = |\widehat{\varphi(ae_\alpha)}(\gamma) - \widehat{\varphi(a)}(\gamma)| \\ &\le \|\varphi\| \cdot \|ae_\alpha - a\|.\end{aligned}$$

Thus the net of functions $\widehat{\varphi(e_\alpha)}$ converges uniformly to 1 on V, and hence it converges uniformly to 1 on every compact subset of E_φ.

2. Extending homomorphisms to multiplier algebras. We continue to let A and B be semisimple commutative Banach algebras. Suppose that both A and B have bounded approximate identities. The main purpose of this section is to investigate the problem of when a homomorphism $\varphi : A \to B$ extends to a homomorphism from $M(A)$ to $M(B)$. Let $(e_\alpha)_\alpha$ be a bounded approximate identity of A and e be a w^*-cluster point of the net $(e_\alpha)_\alpha$ in A^{**}. We first show that the condition that $B\varphi^{**}(e) \subseteq B$ (which by Theorem 1.4 is independent of the choice of the bounded approximate identity) guarantees the existence of an extension $\phi : M(A) \to M(B)$ of φ.

THEOREM 2.1. *Let $\varphi : A \to B$ be a homomorphism and suppose that $B\varphi^{**}(e) \subseteq B$. Then*:

(i) *For all $T \in M(A)$, $B\varphi^{**}(T^{**}(e)) \subseteq B$.*
(ii) *The map $\phi : M(A) \to M(B)$ defined by $\phi(T)(b) = b\varphi^{**}(T^{**}(e))$ is a homomorphism that extends φ. Moreover, $\phi(M(A)) \subseteq B$ if and only if E_φ is compact.*

Proof. (i) Let $T \in M(A)$ and $b \in I_\varphi$. Since $\|b - b\varphi(e_\alpha)\| \to 0$, we get

$$\|b\varphi^{**}(T^{**}(e)) - b\varphi(e_\alpha)\varphi^{**}(T^{**}(e))\| \to 0.$$

On the other hand,

$$\varphi(e_\alpha)\varphi^{**}(T^{**}(e)) = \varphi^{**}(e_\alpha T^{**}(e)) = \varphi^{**}(T^{**}(e_\alpha)) = \varphi(T(e_\alpha))$$

belongs to B since $T(e_\alpha) \in A$. It follows that $b\varphi(e_\alpha)\varphi^{**}(T^{**}(e)) \in B$ and hence $b\varphi^{**}(T^{**}(e) \in B$. Now let $b \in J_\varphi$. Then $b\varphi^{**}(T^{**}(e)) = 0$. Since $B = I_\varphi \oplus J_\varphi$ by Theorem 1.4, we conclude that $B\varphi^{**}(T^{**}(e)) \subseteq B$ for all $T \in M(A)$.

(ii) By (i), $B\varphi^{**}(T^{**}(e)) \subseteq B$ for every $T \in M(A)$. Define $\phi : M(A) \to M(B)$ by $\phi(T)(b) = b\varphi^{**}(T^{**}(e))$. Then, for $T, S \in M(A)$ and $b \in B$,

$$\begin{aligned}\phi(T \circ S)(b) &= b\varphi^{**}(T^{**} \circ S^{**}(e)) = b\varphi^{**}(T^{**}(e)S^{**}(e)) \\ &= b\varphi^{**}(T^{**}(e))\varphi^{**}(S^{**}(e)) = \phi(T) \circ \phi(S)(b).\end{aligned}$$

Similarly, $\phi(T + S) = \phi(T) + \phi(S)$. Thus ϕ is a homomorphism, and ϕ extends φ since, by definition of ϕ,

$$\phi(L_a)(b) = b\varphi^{**}(L_a^{**}(e)) = b\varphi(a) = L_{\varphi(a)}(b),$$

that is, $\phi(L_a) = L_{\varphi(a)}$.

To prove that $\phi(M(A)) \subseteq B$ if and only if E_φ is compact, recall that

$$\langle \varphi^{**}(I^{**}(e)), \gamma\rangle = \langle \varphi^*(\gamma), e\rangle = \lim_\alpha \widehat{\varphi(e_\alpha)}(\gamma) = 1_{E_\varphi}(\gamma)$$

for all $\gamma \in \Delta(B)$, and let m denote the multiplier of B defined by $\varphi^{**}(I^{**}(e))$.

If E_φ is compact then, since E_φ is also open, by the Shilov idempotent theorem there exists $b \in B$ such that $\widehat{b} = 1_{E_\varphi}$. Thus $\widehat{m}$ and $\widehat{b}$ agree on $\Delta(B)$, whence $m = L_b$. Since B is an ideal of $M(B)$, from the definition of ϕ we conclude that $\phi(M(A)) = \phi(M(A)) \circ \phi(m) \subseteq B$.

Conversely, if $m = L_b$ for some $b \in B$, then $\widehat{m}$ vanishes on $\Delta(M(B)) \setminus \Delta(B)$ and, since m is an idempotent, $\widehat{m} = 1_E$ for some compact open subset E of $\Delta(M(B))$. On the other hand, $E \cap \Delta(B) = E_\varphi$. It follows that $E_\varphi = E$, which is compact. ∎

The preceding theorem will be applied several times in this and the following section.

Corollary 2.2. *Let $\varphi : A \to B$ be a homomorphism. If $B = I_\varphi \oplus J_\varphi$, then φ extends to a homomorphism from $M(A)$ to $M(B)$.*

Proof. The statement is an immediate consequence of Theorem 2.1 and the implication (ii)⇒(i) of Theorem 1.4. ∎

We do not know whether conversely, given a homomorphism $\varphi : A \to B$, the existence of an extension $\phi : M(A) \to M(B)$ of φ implies that $B = I_\varphi \oplus J_\varphi$.

Let G and H be locally compact abelian groups and $\varphi : L^1(G) \to M(H)$ a homomorphism. As remarked in [5, p. 219, last two lines], $Z_\varphi \cap \widehat{H}$ is open in $\widehat{H}$. In particular, if $\varphi(L^1(G)) \subseteq L^1(H)$ then Z_φ is open in $\widehat{H}$. We observe next that the same conclusion is true when φ is a weakly compact homomorphism.

Corollary 2.3. *Let $\varphi : A \to B$ be a weakly compact homomorphism. Then Z_φ is open in $\Delta(B)$ and φ extends to a homomorphism from $M(A)$ into $M(B)$.*

Proof. Since $e_\alpha \to e$ in the w^*-topology of A^{**}, we have $\varphi(e_\alpha) = \varphi^{**}(e_\alpha) \to \varphi^{**}(e)$ in the w^*-topology of B^{**}. On the other hand, since φ is weakly compact, we can assume that $(\varphi(e_\alpha))_\alpha$ converges weakly in B. It follows that $\varphi^{**}(e) \in B$ and hence, by Theorem 2.1, φ extends to a homomorphism from $M(A)$ into $M(B)$.

To show that E_φ is closed in $\Delta(B)$, let $(\gamma_\beta)_\beta$ be a net in E_φ and let $\gamma \in \Delta(B)$ be a w^*-limit point of $(\gamma_\beta)_\beta$. Since $E_\varphi = \Delta(I_\varphi)$ and $(\varphi(e_\alpha))_\alpha$ is an approximate identity for I_φ, we have $\delta(\varphi(e_\alpha)) \to 1$ for each $\delta \in E_\varphi$. Thus $\langle \delta, \varphi^{**}(e)\rangle = 1$ for all $\delta \in E_\varphi$. Because $\varphi^{**}(e) \in B$, it follows that

$$\langle \gamma, \varphi^{**}(e)\rangle = \lim_\beta \langle \gamma_\beta, \varphi^{**}(e)\rangle = 1.$$

Consequently, $\langle \gamma, \varphi^{**}(e_\alpha)\rangle \neq 0$ eventually, whence $\gamma \in E_\varphi$. ∎

The first to study the problem of extending homomorphisms to multiplier algebras and the uniqueness of such extensions was Cohen. In [5, Theorem 3] a complete solution was given when $A = L^1(G)$ and $B = M(H)$, where G and H are locally compact abelian groups. The proof of the following proposition is an adaptation of the one in [5].

In the next results, the reader should not confuse the sets Z_φ and Z_ϕ.

Proposition 2.4. *Let $\varphi : A \to B$ be a homomorphism and suppose that $\phi : M(A) \to M(B)$ is a homomorphism extending φ. Then the following two conditions are equivalent*:

(i) *ϕ is the only extension of φ.*
(ii) *Either $A = M(A)$ or $Z_\phi \cap \Delta(B) = \emptyset$.*

Proof. Of course, we do not have to consider the case that $A = M(A)$.

Suppose first that $Z_\phi \cap \Delta(B) = \emptyset$. Fix $\gamma \in \Delta(B)$ and let $\widetilde{\gamma}$ denote its extension to $M(B)$. Then the function

$$T \mapsto \langle \phi(T), \widetilde{\gamma} \rangle = \langle \phi^*(\widetilde{\gamma}), T \rangle$$

on $M(A)$ is multiplicative. On A, this function coincides with the function $a \mapsto \langle \phi^*(\gamma), a \rangle$, $a \in A$. Since $Z_\phi \cap \Delta(B) = \emptyset$, we have $\varphi^*(\gamma) \neq 0$ and hence $\phi^*(\widetilde{\gamma})$ is the unique element of $\Delta(M(A))$ extending $\varphi^*(\gamma)$. Thus, if $\psi : M(A) \to M(B)$ is another homomorphism extending φ, then for all $\gamma \in \Delta(B)$, $\psi^*(\widetilde{\gamma}) = \phi^*(\widetilde{\gamma})$ and hence $\langle \widetilde{\gamma}, \psi(b) \rangle = \langle \widetilde{\gamma}, \phi(b) \rangle$ for all $b \in B$. This implies that $\psi = \phi$.

Conversely, assume that $Z_\phi \cap \Delta(B) \neq \emptyset$ and $A \neq M(A)$. Since Z_ϕ is open and closed in $\Delta(M(B))$, by Shilov's idempotent theorem there exists $S \in M(B)$ such that $\widehat{S} = 1_{Z_\phi}$. By hypothesis, $M(A)/A$ is a non-trivial unital commutative Banach algebra and hence $\Delta(M(A)) \setminus \Delta(A) = \Delta(M(A)/A) \neq \emptyset$. Choose any $\varrho \in \Delta(M(A)) \setminus \Delta(A)$ and define $\psi : M(A) \to M(B)$ by $\psi(T) = \phi(T) + \varrho(T)S$. Clearly, ψ is linear and extends ϕ. For $T_1, T_2 \in M(A)$ we have

$$\begin{aligned}\psi(T_1)\psi(T_2) &= \phi(T_1T_2) + \varrho(T_1T_2) + \varrho(T_2)\phi(T_1)S + \varrho(T_1)\phi(T_2)S \\ &= \phi(T_1T_2) + \varrho(T_1T_2)S = \psi(T_1T_2),\end{aligned}$$

since $\widehat{S}(\gamma) = 0$ for $\gamma \notin Z_\phi$ and $\phi^*(\gamma) = 0$ for $\gamma \in Z_\phi$. Finally, $\psi \neq \phi$ since $\varrho(T) \neq 0$ for some $T \in M(A)$ and $\widehat{S}(\gamma) \neq 0$ for some γ in the non-empty set $Z_\phi \cap \Delta(B)$. ∎

Proposition 2.5. *Let A_1 and B_1 be unital commutative Banach algebras containing A and B as closed ideals, respectively. Let $\varphi : A \to B$ be a homomorphism, and suppose that φ extends to a homomorphism $\phi : A_1 \to B_1$. Then the following conditions are equivalent*:

(i) *Z_φ is open in $\Delta(B)$.*
(ii) *$h_{\Delta(B)}(\varphi(A)) \cap E_\phi$ is open in $\Delta(B)$.*

In particular, if the ideal of B generated by $\varphi(A)$ is dense in I_φ, then Z_φ is open in $\Delta(B)$ if and only if $Z_\varphi \cap E_\phi$ is open in $\Delta(B)$.

Proof. Since $\varphi^*(\gamma) = \phi^*(\gamma)|_A$ for $\gamma \in \Delta(B)$, we have

$$Z_\varphi = (Z_\phi \cap \Delta(B)) \cup (E_\phi \cap h_{\Delta(B)}(\varphi(A))).$$

Since A_1 is unital, Z_ϕ and E_ϕ are both open in $\Delta(M(B))$ and hence $\Delta(B)$ is the disjoint union of the open subsets $Z_\phi \cap \Delta(B)$ and $E_\phi \cap \Delta(B)$. Thus Z_φ is open in $\Delta(B)$ if and only if $h_{\Delta(B)}(\varphi(A)) \cap E_\phi$ is open in $E_\phi \cap \Delta(B)$ (equivalently, open in $\Delta(B)$).

Finally, note that if the ideal generated by $\varphi(A)$ in B is dense in I_φ, then $h_{\Delta(B)}(\varphi(A)) = h_{\Delta(B)}(I_\varphi) = Z_\varphi$. ∎

As we have seen in Theorem 1.4, $B\varphi^{**}(e) \subseteq B$ holds if and only if $B = I_\varphi \oplus J_\varphi$, and this latter condition implies (but is not equivalent to unless B is Tauberian) that Z_φ is open in $\Delta(B)$ (Lemma 1.3). Thus the question arises of whether, at least for a large class of algebras B, openness of Z_φ already suffices to show the existence of an extension of a given homomorphism. This question leads to the class of BSE-algebras (named after Bochner–Schoenberg–Eberlein), and it actually turns out that for such B all the extensions $\phi : M(A) \to M(B)$ of a homomorphism $\varphi : A \to B$ can be described.

Let A be a commutative Banach algebra. A complex-valued function σ on $\Delta(A)$ is said to satisfy the *BSE-condition* if there exists $C > 0$ such that, for every finite collection $c_1, \dots, c_n$ of complex numbers and $\gamma_1, \dots, \gamma_n$ in $\Delta(A)$,

$$\Big|\sum_{j=1}^{n} c_j \sigma(\gamma_j)\Big| \le C \Big\|\sum_{j=1}^{n} c_j \gamma_j\Big\|_{A^*}.$$

This condition is motivated by the Bochner–Schoenberg–Eberlein theorem, which characterizes the Fourier–Stieltjes transforms of measures on a locally compact abelian group. The algebra A is called a *BSE-algebra* if the continuous functions on $\Delta(A)$ satisfying the BSE-condition are precisely the functions of the form $\widehat{T}$ where $T \in M(A)$.

Recall that, by Theorem 4 and Corollary 5 of [27], a semisimple commutative Banach algebra A with a bounded approximate identity is a BSE-algebra if every element $u \in A^{**}$ for which the Gelfand transform $\widehat{u} : \Delta(A) \to \mathbb{C}$ is continuous is a multiplier of A, that is, $Au \subseteq A$ [27, Theorem 4].

THEOREM 2.6. *Let $\varphi : A \to B$ be a homomorphism, and suppose that B is a BSE-algebra and that Z_φ is open in $\Delta(B)$. Then φ extends to a homomorphism from $M(A)$ to $M(B)$.*

Proof. Since Z_φ is open in $\Delta(B)$, $\widehat{\varphi^{**}(e)}$ is a continuous function on $\Delta(B)$. Since B is a BSE-algebra, it follows that $B\varphi^{**}(e) \subseteq B$ and this in turn implies that there exists a homomorphism from $M(A)$ to $M(B)$ extending φ (Theorem 2.1). ∎

THEOREM 2.7. *Let $\varphi : A \to B$ be a homomorphism and suppose that Z_φ is open and B is a BSE-algebra. Let $\phi : M(A) \to M(B)$ be a homomorphism extending φ, and let $\mathcal{X}$ be the set of all continuous mappings $\chi : \gamma \mapsto \chi_\gamma$ from Z_φ into $h(A) \cup \{0\} \subseteq M(A)^*$, where $h(A) \cup \{0\}$ is endowed with the w^*-topology. For each $\chi \in \mathcal{X}$ and $T \in M(A)$,*

$$\widehat{\phi_\chi(T)}(\gamma) = \begin{cases} \widehat{\phi(T)}(\gamma) & \text{if } \gamma \in E_\varphi, \\ \chi_\gamma(T) & \text{if } \gamma \in Z_\varphi, \end{cases}$$

defines an element $\phi_\chi(T)$ *of* $M(B)$, *and the map*

$$\phi_\chi : M(A) \to M(B), \qquad T \mapsto \phi_\chi(T),$$

is a homomorphism extending φ. *Moreover*, *the assignment* $\chi \mapsto \phi_\chi$ *is a bijection between* $\mathcal{X}$ *and the set of all homomorphisms from* $M(A)$ *to* $M(B)$ *extending* φ.

Proof. For $\chi \in \mathcal{X}$ and $T \in M(A)$, define a function $f_{\chi,T}$ on $\Delta(B)$ by $f_{\chi,T}(\gamma) = \chi_\gamma(T)$ for $\gamma \in Z_\varphi$ and $f_{\chi,T}(\gamma) = \widehat{\phi(T)}(\gamma)$ for $\gamma \in E_\varphi$. Then $f_{\chi,T}$ is continuous since $\Delta(B)$ is the disjoint union of the open sets Z_φ and E_φ and the functions $\gamma \mapsto \chi_\gamma(T)$ and $\widehat{\phi(T)}$ are continuous on Z_φ and E_φ, respectively. Since B is a BSE-algebra, there exists a unique element $\phi_\chi(T) \in M(B)$ such that $\widehat{\phi_\chi(T)} = f_{\chi,T}$.

It is clear that $\widehat{\phi(TS)}(\gamma) = \widehat{\phi(T)}(\gamma)\widehat{\phi(S)}(\gamma)$ for all $T, S \in M(A)$ and $\gamma \in \Delta(B)$ and therefore $\phi_\chi(TS) = \phi_\chi(T)\phi_\chi(S)$. Moreover, for $a \in A$, $\widehat{\phi_\chi(a)}(\gamma) = \widehat{\varphi(a)}(\gamma)$ for all $\gamma \in \Delta(B)$ since $\phi|_A = \varphi$, $\widehat{\varphi(a)}(\gamma) = 0$ for $\gamma \in Z_\varphi$ and $\chi_\gamma \in h(A)$. This shows that $\phi_\chi : T \mapsto \phi_\chi(T)$ is a homomorphism from $M(A)$ to $M(B)$ extending φ.

Obviously, the mapping $\chi \mapsto \phi_\chi$ is injective. Finally, let $\psi : M(A) \to M(B)$ be an arbitrary homomorphism extending φ. Let $\gamma \in E_\varphi$. Then, since $\psi|_A = \phi|_A = \varphi$ and $\varphi^*(\gamma) \neq 0$, $\psi^*(\gamma)$ and $\phi^*(\gamma)$ are elements of $\Delta(M(A))$ which restrict to the same element $\varphi^*(\gamma)$ of $\Delta(A)$, and this implies that $\psi^*(\gamma) = \phi^*(\gamma)$. On the other hand, for $\gamma \in Z_\varphi$, the function $\chi_\gamma : T \mapsto \widehat{\psi(T)}(\gamma)$ is either zero or an element of $\Delta(M(A))$, which annihilates A since $\varphi^*(\gamma) = 0$. Moreover, the map $\gamma \mapsto \chi_\gamma$ from Z_φ into $h(A) \cup \{0\}$ is continuous since the function $\gamma \mapsto \chi_\gamma(T) = \widehat{\psi(T)}(\gamma)$ is continuous for every $T \in M(A)$. This shows that $\chi : \gamma \mapsto \chi_\gamma$ belongs to $\mathcal{X}$ and that $\psi = \phi_\chi$. ∎

Example 1.8 shows that the condition that Z_φ be open in Theorems 2.6 and 2.7 is far from being necessary for the homomorphism $\varphi : A \to B$ to extend to some homomorphism $\phi : M(A) \to M(B)$.

The Fourier algebras of amenable locally compact groups are known to be BSE-algebras [9, p. 202, Corollaire 1] and so are the disk algebra and Hardy algebras [27]. We continue by adding two more classes of examples to the body of BSE-algebras.

PROPOSITION 2.8. *Let* A *be a semisimple*, *Tauberian*, *commutative Banach algebra with a* (*not necessarily bounded*) *approximate identity and let* E *be a closed subset of* $\Delta(A)$. *Suppose that* $Av \subseteq A$ *for every* $v \in A^{**}$ *such that* $\widehat{v}$ *is continuous on* $\Delta(A)$. *Then* $k(E)u \subseteq k(E)$ *for every* $u \in k(E)^{**}$ *such that* $\widehat{u}$ *is continuous on* $\Delta(k(E))$. *In particular*, *if* $k(E)$ *has a bounded approximate identity*, *then* $k(E)$ *is a BSE-algebra.*

Proof. Since A is Tauberian, the ideal $J = \{a \in A : \widehat{a}$ has compact support$\}$ is dense in A. We claim that the ideal $I = k(E) \cap J$ is dense in $k(E)$. To see this, let $a \in k(E)$, $a \neq 0$, and $\varepsilon > 0$ be given. Since A has an approximate identity and J is dense in A, there exist $u \in A$ such that $\|a - au\| \leq \varepsilon$ and $v \in J$ such that $\|v - u\| \leq \varepsilon/\|a\|$. Then $av \in I$ and $\|a - av\| \leq 2\varepsilon$.

Now let u be an element of $k(E)^{**} \subseteq A^{**}$ such that $\widehat{u}$ is continuous on $\Delta(k(E)) = \Delta(A) \setminus E$. We have to show that $au \in k(E)$ for each $a \in k(E)$. To that end, consider $b \in I$. Since $\widehat{b}$ has compact support and $\widehat{b} = 0$ on E, $\widehat{bu} = \widehat{b}\widehat{u}$ has compact support contained in $\overline{\Delta(A) \setminus E}$ and $\widehat{bu}$ is continuous on $\Delta(A) \setminus E$. Moreover, since $\widehat{bu} = 0$ on E, $\widehat{bu}$ is continuous on $\Delta(A)$ if any net $(\gamma_\alpha)_\alpha$ in $\Delta(A) \setminus E$ converging to some $\gamma \in E$ satisfies $\widehat{bu}(\gamma_\alpha) \to 0$. However, this is clear since $\widehat{b}$ is continuous and vanishes on E and $\|\widehat{u}\| \leq \|u\|$.

Since $ub \in k(E)^{**}$ and $\widehat{ub}$ is continuous on $\Delta(A)$, the hypothesis implies that $Aub \subseteq A$. Since $\widehat{ub}$ vanishes on E, $uba \in k(E)$ for all $a \in A$. It follows that $ub \in A$ since A has an approximate identity and $\|ux\| \leq \|u\| \cdot \|x\|$ for all $x \in A$. This shows that $uI \subseteq k(E)$ and hence $uk(E) = u\overline{I} \subseteq k(E)$. ∎

We now apply Proposition 2.8 to the Fourier algebra $A(G)$ of an amenable locally compact group G. Recall that $A(G)$ is semisimple and Tauberian [9] and has a bounded approximate identity. Moreover, if $u \in A(G)^{**}$, then there is a bounded net $(u_\alpha)_\alpha$ in $A(G)$ such that $u_\alpha(x) \to u(x)$ for all $x \in G = \Delta(A(G))$. If, in addition, u is continuous on G then $u \in B(G)$ (see [9, p. 202, Corollaire 1]) and hence $A(G)u \subseteq A(G)$. Thus the following corollary is an immediate consequence of Proposition 2.8 and [10, Lemma 2.2].

Corollary 2.9. *Let G be a locally compact amenable group and E a closed subset of G such that the ideal $k(E)$ of $A(G)$ has a bounded approximate identity. Then $k(E)$ is a BSE-algebra. In particular*, $k(E)$ *is a BSE-algebra for every $E \in \mathcal{R}_{\mathrm{c}}(G)$.*

In concluding this section we consider L^1-algebras of compact commutative hypergroups. We refrain from repeating the definition of a locally compact hypergroup and instead refer to the literature (see [3] and [26] and the references therein). Every commutative hypergroup K possesses a Haar measure [26], and much of the basic theory of $L^1(K)$ parallels that of L^1-algebras of locally compact abelian groups.

Example 2.10. Let K be a compact commutative hypergroup such that $\widehat{K}$, the set of all bounded characters of K, is a hypergroup with respect to pointwise multiplication. For $\alpha \in \widehat{K}$, let $\gamma_\alpha : L^1(K) \to \mathbb{C}$ be defined by $\gamma_\alpha(f) = \int_K f(x)\alpha(x)\,dx$, $f \in L^1(K)$. Then the map $\alpha \mapsto \gamma_\alpha$ is a homeomorphism between $\widehat{K}$ and $\Delta(L^1(K))$ when $\widehat{K}$ is endowed with the topology of uniform convergence of characters on K [3, Section 2.2]. We claim that $L^1(K)$ is a BSE-algebra.

To verify this, consider the linear span $T(K)$ of trigonometric polynomials on K, that is, the set of all functions of the form $f(x) = \sum_{j=1}^n c_j\alpha_j(x)$, $\alpha_j \in \widehat{K}$, $c_j \in \mathbb{C}$, $n \in \mathbb{N}$. Then $T(K)$ is uniformly dense in $C(K)$ (see [3, Theorem 2.4.5]), and it follows from the orthogonality relations for characters that every $f \in T(K)$ has a unique such decomposition, where the α_j are different and the c_j are non-zero. Now, let σ be a continuous function on $\widehat{K}$ satisfying the BSE-condition. We can define a linear functional L on $T(K)$ by setting $L(f) = \sum_{j=1}^n c_j\sigma(\alpha_j)$ for $f = \sum_{j=1}^n c_j\alpha_j \in T(K)$. Then

$$|L(f)| \le C\Big\|\sum_{j=1}^n c_j\alpha_j\Big\|_\infty = C\|f\|_\infty,$$

and hence L extends uniquely to a bounded linear functional, also denoted by L, on $C(K)$. By the Riesz representation theorem, there exists $\mu \in M(K)$ such that $L(f) = \int_K f(x)\,d\mu(x)$ for all $f \in C(K)$. It follows that $\widehat{\mu}(\alpha) = L(\alpha) = \sigma(\alpha)$ for all $\alpha \in \widehat{K}$. Since μ defines a multiplier of $L^1(K)$ (in fact, $M(K)$ identifies canonically with $M(L^1(K))$, see [3, Theorem 1.6.24]), we conclude that $L^1(K)$ is a BSE-algebra.

3. When homomorphisms always extend to multiplier algebras.

Let G be a locally compact group. Then $B(G)$ is a dual Banach space. In fact, $B(G)$ can be canonically identified with the dual space of the group C^*-algebra $C^*(G)$ via the pairing $\langle u, f\rangle = \int_G f(x)u(x)\,dx$ for $f \in L^1(G)$ and $u \in B(G)$. Moreover, the multiplication of $B(G)$ is separately continuous in the w^*-topology of $B(G)$. However, there are many other Banach algebras sharing these properties. Therefore, it is worthwhile to prove an extension result for homomorphisms into such algebras.

THEOREM 3.1. *Let A and B be semisimple commutative Banach algebras such that A has a bounded approximate identity and B is unital. Suppose that B is the dual space of some Banach space X and that the multiplication of B is separately $\sigma(B,X)$-continuous. Then, for any homomorphism $\varphi : A \to B$, we have $B = I_\varphi \oplus J_\varphi$ and hence φ extends to a homomorphism from $M(A)$ into B.*

Proof. In view of Corollary 2.2 it suffices to show that $B = I_\varphi \oplus J_\varphi$. Let $(e_\alpha)_\alpha$ be a bounded approximate identity for A. Then the net $\varphi(e_\alpha)_\alpha$, being bounded in B, has a $\sigma(B,X)$-cluster point v in B. After passing to a subnet if necessary, we can assume that $\varphi(e_\alpha) \to v$ in $\sigma(B,X)$. Then, by hypothesis, $b\varphi(e_\alpha) \to bv$ for every $b \in B$ in the topology $\sigma(B,X)$. Thus

$$I_\varphi = \{b \in B : bv = b\}, \qquad J_\varphi = \{b \in B : bv = 0\}.$$

We next show that v is an idempotent. To see this, let $a \in A$ and consider the net $(\varphi(a)\varphi(e_\alpha))_\alpha$ in B. Then, on the one hand, $\varphi(a)\varphi(e_\alpha) \to \varphi(a)v$, and on the other hand, $\varphi(a)\varphi(e_\alpha) = \varphi(ae_\alpha) \to \varphi(a)$. This shows that $\varphi(a)v = \varphi(a)$ for all $a \in A$. In particular, $\varphi(e_\alpha)v = \varphi(e_\alpha)$ for all α. Passing to the $\sigma(B,X)$-cluster point v, we get $v^2 = v$. The above description of I_φ and J_φ now yields $B = I_\varphi \oplus J_\varphi$. ∎

Since, in proving Theorem 3.1, we have only used Corollary 2.2, it would have been sufficient to assume that B has a bounded approximate identity rather than being unital. However, as pointed out by the referee, a Banach algebra which is a dual space such that the multiplication is w^*-continuous (such Banach algebras are often termed dual Banach algebras) has to be unital whenever it has a bounded approximate identity (see [24, Proposition 1.2]). Dual Banach algebras have been studied by several authors (see [25] and the references therein).

As an immediate consequence of Theorem 3.1 we obtain

Corollary 3.2. *Let G be a locally compact group and let A be any commutative semisimple Banach algebra with bounded approximate identity. Then every homomorphism $\varphi : A \to B(G)$ extends to a homomorphism $\phi : M(A) \to B(G)$.*

Proof. As mentioned above, $B(G)$ is a dual Banach space and multiplication in $B(G)$ is separately w^*-continuous. ∎

Corollary 3.2 in particular shows that if G and H are locally compact groups and H is amenable, then every homomorphism from $A(H)$ into $B(G)$ extends to a homomorphism from $B(H) = M(A(H))$ into $B(G)$. The reader should compare this with results on extensions of completely bounded homomorphisms in Section 3 of [15].

Our second application of Theorem 3.1 concerns L^1-algebras of commutative hypergroups. There is a wealth of important examples of commutative, non-compact hypergroups which fail to be groups, and they arise in several different contexts, of which we just mention two.

Example 3.3. (1) Let G be a locally compact group and H a compact subgroup of G. Then the set $K = G//H$ of all double cosets HxH, $x \in G$, is a locally compact hypergroup when endowed with the quotient topology. The pair (G,H) is called a *Gelfand pair* if $G//H$ is commutative. Then $L^1(G//H)$ is isomorphic to the subalgebra of $L^1(G)$ consisting of all H-biinvariant functions in $L^1(G)$.

Let H_n denote the $(2n+1)$-dimensional real Heisenberg group, model H_n as $\mathbb{C}^n \times \mathbb{R}$ and let the unitary group $U(n)$ act on $\mathbb{C}^n \times \mathbb{R}$ by $k \cdot (z,t) = (k \cdot z, t)$. Let K be a closed subgroup of $U(n)$ and form the semidirect product $G_n(K) = H_n \rtimes K$. For many such K, for instance $U(n)$ and the n-dimensional torus $T(n)$, $(G_n(K), K)$ is a Gelfand pair [2].

Both $(\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2))$ and $(\mathrm{SL}(2,\mathbb{C}), \mathrm{SU}(2))$ are Gelfand pairs. More generally, if G is a connected semisimple Lie group with finite centre and K is a maximal compact subgroup of G, then (G, K) is a Gelfand pair [14, Chapter IV].

(2) Hypergroups with underlying set $\mathbb{N}_0$ or $\mathbb{R}_+$ arise, for instance, from sequences of polynomials such as Jacobi polynomials and various kinds of Sturm–Liouville functions (compare Chapter 3 of [3]).

Corollary 3.4. *Let K be a commutative hypergroup, and let A be a commutative semisimple Banach algebra with bounded approximate identity. Then every homomorphism $\varphi : A \to L^1(K)$ extends to a homomorphism $\phi : M(A) \to M(L^1(K))$.*

Proof. Let $M(K)$ be the Banach algebra of all bounded Radon measures on K, and for $\mu \in M(K)$, let

$$T_\mu : L^1(K) \to L^1(K), \qquad f \mapsto f * \mu,$$

be the associated convolution operator. Then the map $\mu \mapsto T_\mu$ is an isometric isomorphism between $M(K)$ and the multiplier algebra $M(L^1(K))$ [3, Theorem 1.6.24]. Moreover, the map $\mu \mapsto F_\mu$, where $F_\mu(g) = \int_K g(k)\,d\mu(k)$ for $g \in C_0(K)$, is an isometric isomorphism from $M(K)$ onto $C_0(K)^*$. The multiplication in $M(K)$ is separately continuous in the w^*-topology, so that Theorem 3.1 applies. ∎

For the next two lemmas, assume that A and B are semisimple commutative Banach algebras and that A has a bounded approximate identity. Then, by Cohen's factorization theorem, the subset

$$A \cdot A^* = \{a \cdot f : a \in A,\ f \in A^*\}$$

of A^* is a closed linear subspace of A^*.

Lemma 3.5. *Let $\varphi : A \to B$ be a homomorphism and suppose that $B\varphi^{**}(e) \subseteq B$. Then $\varphi^*(B \cdot B^*) \subseteq A \cdot A^*$.*

Proof. Towards a contradiction, assume that for some $b \in B$ and $g \in B^*$, $\varphi^*(b \cdot g) \notin A \cdot A^*$. Then, by the Hahn–Banach theorem, there exists $m \in A^{**}$ such that

$$\langle m, \varphi^*(b \cdot g)\rangle = \langle \varphi^{**}(m)b, g\rangle \neq 0,$$

but $\langle ma, f\rangle = 0$ for all $a \in A$ and $f \in A^*$. Thus

$$\varphi^{**}(m)b \neq 0 \quad \text{and} \quad em = \lim_\alpha (e_\alpha m) = 0.$$

Now, $em = 0$ implies that $\varphi^{**}(e)\varphi^{**}(m) = 0$ and hence, for every $x \in I_\varphi$,

$$\varphi^{**}(m)x = \varphi^{**}(m)\varphi^{**}(e)x = 0.$$

On the other hand, since $m = me$ and $\varphi^{**}(m)b \neq 0$,

$$\varphi^{**}(m)\varphi^{**}(e)b = \varphi^{**}(m)b \neq 0.$$

By Theorem 1.4, b can be written as $b = x + y$, where $x \in I_\varphi$ and $y \in J_\varphi$. It follows that

$$\varphi^{**}(m)\varphi^{**}(e)b = \varphi^{**}(m)\varphi^{**}(e)x + \varphi^{**}(m)\varphi^{**}(e)y = \varphi^{**}(m)x = 0.$$

This contradiction shows that $\varphi^*(B \cdot B^*) \subseteq A \cdot A^*$. ∎

We remind the reader that a commutative Banach algebra A is said to be *Arens regular* if A^{**}, equipped with the first Arens product, is commutative. The class of Arens regular algebras is quite large. For instance, it contains all uniform algebras and Arens regularity is inherited by quotient algebras and closed subalgebras. We now present another situation in which the conclusion of Lemma 3.5 can be drawn.

LEMMA 3.6. *Let $\varphi : A \to B$ be a homomorphism such that $A/\ker\varphi$ is Arens regular. Then $\varphi^*(B \cdot B^*) \subseteq A \cdot A^*$.*

Proof. Note first that since the quotient algebra $A^{**}/(\ker\varphi)^{**}$ is isomorphic to $(A/\ker\varphi)^{**}$ and $(A/\ker\varphi)^{**}$ is commutative, for any two elements m and n of A^{**}, we have $mn - nm \in (\ker\varphi)^{**}$. Since $(\ker\varphi)^{**} \subseteq \ker(\varphi^{**})$, we get $\varphi^{**}(m)\varphi^{**}(n) = \varphi^{**}(n)\varphi^{**}(m)$. Taking for n a right identity e of A^{**}, it follows that $\varphi^{**}(m) = \varphi^{**}(e)\varphi^{**}(m) = \varphi^{**}(em)$ for all $m \in A^{**}$. Towards a contradiction, assume that there exist $b \in B$ and $g \in B^*$ such that $\varphi^*(b \cdot g) \notin A \cdot A^*$. Now, exactly as in the proof of Lemma 3.5 we see that there exists $q \in A^{**}$ such that $aq = 0$ for all $a \in A$ and $b\varphi^{**}(q) \neq 0$. Since the map $m \mapsto mq$ of A^{**} is w^*-continuous, it follows that $eq = 0$ and hence $\varphi^{**}(q) = \varphi^{**}(eq) = 0$. This contradicts $b\varphi^{**}(q) \neq 0$. ∎

Recall that a Banach space E is *weakly sequentially complete* if every weak Cauchy sequence in E is weakly convergent.

PROPOSITION 3.7. *Suppose that A has a sequential bounded approximate identity and that B is weakly sequentially complete. Then, for any homomorphism $\varphi : A \to B$, $B\varphi^{**}(e) \subseteq B$ if and only if $\varphi^*(B \cdot B^*) \subseteq A \cdot A^*$. If this is the case, then φ extends to a homomorphism $\phi : M(A) \to M(B)$.*

Proof. In view of Lemma 3.5 we only have to show that $\varphi^*(B \cdot B^*) \subseteq A \cdot A^*$ implies that $B\varphi^{**}(e) \subseteq B$. Let $(e_n)_{n\in\mathbb{N}}$ be a bounded approximate identity for A and let b be an arbitrary element of B. We claim that the sequence $(b\varphi(e_n))_n$ is weakly Cauchy in B. To see this, let $f \in B^*$ be given. Then

$$\langle b\varphi(e_n), f\rangle = \langle \varphi(e_n), b \cdot f\rangle = \langle e_n, \varphi^*(b \cdot f)\rangle.$$

Since, by hypothesis, $\varphi^*(b \cdot f) = a \cdot g$ for some $a \in A$ and $g \in A^*$,

$$\langle e_n, \varphi^*(b \cdot f)\rangle = \langle e_n, a \cdot g\rangle = \langle ae_n, g\rangle \to \langle a, g\rangle.$$

This proves that the sequence $(b\varphi(e_n))_n$ is weakly Cauchy in B. Since B is weakly sequentially complete, $(b\varphi(e_n))_n$ converges weakly to some element b_0 of B. On the other hand, the element $b\varphi^{**}(e)$ is a w^*-cluster point of $(b\varphi(e_n))_n$ in B^{**}. It follows that $b_0 = b\varphi^{**}(e)$. Since b was arbitrary, this proves that $B\varphi^{**}(e) \subseteq B$. In this case the last claim follows from Theorem 2.1. ■

As an immediate consequence of Proposition 3.7 and Theorem 2.1 we obtain

THEOREM 3.8. *Let A and B be semisimple commutative Banach algebras with bounded approximate identities. Suppose that B is weakly sequentially complete and that A has a sequential bounded approximate identity. In addition, assume that A satisfies $A^* = A \cdot A^*$. Then every homomorphism from A to B extends to a homomorphism from $M(A)$ to $M(B)$.*

Note that $A^* = A\cdot A^*$ when, for instance, A is Arens regular [30, Theorem 3.1].

To demonstrate the applicability of Theorem 3.8, we list some classes of semisimple commutative Banach algebras that are weakly sequentially complete.

(1) For any locally compact group G, $L^1(G)$ and $M(G)$ are weakly sequentially complete. Similarly, for any locally compact group G, $A(G)$ and $B(G)$ are weakly sequentially complete. This follows from the well known fact that preduals of von Neumann algebras are weakly sequentially complete (see [29, Chapter III, Corollary 5.2]).

(2) For a compact group G and $1 < p < \infty$, the Figà-Talamanca–Herz algebra $A_p(G)$ is weakly sequentially complete. This follows from Lemma 18 of [12].

(3) The projective tensor product $A \widehat{\otimes} B$ of A and B is weakly sequentially complete if both A and B are and at least one of the spaces has an unconditional basis [20, Théorème 1].

(4) If E is a weakly sequentially complete Banach space, then so is every closed subspace F of E. This follows readily from the Hahn–Banach extension theorem.

As in the previous sections, let A and B be semisimple commutative Banach algebras, and suppose that both A and B have bounded approximate identities. Let $e \in E(A)$ and $u \in E(B)$. Recall that for $a \in A$ and $f \in A^*$, $a \cdot f \in A^*$ is defined by $\langle a \cdot f, x\rangle = \langle f, ax\rangle$, $x \in A$. We embed $M(A)$ into A^{**} and $M(B)$ into B^{**} by

$$T(a) = a \cdot T^{**}(e) \quad \text{and} \quad S(b) = b \cdot S^{**}(u),$$

$a \in A$, $b \in B$, respectively. Moreover, suppose that there exists a closed subspace X of B^* with the following properties:

(1) $B \cdot X \subseteq X$ and X is w^*-dense in B^*.
(2) X^* identifies naturally with $M(B)$ in the sense that given any $F \in X^*$, there exists a unique $S \in M(B)$ such that

$$\langle F, f\rangle = \langle S^{**}(u), f\rangle \quad (f \in X).$$

Note that $B \cdot X = X$ since B has a bounded approximate identity. If $f \in X$ is written as $f = b \cdot g$ with $b \in B$ and $g \in X$, then

$$\langle S^{**}(u), f\rangle = \langle b \cdot S^{**}(u), g\rangle = \langle S(b), g\rangle.$$

The next theorem is less general than Theorem 3.1, but it has the advantage of being constructive. This is its main feature.

THEOREM 3.9. *Let A and B be as before and suppose that there exists a subspace X of B^* such that $M(B)$ identifies with X^* in the above sense. Then every homomorphism $\varphi : A \to B$ extends to a homomorphism $\phi : M(A) \to M(B)$.*

Proof. For $T \in M(A)$, define $F_T : X \to \mathbb{C}$ by

$$F_T(f) = \langle \varphi^{**}(T^{**}(e)), f\rangle \quad (f \in X).$$

Since $\varphi^{**}(T^{**}(e)) \in B^{**}$, F_T is a continuous linear functional on X. Thus, by (2), there exists a unique $S_T \in M(B)$ such that

$$\langle S_T^{**}(u), f\rangle = \langle \varphi^{**}(T^{**}(e)), f\rangle$$

for all $f \in X$. We claim that the mapping

$$\phi : M(A) \to M(B), \quad T \mapsto S_T,$$

is a homomorphism extending φ. The following proof is similar to that of Theorem 2.1(ii). Notice first that, since $B \cdot X \subseteq X$, for $a \in A$, $b \in B$ and $f \in X$ we have

$$\begin{aligned}\langle \phi(L_a)(b), f\rangle &= \langle S_{L_a}(b), f\rangle = \langle S_{L_a}^{**}(u), b \cdot f\rangle = \langle \varphi^{**}(L_a^{**}(e)), b \cdot f\rangle \\ &= \langle b \cdot \varphi^{**}(L_a^{**}(e)), f\rangle = \langle \varphi(a)b, f\rangle = \langle L_{\varphi(a)}(b), f\rangle.\end{aligned}$$

Since X is w^*-dense in B^*, we conclude that $\phi(L_a) = L_{\varphi(a)}$. For $T_1, T_2 \in M(A)$ and $b \in B$, since $b \cdot \varphi^{**}(T_1^{**}(e)) = S_{T_1}(b) \in B$,

$$\begin{aligned}S_{T_1 \circ T_2}(b) &= b \cdot S_{T_1 \circ T_2}^{**}(u) = b \cdot \varphi^{**}((T_1 \circ T_2)^{**}(e)) = b \cdot \varphi^{**}(T_1^{**}(e)T_2^{**}(e)) \\ &= S_{T_2}(b \cdot \varphi^{**}(T_1^{**}(e))) = S_{T_2}(S_{T_1}(b)) = S_{T_1} \circ S_{T_2}(b).\end{aligned}$$

It is very easy to check that $S_{T_1+T_2} = S_{T_1} + S_{T_2}$. Thus ϕ is a homomorphism extending φ. ∎

The following corollary is an interesting application of Theorem 3.9.

COROLLARY 3.10. *Suppose that A and B are semisimple commutative Banach algebras with bounded approximate identities and that B is an ideal in B^{**}. Then every homomorphism $\varphi : A \to M(B)$ extends to a homomorphism $\phi : M(A) \to M(B)$.*

Proof. Let $X = B \cdot B^*$. Then X is closed and w^*-dense in B^* since B has a bounded approximate identity. As above, fix $u \in E(B)$ and embed $M(B)$ into B^{**} by $S(b) = b \cdot S^{**}(u)$, $b \in B$. Given $F \in X^*$, define $S : B \to B^{**}$ by

$$\langle S(b), f\rangle = \langle F, b \cdot f\rangle, \quad f \in B^*,\ b \in B.$$

If $\widetilde{F}$ is any element of B^{**} extending F, then $\langle F, b \cdot f\rangle = \langle b \cdot \widetilde{F}, f\rangle$ and $b \cdot \widetilde{F} \in B$ since B is an ideal in B^{**}. Thus $S(B) \subseteq B$, and it is easily verified that S is a multiplier of B and

$$\langle S^{**}(u), b \cdot f\rangle = \langle F, b \cdot f\rangle$$

for all $f \in B^*$ and $b \in B$. ■

We finish this section by briefly mentioning two further sets of conditions on A and B which guarantee that every homomorphism from A into B extends to a homomorphism from $M(A)$ to $M(B)$.

THEOREM 3.11. *Suppose that one of the following two conditions is satisfied*:

- (i) *B is Arens regular and weakly sequentially complete.*
- (ii) *A is regular and Arens regular, and B is weakly sequentially complete.*

Then, for any homomorphism $\varphi : A \to B$, we have $B = I_\varphi \oplus J_\varphi$ and hence φ extends to a homomorphism from $M(A)$ into $M(B)$.

Proof. In view of Corollary 2.2 it suffices to show that $B = I_\varphi \oplus J_\varphi$. If (i) holds, then by [31, Theorem 3.3] or [6, Theorem 2.9.39], the ideal I_φ is unital. If (ii) holds, then the subalgebra $\overline{\varphi(A)}$ of B is unital by Theorem 4.1 of [31]. Denoting in either case the identity by ε, necessarily the net $(\varphi(e_\alpha))_\alpha$ converges to ε in norm. This readily implies that

$$I_\varphi = \{b \in B : b\varepsilon = b\} \quad \text{and} \quad J_\varphi = \{b \in B : b\varepsilon = 0\}.$$

Since ε is an idempotent, it follows that $B = I_\varphi \oplus J_\varphi$. ■

4. Homomorphisms from A into A. Several authors have studied multipliers with closed range on semisimple commutative Banach algebras A (compare [1], [18], [32] and [33]). In particular, it is shown in [33] that, under a certain assumption on the ideal structure of A, a multiplier $T : A \to A$ with closed range factors as a product of an idempotent multiplier and an invertible multiplier. Theorem 4.2 below, which may be viewed as a polar decomposition theorem, provides an analogous factorization for homomorphisms with closed range and will be applied to completely bounded homomorphisms of Fourier algebras of amenable locally compact groups. In preparation we need the following lemma.

LEMMA 4.1. *Let A be a semisimple commutative Banach algebra with bounded approximate identity and let $\varphi : A \to A$ be a homomorphism satisfying $A = I_\varphi \oplus J_\varphi$. If $\varphi(A)$ is closed in A, then so is $\varphi(I_\varphi)$.*

Proof. Let $(a_n)_n$ be a sequence in I_φ such that $\varphi(a_n) \to y$ for some $y \in A$. As $\varphi(A)$ is closed in A, $y = \varphi(a)$ for some $a \in A$. Let $a = b + c$ be the decomposition of a in the direct sum $A = I_\varphi \oplus J_\varphi$. Then $a_n - b \in I_\varphi$ and

$$\varphi(a_n - b) = \varphi(a_n) + \varphi(c) - \varphi(a) \to \varphi(c).$$

We claim that $\varphi(c) = 0$. Indeed, since A is semisimple, it suffices to show that $\langle \varphi(c), \gamma\rangle = 0$ for every $\gamma \in \Delta(A)$. Recall from Section 1 that $\Delta(A) = Z_\varphi \cup E_\varphi$, $I_\varphi \subseteq k(Z_\varphi)$ and $J_\varphi = k(E_\varphi)$. Now, if $\gamma \in Z_\varphi$ then $\varphi^*(\gamma) = 0$ and hence $\langle \varphi(c), \gamma\rangle = 0$. For $\gamma \in E_\varphi$ we have to distinguish the two cases: $\varphi^*(\gamma) \in E_\varphi$ and $\varphi^*(\gamma) \in Z_\varphi$. In the first case, $\langle \varphi(c), \gamma\rangle = \langle c, \varphi^*(\gamma)\rangle = 0$ since $c \in J_\varphi$. In the second case, $\langle \varphi(a_n - b), \gamma\rangle = \langle a_n - b, \varphi^*(\gamma)\rangle = 0$ since $a_n - b \in I_\varphi$ and hence $\langle \varphi(c), \gamma\rangle = \lim_{n\to\infty}\langle \varphi(a_n - b), \gamma\rangle = 0$.

Finally, $\varphi(c) = 0$ implies that $\varphi(a_n) \to \varphi(b)$. Since $b \in I_\varphi$, we conclude that $\varphi(I_\varphi)$ is closed in A. ∎

THEOREM 4.2. *Let A be a semisimple commutative Banach algebra with bounded approximate identity, and let $\varphi : A \to A$ be a homomorphism such that $A = I_\varphi \oplus J_\varphi$. Then φ decomposes as $\varphi = S \circ \varrho$, where $S : A \to A$ is an idempotent multiplier and ϱ is a homomorphism such that $\varrho^*(\gamma) \neq 0$ for all $\gamma \in \Delta(A)$. Moreover:*

(i) *If $\ker \varphi \subseteq J_\varphi$, then ϱ is a one-to-one homomorphism.*
(ii) *φ has closed range if and only if ϱ has closed range.*

Proof. Let $S : A \to A$ be the idempotent multiplier with $S(A) = I_\varphi$ and $\ker S = J_\varphi$ corresponding to the decomposition $A = I_\varphi \oplus J_\varphi$. To φ and S we associate the mapping $\varrho : A \to A$ defined by

$$\varrho(a) = \varphi(a) + a - S(a), \qquad a \in A.$$

Then, for every $a \in A$, since $\varphi(A) \subseteq I_\varphi$,

$$S(\varrho(a)) = S(\varphi(a) + a - S(a)) = \varphi(a).$$

We claim that ϱ is a homomorphism. Linearity being obvious, let us check multiplicativity. For $a, b \in A$, we have

$$aS(b) = S(a)b = S(ab) = S(a)S(b),$$

since S is an idempotent, and

$$\varphi(a)S(b) = S(\varphi(a)b) = \varphi(a)S(b) \quad \text{and} \quad S(a)\varphi(b) = S(a\varphi(b)) = a\varphi(b),$$

since $\varphi(a)b \in I_\varphi$ and $a\varphi(b) \in I_\varphi$. From these equations, it follows that

$$\begin{aligned}\varrho(a)\varrho(b) &= \varphi(a)\varphi(b) + \varphi(a)b - \varphi(a)S(b) + a\varphi(b) \\ &\qquad + ab - aS(b) - S(a)\varphi(b) - S(a)b + S(a)S(b) \\ &= \varphi(ab) + ab - S(ab) = \varrho(ab).\end{aligned}$$

Since S is a multiplier, $S^*(\gamma) = \widehat{S}(\gamma)\gamma$ for every $\gamma \in \Delta(A)$, where $\widehat{S}$ is the Gelfand transform of S. Then we get

$$\varrho^*(\gamma) = \varphi^*(\gamma) + \gamma - \widehat{S}(\gamma)\gamma, \quad \gamma \in \Delta(A).$$

Since S is an idempotent which is zero on J_φ and the identity on I_φ, we have $\widehat{S}(\gamma) = 0$ if $\gamma \in Z_\varphi$ and $\widehat{S}(\gamma) = 1$ for $\gamma \in E_\varphi$. This implies that $\varrho^*(\gamma) = \varphi^*(\gamma)$ for $\gamma \in E_\varphi$ and $\varrho^*(\gamma) = \gamma$ for $\gamma \in Z_\varphi$. In particular, $\varrho^*(\gamma) \neq 0$ for all $\gamma \in \Delta(A)$.

Now assume that $\ker\varphi \subseteq J_\varphi$. To show that ϱ is one-to-one, let $a \in A$ be such that $\varphi(a) + a - S(a) = 0$. Applying S, we get $\varphi(a) = 0$ and hence $a \in J_\varphi$ by hypothesis. Since $\varphi(a) = 0$, we get $a = S(a)$, which implies that $a \in I_\varphi$. It follows that $a = 0$ since $I_\varphi \cap J_\varphi = \{0\}$. This shows (i).

For (ii), suppose that φ has closed range. To show that ϱ has closed range, let $(a_n)_n$ be a sequence in A such that

$$\varrho(a_n) = \varphi(a_n) + a_n - S(a_n) \to a$$

for some $a \in A$. Applying S, we get $\varphi(a_n) \to S(a)$. Since $\varphi(A)$ is closed in A, there exists $b \in A$ such that $S(a) = \varphi(b)$. Now, for an arbitrary element x of A, let $x = x' + x''$ be the decomposition of x in the direct sum $A = I_\varphi \oplus J_\varphi$. Then, by the definition of S,

$$\varrho(a_n) = \varphi(a_n) + a_n' + a_n'' - S(a_n' + a_n'') = \varphi(a_n) + a_n'',$$

and hence, since $\varrho(a_n) \to a$ and $\varphi(a_n) \to S(a)$,

$$\varrho(a_n'') = \varrho(\varrho(a_n) - \varphi(a_n)) \to \varrho(a - S(a)) = \varrho(a - \varphi(b)).$$

This in turn implies that

$$\varphi(a_n') = \varrho(a_n') = \varrho(a_n) - \varrho(a_n'')$$

converges. Since $a_n' \in I_\varphi$ and the range of φ is closed by hypothesis, by the preceding lemma there exists $c \in I_\varphi$ such that $\varrho(a_n') \to \varphi(c) = \varrho(c)$. It follows that

$$\varrho(a_n) = \varrho(a_n') + \varrho(a_n'') \to \varrho(c) + \varrho(a - \varphi(b)) = \varrho(c + a - \varphi(b)),$$

as required.

Conversely, if ϱ has closed range, then so does $\varphi = S \circ \varrho$ since S is an idempotent. ∎

Let G be an amenable locally compact group. Given any homomorphism $\varphi : A(G) \to A(G)$, by Theorem 3.1 there exists an idempotent $u \in B(G)$

such that $I_\varphi = A(G)u$, and conversely, given an idempotent $u \in B(G)$, the homomorphism $\varphi : v \mapsto vu$ has the property that $I_\varphi = A(G)u$. If the homomorphism is completely bounded, we have a more precise result (Corollary 4.3 below). Before passing on to this, we briefly recall the notion of a completely bounded homomorphism. A linear map $T : C \to D$ between C^*-algebras is *completely bounded* if all its amplifications $T^{(n)} : M_n(C) \to M_n(D)$, defined by $T^{(n)}([c_{ij}]) = [T(c_{ij})]$, give a bounded family of norms $\|T^{(n)}\|$, $n \in \mathbb{N}$. Now, $A(G)$ being the predual of the group von Neumann algebra $\mathrm{VN}(G)$, a homomorphism $\varphi : A(G) \to A(G)$ is called *completely bounded* if $\varphi^* : \mathrm{VN}(G) \to \mathrm{VN}(G)$ is completely bounded. When G is amenable, in Theorem 3.13 of [15] the range of a completely bounded homomorphism has been identified as the set

$$L_\varphi = \{a \in I_\varphi : \text{for } \gamma_1, \gamma_2 \in E_\varphi,\ \varphi^*(\gamma_1) = \varphi^*(\gamma_2) \Rightarrow \widehat{a}(\gamma_1) = \widehat{a}(\gamma_2)\}.$$

This description shows that the range of φ is closed and it generalizes considerably the corresponding result for homomorphisms between L^1-algebras of locally compact abelian groups [16].

Corollary 4.3. *Let G be an amenable locally compact group and let $\varphi : A(G) \to A(G)$ be a homomorphism such that* $\ker\varphi \subseteq J_\varphi$. *Then*:

(i) *Suppose that φ has closed range. Then φ is of the form $\varphi(u) = w\varrho(u)$, $u \in A(G)$, where w is an idempotent in $B(G)$ and $\varrho : A(G) \to A(G)$ is a one-to-one homomorphism with closed range.*

(ii) *If φ is completely bounded, then ϱ can be chosen to be completely bounded and hence to have closed range.*

Proof. (i) If we view φ as a homomorphism, ψ say, of $A(G)$ into $B(G)$, Theorem 3.1 shows that

$$B(G) = I_\psi \oplus J_\psi = B(G)u \oplus B(G)(1-u),$$

where u is a w^*-cluster point of the net $(\varphi(e_\alpha))_\alpha$. This implies that

$$A(G) = A(G)u \oplus A(G)(1-u) = I_\varphi \oplus J_\varphi,$$

and hence Theorem 4.2 applies.

(ii) Let w be the idempotent in $B(G)$ corresponding to the decomposition $A(G) = I_\varphi \oplus J_\varphi$. Then $I_\varphi = A(G)w$ and $J_\varphi = A(G)(1-w)$. Now, let ϱ be defined as in the proof of Theorem 4.2, that is, $\varrho(u) = \varphi(u) + u - uw$, $u \in A(G)$. Then ϱ is completely bounded since both φ and the homomorphism $u \mapsto uw$ are completely bounded. Moreover, since φ has closed range, so does ϱ by Theorem 4.2. ∎

Remark 4.4. Let A be a semisimple commutative Banach algebra with bounded approximate identity, and let $\varphi : A \to A$ be a homomorphism with closed range such that $A = I_\varphi \oplus \ker\varphi$. The same proof as for Theorem 4.2 shows that φ factors as $\varphi = S \circ \varrho$ where S is an idempotent multiplier and $\varrho : A \to A$ is a one-to-one homomorphism with closed range such that $\varrho^*(\gamma) \neq 0$ for all $\gamma \in \Delta(A)$.

In the following we characterize certain properties of a homomorphism $\varphi : A \to A$ in terms of the set E_φ and the adjoint mapping φ^*.

LEMMA 4.5. *Let $\varphi : A \to A$ be a homomorphism and suppose that A is regular. Then*:

(i) $\ker \varphi = J_\varphi$ *if and only if* $\varphi^*(E_\varphi) = \overline{E}_\varphi$.
(ii) φ *is injective if and only if* $\varphi^*(E_\varphi) = \Delta(A)$.

Proof. Recall that by Lemma 1.2(ii), $\varphi^*(E_\varphi)$ is closed in $\Delta(A)$ and $k(\varphi^*(E_\varphi)) = \ker \varphi$. Thus, since $J_\varphi = k(E_\varphi)$ (Lemma 1.2(i)), $J_\varphi = \ker \varphi$ if and only if $k(\overline{E}_\varphi) = k(E_\varphi) = k(\varphi^*(E_\varphi))$, and this in turn is equivalent to $\overline{E}_\varphi = \varphi^*(E_\varphi)$ since A is regular. This shows (i).

As for (ii), if $\varphi^*(E_\varphi) = \Delta(A)$ then $\ker \varphi = k(\Delta(A)) = \{0\}$ since A is semisimple. Conversely, if $\{0\} = \ker \varphi = k(\varphi^*(E_\varphi))$ then $\varphi^*(E_\varphi) = \Delta(A)$ since $\varphi^*(E_\varphi)$ is closed in $\Delta(A)$ and A is regular. ∎

LEMMA 4.6. *Let $\varphi : A \to A$ be a homomorphism satisfying $I_\varphi \oplus J_\varphi = A$, and let*

$$L_\varphi = \{a \in I_\varphi : \textit{for } \gamma_1, \gamma_2 \in E_\varphi,\ \varphi^*(\gamma_1) = \varphi^*(\gamma_2) \Rightarrow \widehat{a}(\gamma_1) = \widehat{a}(\gamma_2)\}.$$

Then:

(i) $L_\varphi = I_\varphi$ *if and only if φ^* is one-to-one on E_φ.*
(ii) $L_\varphi = A$ *if and only if $E_\varphi = \Delta(A)$ and φ^* is one-to-one on $\Delta(A)$.*

Proof. (i) Suppose first that $L_\varphi = I_\varphi$ and let $\gamma_1, \gamma_2 \in E_\varphi$ be such that $\varphi^*(\gamma_1) = \varphi^*(\gamma_2)$. Then either both γ_1 and γ_2 are in E_φ or both belong to Z_φ. In the first case, $\gamma_1 = \gamma_2$ by hypothesis and hence $\widehat{a}(\gamma_1) = \widehat{a}(\gamma_2)$. In the second case, $\varphi^*(\gamma_1) = \varphi^*(\gamma_2) = 0$ and this implies

$$\widehat{a}(\gamma_j) = \lim_\alpha \widehat{a}(\gamma_j)\varphi^*(\gamma_j)(e_\alpha) = 0$$

for $j = 1, 2$. This shows that $a \in L_\varphi$.

(ii) If $L_\varphi = A$ then $I_\varphi = A$ and hence $Z_\varphi = h(I_\varphi) = \emptyset$, so that $E_\varphi = \Delta(A)$. If $\gamma_1, \gamma_2 \in E_\varphi$ are such that $\varphi^*(\gamma_1) = \varphi^*(\gamma_2)$ then, by hypothesis, $\widehat{a}(\gamma_1) = \widehat{a}(\gamma_2)$ for all $a \in A$, whence $\gamma_1 = \gamma_2$.

Conversely, let $E_\varphi = \Delta(A)$ and let φ^* be one-to-one on $\Delta(A)$. Then, by (i), $L_\varphi = I_\varphi$. Moreover, $J_\varphi = k(E_\varphi) = \{0\}$ since A is semisimple and hence $L_\varphi = I_\varphi \oplus J_\varphi = A$. ∎

Let $\varphi : A \to A$ be a homomorphism. To avoid long paraphrasing, let us say that φ is *similar to a multiplier* $T : A \to A$ if $\varphi(A) = T(A)$ and $\ker \varphi = \ker T$.

Corollary 4.7. *Let G be an amenable locally compact group and $\varphi : A(G) \to A(G)$ a completely bounded homomorphism. Then*:

(i) *φ is similar to an idempotent multiplier if and only if φ^* is one-to-one on the set $E_\varphi = \{x \in G : \varphi^*(\gamma_x) \neq 0\}$ and $\varphi^*(E_\varphi) = E_\varphi$.*
(ii) *φ is surjective if and only if $E_\varphi = G$ and φ^* is one-to-one on G.*
(iii) *φ is injective if and only if $\varphi^*(E_\varphi) = G$.*

Proof. Suppose φ^* is one-to-one on the set E_φ and $\varphi^*(E_\varphi) = E_\varphi$. Then $L_\varphi = I_\varphi$ by Lemma 4.6(i), and hence $\varphi(A(G)) = I_\varphi$. Since $I_\varphi \oplus J_\varphi = A(G)$, the set E_φ is closed. Since by hypothesis $\varphi^*(E_\varphi) = E_\varphi$, Lemma 4.5(i) shows that $\ker\varphi = J_\varphi$. Now both $\varphi(A(G)) = I_\varphi$ and $\ker\varphi = J_\varphi$ are closed ideals and $I_\varphi \oplus J_\varphi = A(G)$. Hence any projection inducing this decomposition is an idempotent multiplier. So φ is similar to an idempotent multiplier on $A(G)$. The reverse implication is obvious.

The assertions (ii) and (iii) are immediate consequences of Lemmas 4.5 and 4.6. ∎

When G is abelian, every homomorphism of $L^1(G) = A(\widehat{G})$ is completely bounded, and hence Corollaries 4.3 and 4.7 apply to homomorphisms from $L^1(G)$ into $L^1(G)$.

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