STUDIA MATHEMATICA 157 (1) (2003)

The Hypercyclicity Criterion for sequences of operators

by

L. Bernal-González (Sevilla) and K.-G. Grosse-Erdmann (Hagen)

Abstract. We show that under no hypotheses on the density of the ranges of the mappings involved, an almost-commuting sequence (T_n) of operators on an F-space X satisfies the Hypercyclicity Criterion if and only if it has a hereditarily hypercyclic subsequence (T_{n_k}), and if and only if the sequence $(T_n \oplus T_n)$ is hypercyclic on $X \times X$. This strengthens and extends a recent result due to Bès and Peris. We also find a new characterization of the Hypercyclicity Criterion in terms of a condition introduced by Godefroy and Shapiro. Finally, we show that a weakly commuting hypercyclic sequence (T_n) satisfies the Hypercyclicity Criterion whenever it has a dense set of points with precompact orbits. We remark that some of our results are new even in the case of iterates (T^n) of a single operator T.

1. Introduction. Throughout this paper, X will denote a separable F-space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, where an *F-space* is a topological vector space whose topology is induced by a complete translation-invariant metric. Let $L(X)$ denote the space of all operators on X, that is, all continuous linear mappings $X \to X$. Then an operator $T \in L(X)$ is called *hypercyclic* whenever there exists some $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x under T is dense in X. In this case the vector x is also called *hypercyclic*. The theory of hypercyclic operators has recently been studied intensively. We refer to the comprehensive survey [19]; see also [15, Section 1].

In this paper we shall study, more generally, an arbitrary sequence (T_n) of operators on X. Then (T_n) is called *hypercyclic* provided there exists some $x \in X$ such that $\{T_n x : n \in \mathbb{N}\}$ is dense in X. Observe that X must be separable in order to support such a sequence. We continue to refer to the set $\{T_n x : n \in \mathbb{N}\}$ as the *orbit* of x under (T_n). A vector x with dense orbit under (T_n) is called *hypercyclic for* (T_n). This more general

2000 *Mathematics Subject Classification*: Primary 47A16; Secondary 46A16, 47B33, 54H20.

Key words and phrases: hypercyclic operator, hypercyclic sequence, hereditarily hypercyclic sequence, Hypercyclicity Criterion, almost-commuting sequence, chaotic sequence.

Research of L. Bernal-González partially supported by Plan Andaluz de Investigación de la Junta de Andalucía.

notion of hypercyclicity is also sometimes referred to as *universality* (see [19, Section 1]).

Moreover, the sequence $(T_n) \subset L(X)$ is called *densely hypercyclic* whenever the set of its hypercyclic vectors is dense in X. It is called *hereditarily hypercyclic* whenever each subsequence (T_{n_k}) is hypercyclic, and *densely hereditarily hypercyclic* whenever each subsequence is densely hypercyclic; cf. [5] and [19, Section 2], but note that Bès and Peris [11] use a different notion of hereditary hypercyclicity. Corresponding concepts can be defined for a single operator $T \in L(X)$ by looking at its sequence of iterates.

1.1. *The Hypercyclicity Criterion.* This criterion, which gives sufficient conditions under which a sequence (T_n) is hypercyclic, has turned out to be extremely useful in applications.

DEFINITION 1.1. A sequence $(T_n) \subset L(X)$ satisfies the *Hypercyclicity Criterion* provided there exist dense subsets X_0 and Y_0 of X and an increasing sequence (n_k) of positive integers satisfying the following two conditions:

(i) $T_{n_k}x \to 0$ $(k \to \infty)$ for all $x \in X_0$;

(ii) for any $y \in Y_0$ there is a sequence (u_k) in X such that $u_k \to 0$ and $T_{n_k}u_k \to y$ $(k \to \infty)$.

Note that this is an equivalent reformulation of the Hypercyclicity Criterion as stated in [11, Definition 1.2 and Remark 2.6]. The criterion evolved from earlier versions due to Kitai [23] and Gethner and Shapiro [14, Remark 2.3]; see also [18] and [15, Corollary 1.4].

As before, an operator T is said to satisfy the *Hypercyclicity Criterion* provided the sequence (T^n) of its iterates satisfies it. It is worth to comment here that H. Salas [29] and D. Herrero [20] have shown that there are hypercyclic operators (on Hilbert space) that do not satisfy the Hypercyclicity Criterion for the *full* sequence $(n_k) = (1, 2, 3, \ldots)$, but so far no hypercyclic operator has been found that does not satisfy the Hypercyclicity Criterion as stated in Definition 1.1. This has led to the following question (see [11] and [25]), which can be distinguished as "the great open problem" in hypercyclicity: *Does every hypercyclic operator satisfy the Hypercyclicity Criterion?*

REMARK 1.2. Formally, one may further weaken the Hypercyclicity Criterion. The criterion remains a sufficient condition for hypercyclicity when property (i) is replaced by the following weaker property:

(i′) for each $x \in X_0$, $(T_{n_k}x)$ has a convergent subsequence.

Furthermore, if (i′) and (ii) are satisfied then (T_n) has a dense set of hypercyclic vectors. We shall need this result, which can be found in [19, Theorem 2 with Remark 2], in Section 4. For the sake of completeness, we provide the following proof. Assume that a pair U, V of non-empty open subsets of X is given. Since X_0 is dense, we can select a vector $x \in X_0 \cap U$, so that $T_{m_k}x \to a$ as $k \to \infty$ for some subsequence (m_k) of (n_k) and some $a \in X$. By the density of Y_0 there exists $y \in Y_0 \cap (V - a)$. Hence we can find a sequence $(u_k) \subset X$ such that $u_k \to 0$ and $T_{m_k}u_k \to y$ as $k \to \infty$. By linearity we obtain $T_{m_k}(x + u_k) \to a + y$. Since U and V are open sets with $x \in U$ and $a + y \in V$ we can find some $k_0 \in \mathbb{N}$ with $x + u_k \in U$ and $T_{m_k}(x + u_k) \in V$, hence $T_{m_k}(U) \cap V \neq \emptyset$, for $k \geq k_0$. The result is then derived via an application of Lemma 2.1 (see Section 2).

We note that by a recent result of Bermúdez, Bonilla and Peris [4, Theorem 2.2] the above weakened form of the Hypercyclicity Criterion is equivalent to the original criterion in the case of iterates (T^n) of a single operator T.

Bès and Peris [11, Theorem 2.3] have shown that an operator T satisfies the Hypercyclicity Criterion if and only if some subsequence (T^{n_k}) is hereditarily hypercyclic (in the sense of the introduction), and if and only if $T \oplus T$ is hypercyclic, where

$$T \oplus T : X \times X \to X \times X, \qquad (T \oplus T)(x_1, x_2) = (Tx_1, Tx_2).$$

This is of great interest because it shows the equivalence of Herrero's problem [21] of whether $T \oplus T$ is hypercyclic whenever T is to the problem, mentioned above, of whether every hypercyclic operator satisfies the Hypercyclicity Criterion.

Bès and Peris have generalized their result to certain sequences (T_n) of operators [11, Remark 2.6(3)].

THEOREM 1.3 (Bès, Peris). *Let (T_n) be a commuting sequence (that is, $T_nT_m = T_mT_n$ for all $m, n \in \mathbb{N}$) of operators in $L(X)$ with dense range. Then the following assertions are equivalent*:

(A) *(T_n) satisfies the Hypercyclicity Criterion.*

(B) *(T_n) has a hereditarily hypercyclic subsequence.*

(C) *$(T_n \oplus T_n)$ is hypercyclic on $X \times X$.*

Note that, trivially, if $T \in L(X)$ is hypercyclic then its sequence of iterates (T^n) is commuting and each power T^n has dense range. Peris has shown (cf. [19, Proposition 1]) that if (T_n) is a hypercyclic commuting sequence of operators with dense range then it is in fact densely hypercyclic. Since $(T_n \oplus T_n)$ is a commuting sequence of operators with dense range whenever (T_n) is, we conclude that the following assertions are also equivalent to the assertions in the theorem:

(B′) *(T_n) has a densely hereditarily hypercyclic subsequence.*

(C′) *$(T_n \oplus T_n)$ is densely hypercyclic on $X \times X$.*

1.2. *Chaos.* Following Godefroy and Shapiro [15], Devaney's notion of chaos [13, p. 50] has been generally accepted in the theory of hypercyclicity. Thus, an operator T on an F-space is called *chaotic* if it has a dense orbit, a dense set of periodic points, and a property called sensitive dependence on initial conditions. We propose here the following definition of chaos for sequences of operators. By d we denote a translation-invariant metric on X.

Definition 1.4. Let $(T_n) \subset L(X)$. Then (T_n) is called *chaotic* if the following three conditions are satisfied:

(i) (T_n) is densely hypercyclic;

(ii) (T_n) has a dense set of *periodic points*, that is, points $x \in X$ for which there is some $p \in \mathbb{N}$ with $T_{n+p}x = T_n x$ for all $n \in \mathbb{N}$;

(iii) (T_n) has *sensitive dependence on initial conditions*, that is, there exists a $\delta > 0$ such that for all $x \in X$ and $\varepsilon > 0$ there is a point $y \in X$ with $d(x, y) < \varepsilon$ such that $d(T_n x, T_n y) > \delta$ for some $n \in \mathbb{N}$.

Note that this reduces to Devaney's notion of chaos for sequences (T^n) of iterates of a single operator T. Godefroy and Shapiro [15, Proposition 6.1] have shown that, in fact, every hypercyclic operator on an F-space has sensitive dependence on initial conditions (cf. also [3]). Since their proof also works for general densely hypercyclic sequences (T_n) we see that *a sequence (T_n) is chaotic if and only if it is densely hypercyclic and has a dense set of periodic points.*

The reader is referred to [12] for a concept of chaos for operators on locally convex spaces that are not necessarily F-spaces.

1.3. *Outline of the paper.* In Section 2 we characterize the sequences (T_n) of operators that satisfy the Hypercyclicity Criterion without assuming that the T_n commute or have dense range. The result will then be applied to composition operators on spaces of holomorphic functions.

In Section 3 we characterize the Hypercyclicity Criterion for almost-commuting sequences of operators. In particular we shall extend the result of Bès and Peris to this class of operators, without any assumption on the density of ranges. In addition, we obtain a new characterization of the Hypercyclicity Criterion in terms of a condition studied by Godefroy and Shapiro.

Finally, in Section 4 we show that every hypercyclic weakly commuting sequence (T_n) having a dense set of points with precompact orbit satisfies the Hypercyclicity Criterion. As a special case we see that every weakly commuting chaotic sequence (T_n) satisfies the Hypercyclicity Criterion.

We want to emphasize that some of our results are new even in the special case of a (single) hypercyclic operator T.

2. The Hypercyclicity Criterion for arbitrary sequences. We begin with a characterization of the Hypercyclicity Criterion for general sequences of operators. For the proof of this result we need the following characterization of dense hypercyclicity for sequences of operators [17, Satz 1.2.2] (see also [15, Theorem 1.2] and [11, Lemma 2.5]).

LEMMA 2.1. *For any sequence (T_n) of operators on X the following assertions are equivalent*:

(i) *The sequence (T_n) is densely hypercyclic.*

(ii) *For every pair U, V of non-empty open subsets of X there are infinitely many $n \in \mathbb{N}$ with $T_n(U) \cap V \neq \emptyset$.*

(iii) *The set of hypercyclic vectors for (T_n) is residual*, *that is*, *its complement is of first category.*

We remark that assertion (ii) is equivalent to the apparently weaker assertion that for all non-empty open sets U and V there is *some* $n \in \mathbb{N}$ with $T_n(U) \cap V \neq \emptyset$.

We are now ready to state the main result of this section.

THEOREM 2.2. *Let (T_n) be a sequence of operators on X. Then the following assertions are equivalent*:

(A) *(T_n) satisfies the Hypercyclicity Criterion.*

(B) *(T_n) has a densely hereditarily hypercyclic subsequence.*

(C_{fin}) *For every $N \in \mathbb{N}, (T_n \oplus \ldots \oplus T_n)$ (N-fold) is densely hypercyclic on X^N.*

Proof. (A)$\Rightarrow$(B). If (n_k) is the sequence of positive integers appearing in the Hypercyclicity Criterion then, obviously, every subsequence of (T_{n_k}) also satisfies the criterion. Hence, (T_{n_k}) is densely hereditarily hypercyclic (cf. Remark 1.2).

(B)$\Rightarrow$(C_{fin}). Fix $N \in \mathbb{N}$ and let $U_1, \ldots, U_N$ and $V_1, \ldots, V_N$ be non-empty open subsets of X. Let (T_{n_k}) be densely hereditarily hypercyclic. By Lemma 2.1 there is a subsequence $(n_k^{(1)})$ of (n_k) such that $T_{n_k^{(1)}}(U_1) \cap V_1 \neq \emptyset$ for all k. Since, by heredity, $(T_{n_k^{(1)}})$ is also densely hypercyclic there is a subsequence $(n_k^{(2)})$ of $(n_k^{(1)})$ such that $T_{n_k^{(2)}}(U_2) \cap V_2 \neq \emptyset$ for all k, and we also have $T_{n_k^{(2)}}(U_1) \cap V_1 \neq \emptyset$ for all k. Continuing in the same way we obtain a subsequence $(m_k) = (n_k^{(N)})$ of (n_k) with $T_{m_k}(U_i) \cap V_i \neq \emptyset$ for $1 \leq i \leq N$ and all k, hence $(T_{m_k} \oplus \ldots \oplus T_{m_k})(U_1 \times \ldots \times U_N) \cap (V_1 \times \ldots \times V_N) \neq \emptyset$ for all k. Now, Lemma 2.1 implies (C_{fin}).

(C_{fin})$\Rightarrow$(A). We fix a base (U_k) of neighbourhoods of zero in X with $U_{k+1} \pm U_{k+1} \subset U_k$ for all k and a sequence (y_k) that is dense in X. By induction on k we shall show that there is an increasing sequence (n_k) of positive integers, neighbourhoods V_k and W_k of zero ($k \in \mathbb{N}$) and vectors $x_{j,k}, u_{j,k} \in X$ for $k \in \mathbb{N}$ and $1 \le j \le k$ such that

$$W_k + W_k \subset V_k \cap W_{k-1} \quad \text{for } k \ge 2, \tag{1}$$

$$\overline{V}_k \subset U_k, \tag{2}$$

$$x_{j,k} \in x_{j,k-1} + W_{k-1} \quad \text{if } j \ne k, \quad x_{k,k} \in y_k + U_k, \tag{3}$$

$$u_{j,k} \in U_k, \tag{4}$$

$$T_{n_k}(x_{j,k} + \overline{V}_k) \subset U_k, \tag{5}$$

$$T_{n_k} u_{j,k} \in x_{j,k} + U_k. \tag{6}$$

We only show how to construct the objects with index k from those with index $k-1$, $k \ge 2$; the initial construction for $k = 1$ is similar.

Thus, fix $k \ge 2$. Since $(T_n \oplus \ldots \oplus T_n)$ ($2k$-fold) is densely hypercyclic on X^{2k} there is a hypercyclic vector $(x_{1,k}, \ldots, x_{k,k}, u_{1,k}, \ldots, u_{k,k})$ such that (3) and (4) hold for $1 \le j \le k$. By hypercyclicity of this vector there is an integer $n_k > n_{k-1}$ such that

$$T_{n_k} x_{j,k} \in U_k$$

and (6) hold for $1 \le j \le k$. Since T_{n_k} is continuous we can find a neighbourhood V_k of zero satisfying (2) and (5) for $1 \le j \le k$. As a final step we define W_k as a neighbourhood of zero that satisfies (1).

We shall now show that for each $j \in \mathbb{N}$ the sequence $(x_{j,k})_{k \ge j}$ is a Cauchy sequence in X. For $l > k \ge j$ we have, by (3),

$$x_{j,l} \in x_{j,l-1} + W_{l-1} \subset \ldots \subset x_{j,k} + W_k + W_{k+1} + \ldots + W_{l-1}.$$

It is easy to deduce from (1) that $W_k + \ldots + W_{l-1} \subset W_k + W_k \subset V_k$, which implies with (2) that

$$x_{j,l} - x_{j,k} \in V_k \subset U_k$$

for $l > k \ge j$. Since (U_k) is a local base we see that $(x_{j,k})_{k \ge j}$ is a Cauchy sequence and hence converges to some $x_j \in X$. We then have

$$x_j \in x_{j,k} + \overline{V}_k \quad \text{for } k \ge j, \tag{7}$$

hence by (5),

$$T_{n_k} x_j \to 0 \quad \text{as } k \to \infty, \text{ for all } j \in \mathbb{N},$$

giving us condition (i) in the Hypercyclicity Criterion when we set $X_0 := \{x_j : j \in \mathbb{N}\}$.

Next we find that, by (7), (3) and (2),

$$x_j \in x_{j,j} + \overline{V}_j \subset y_j + U_j + \overline{V}_j \subset y_j + U_j + U_j \subset y_j + U_{j-1},$$

which implies that X_0 is a dense subset of X.

Finally, by (6), (7) and (2) we have for $k \ge j$,

$$T_{n_k} u_{j,k} \in x_{j,k} + U_k \subset x_j - \overline{V}_k + U_k \subset x_j - U_k + U_k \subset x_j + U_{k-1},$$

so that also

$$T_{n_k} u_{j,k} \to x_j \quad \text{as } k \to \infty, \text{ for all } j \in \mathbb{N},$$

while (4) implies that

$$u_{j,k} \to 0 \quad \text{as } k \to \infty, \text{ for all } j \in \mathbb{N}.$$

This shows that also condition (ii) in the Hypercyclicity Criterion is satisfied when we set $Y_0 := X_0$. Thus, ($\mathrm{C_{fin}}$) implies (A). ∎

REMARKS 2.3. (a) The proof of the theorem shows that the sequence (n_k) appearing in the Hypercyclicity Criterion produces a densely hereditarily hypercyclic subsequence (T_{n_k}) for assertion (B). Also, if $(T_n) \subset L(X)$ satisfies the Hypercyclicity Criterion then one may suppose that $Y_0 = X_0$; cf. [11, Remark 2.6]. In fact, allowing $Y_0 \neq X_0$ one can slightly simplify the proof by replacing condition (6) by $T_{n_k} u_{j,k} \in y_j + U_k$ and setting $Y_0 = \{y_j : j \in \mathbb{N}\}$.

(b) Condition ($\mathrm{C_{fin}}$) can be restated, perhaps more elegantly, in the following form:

(C_∞) $(T_n \oplus T_n \oplus \ldots)$ *is densely hypercyclic on* $X^{\mathbb{N}}$, *equipped with the product topology.*

This follows easily from the equivalence of (i) and (ii) in Lemma 2.1 since the sets of the form $U_1 \times \ldots \times U_N \times X \times X \times \ldots$ with non-empty open subsets $U_i \subset X$ ($1 \leq i \leq N$, $N \in \mathbb{N}$) form a base for the topology of $X^{\mathbb{N}}$.

(c) One cannot drop the density assumptions in conditions (B) and ($\mathrm{C_{fin}}$). Indeed, let S be the scaled backward shift $S : (x_0, x_1, \ldots) \mapsto (2x_1, 2x_2, \ldots)$ on $X = l^2$ with the usual norm $\|\cdot\|$ (see [27]). Then it is well known, and easy to see, that S satisfies the Hypercyclicity Criterion for the full sequence, hence is hereditarily hypercyclic. Now, to each vector $x = (x_0, x_1, x_2, \ldots) \in X$ we associate the vectors $y = (x_1, x_2, \ldots) \in X$ and $x_0 e_0 \in X$, where $e_0 = (1, 0, 0, 0, \ldots)$. Let us define the operators $T_n : X \to X$, $x \mapsto S^n y + 3^n x_0 e_0$, $n \in \mathbb{N}$. If y is hypercyclic for a subsequence (S^{n_k}) then it is clear that $x := (0, y_0, y_1, \ldots)$ is hypercyclic for (T_{n_k}), so (T_n) is hereditarily hypercyclic. On the other hand, by the result of Bès and Peris, $S \oplus \ldots \oplus S$ (N-fold) is hypercyclic on X^N for each N, and introducing zeros as respective first coordinates it is easy to see that $(T_n \oplus \ldots \oplus T_n)$ is also hypercyclic. However, (T_n) does not satisfy the Hypercyclicity Criterion because the set of hypercyclic vectors for (T_n) is not dense in X. In fact, if x is such a vector then its first coordinate satisfies $x_0 = 0$ because, otherwise, we would have

$$\|T_n x\| \geq 3^n |x_0| - 2^n \|x\| \to \infty \quad \text{as } n \to \infty,$$

which is clearly absurd.

(d) In this paper we are primarily interested in self-mappings. We want to note, however, that under the obvious extensions of the notions appearing in

Theorem 2.2 the result remains true for sequences (T_n) of continuous linear mappings $T_n : X \to Y$ between different separable F-spaces X and Y; the proof is an obvious modification of the simplified proof mentioned in (a).

(e) The characterization of the Hypercyclicity Criterion for arbitrary sequences of operators obtained in Theorem 2.2 obviously poses the problem if it suffices in condition ($\mathrm{C_{fin}}$) to take $N = 2$ only. We do not know the answer to this question. However, under the assumption of some commutativity on the sequence (T_n) this improvement is possible (see Section 3).

As an application of the theorem we want to consider operators that have been much studied in the theory of hypercyclicity, namely composition operators on spaces of holomorphic functions (cf. [19, Section 4a]). Let $G \subset \mathbb{C}$ be a non-empty open subset of the complex plane and let $\varphi_n : G \to G$, $n \in \mathbb{N}$, be automorphisms on G, that is, each φ_n is an invertible holomorphic function from G onto itself. Then we consider the sequence (T_n) of composition operators on the space $H(G)$ of holomorphic functions on G given by

$$T_n f = f \circ \varphi_n, \qquad f \in H(G),$$

where $H(G)$ carries its usual topology of locally uniform convergence. It was shown by Bernal and Montes [8], [26] that if G is not conformally equivalent to $\mathbb{C} \setminus \{0\}$ then the sequence (T_n) is hypercyclic if and only if (φ_n) is a *run-away sequence*, that is, if for every compact subset $K \subset G$ there exists some $n \in \mathbb{N}$ with $K \cap \varphi_n(K) = \emptyset$. We shall now show that the sequence always satisfies the Hypercyclicity Criterion if it is hypercyclic.

Proposition 2.4. *Let (T_n) be a sequence of composition operators $T_n f = f \circ \varphi_n$ on a non-empty open subset G of $\mathbb{C}$ that is not conformally equivalent to $\mathbb{C} \setminus \{0\}$. Then the following assertions are equivalent*:

(i) *(T_n) is hypercyclic.*
(ii) *(T_n) has a densely hereditarily hypercyclic subsequence.*
(iii) *(T_n) satisfies the Hypercyclicity Criterion.*
(iv) *(φ_n) is a run-away sequence.*

Proof. The proof of (i)⇒(iv) is given in [26, p. 197], and (ii)⇒(i) is trivial. Since the equivalence of (ii) and (iii) follows from Theorem 2.2 it suffices to prove that (iv)⇒(ii). Now, there exists an increasing sequence (K_n) of compact subsets of G such that each compact subset of G is contained in some K_n. By (iv) there is an increasing sequence (n_j) of positive integers such that $K_j \cap \varphi_{n_j}(K_j) = \emptyset$ for all $j \in \mathbb{N}$. It then follows easily that each subsequence (φ_{m_j}) of (φ_{n_j}) is run-away. By [26, Remark 2] this shows that (T_{n_j}) is densely hereditarily hypercyclic, proving (ii). ■

This result is particularly interesting because, in contrast to most other classes of operators, the proofs of the hypercyclicity of composition operators on spaces $H(G)$ have so far usually relied on Lemma 2.1 or on explicit constructions; the Hypercyclicity Criterion had turned out to be more cumbersome for these operators (see, for example, the proof of [14, 3.2]). We thus see that at least in principle one could also use the Hypercyclicity Criterion for these operators.

As an immediate consequence of this result and of [5, Theorem 2 and following remark] we obtain the following.

COROLLARY 2.5. *Under the assumptions of Proposition* 2.4, *every hypercyclic sequence of composition operators $T_n f = f \circ \varphi_n$ has a dense vector subspace $L \subset H(G)$ so that each non-zero vector in L is hypercyclic for (T_n).*

Incidentally, Theorem 2 of [5] should be compared with a remarkable recent result of S. Grivaux [16] which asserts that for *any* sequence (T_n) of hypercyclic operators on a Banach space X, there is a dense vector subspace L such that every non-zero vector in L is T_n-hypercyclic for every n. Compare also with [7, Theorem 3.1].

3. The Hypercyclicity Criterion for almost-commuting sequences. In this section we study the Hypercyclicity Criterion under mild commutativity assumptions on the sequence (T_n).

DEFINITION 3.1. (a) A sequence (T_n) in $L(X)$ is called *almost-commuting* if

$$\lim_{n\to\infty} (T_n T_m - T_m T_n)x = 0$$

for every $m \in \mathbb{N}$ and every $x \in X$.

(b) A sequence (T_n) in $L(X)$ is called *weakly commuting* if

$$(T_n T_m - T_m T_n)_n$$

is equicontinuous on X for every $m \in \mathbb{N}$.

The concept of almost-commutativity was introduced by the first author in [6]. It follows from the Banach–Steinhaus Theorem [28, Chapter 2] that every almost-commuting sequence is weakly commuting, while almost-commutativity is strictly weaker than commutativity. For example, if S and T are operators on a Banach space with $\|S\| < 1$ and $\|T\| < 1$ then the sequence $(S, T, S^2, T^2, S^3, \ldots)$ is almost-commuting even if S and T do not commute, in which case the sequence is not commuting.

The following property of weakly commuting and hence also of almost-commuting sequences of operators will be crucial.

LEMMA 3.2. *Let (T_n) be a weakly commuting sequence of operators on X. If $u_k \to 0$ in X and $T_{n_k} u_k \to x_0 \in X$ for some (n_k) then $T_{n_k}(T_m u_k) \to T_m x_0$ for all $m \in \mathbb{N}$.*

Proof. By weak commutativity the sequence $(T_nT_m - T_mT_n)_n$ is equicontinuous for each m, hence so is the sequence $(T_{n_k}T_m - T_mT_{n_k})_k$. Thus it follows from $u_k \to 0$ that

$$(T_{n_k}T_m - T_mT_{n_k})u_k \to 0 \quad \text{as } k \to \infty.$$

Since T_m is continuous and $T_{n_k}u_k \to x_0$ we conclude that

$$T_{n_k}(T_m u_k) = (T_{n_k}T_m u_k - T_mT_{n_k}u_k) + T_mT_{n_k}u_k \to T_m x_0$$

as $k \to \infty$. ∎

From this we obtain the following improvement of Theorem 2.2 for almost-commuting sequences of operators.

Theorem 3.3. *Let (T_n) be an almost-commuting sequence of operators on X. Then the following assertions are equivalent*:

(A) *(T_n) satisfies the Hypercyclicity Criterion.*
(B) *(T_n) has a* (*densely*) *hereditarily hypercyclic subsequence.*
(C) *$(T_n \oplus T_n)$ is* (*densely*) *hypercyclic on $X \times X$.*
(D) *There exists a hypercyclic vector x_0 for (T_n), a sequence (u_k) in X with $u_k \to 0$, and an increasing sequence (n_k) of positive integers such that $T_{n_k}x_0 \to 0$ and $T_{n_k}u_k \to x_0$ as $k \to \infty$.*
(E) *There exists a hypercyclic vector x_0 for (T_n) and an increasing sequence (n_k) of positive integers such that $T_{n_k}x_0 \to 0$ as $k \to \infty$ and (T_{n_k}) is hypercyclic.*

Proof. The implication (A)⇒(B) follows from Theorem 2.2, and condition (B) without density clearly implies (E).

(E)⇒(D). Suppose that x_0 is hypercyclic for (T_n), that $T_{n_k}x_0 \to 0$ and that (T_{n_k}) has a hypercyclic vector u. Since $u_j := j^{-1}u$ is also hypercyclic for (T_{n_k}) for each $j \in \mathbb{N}$ one can find a subsequence (m_j) of (n_k) with $T_{m_j}u_j \to x_0$ as $j \to \infty$. Since $u_j \to 0$ and $T_{m_j}x_0 \to 0$ as $j \to \infty$ we see that condition (D) holds.

(D)⇒(A). Let $X_0 := \{T_n x_0 : n \in \mathbb{N}\}$, which is dense in X by hypercyclicity of x_0. Fix $x = T_m x_0 \in X_0$. By hypothesis, we have

$$T_nT_m x_0 - T_mT_n x_0 \to 0 \quad (n \to \infty).$$

Thus we obtain

$$T_{n_k}x = (T_{n_k}T_m x_0 - T_mT_{n_k}x_0) + T_mT_{n_k}x_0 \to 0$$

because $T_{n_k}x_0 \to 0$ and T_m is continuous. This gives us condition (i) in the Hypercyclicity Criterion.

Next we define

$$u_k^{(m)} := T_m u_k \quad (k, m \in \mathbb{N}).$$

Since $u_k \to 0$ and $T_{n_k} u_k \to x_0$ as $k \to \infty$, Lemma 3.2 implies that

$$T_{n_k} u_k^{(m)} \to T_m x_0$$

as $k \to \infty$ for all $m \in \mathbb{N}$, and by continuity of T_m we have $u_k^{(m)} \to 0$ as $k \to \infty$. Hence also condition (ii) in the Hypercyclicity Criterion is satisfied when we set $Y_0 = X_0 = \{T_n x_0 : n \in \mathbb{N}\}$.

So far we have seen that conditions (A), (B), (D) and (E) are equivalent. For the equivalence of (C) with these conditions it suffices, in view of Theorem 2.2, to show that condition (C) without density implies (D). Thus, suppose that there exists a vector $(x_0, u) \in X \times X$ that is hypercyclic for $(T_n \oplus T_n)$. Let (U_k) be a base of neighbourhoods of zero in X and set $u_k = k^{-1} u$. Since every vector (x_0, u_k) is hypercyclic for $(T_n \oplus T_n)$ there are positive integers n_k such that

$$T_{n_k} x_0 \in U_k \quad \text{and} \quad T_{n_k} u_k \in x_0 + U_k$$

for all $k \in \mathbb{N}$. Thus we have $u_k \to 0$, $T_{n_k} x_0 \to 0$ and $T_{n_k} u_k \to x_0$ as $k \to \infty$. Since x_0 is necessarily hypercyclic for (T_n) condition (D) is satisfied. ∎

While condition (E) was taken from [9, Chapter 1] we had originally introduced condition (D) as an auxiliary property for the proof of Theorem 3.3. However, it turned out to have another, unexpected consequence for almost-commuting sequences of operators: An interesting condition that was shown by Godefroy and Shapiro [15, Corollary 1.3] to be sufficient for hypercyclicity is in fact equivalent to the Hypercyclicity Criterion. This condition should be compared with condition (ii) in Lemma 2.1 that is equivalent to dense hypercyclicity.

THEOREM 3.4. *An almost-commuting sequence (T_n) of operators on X satisfies the Hypercyclicity Criterion if and only if the following condition holds*:

(GS) *For every pair U, V of non-empty open subsets of X and every neighbourhood W of zero there is some $n \in \mathbb{N}$ with*

$$T_n(U) \cap W \neq \emptyset \quad \textit{and} \quad T_n(W) \cap V \neq \emptyset.$$

As in Lemma 2.1, one may equivalently require in condition (GS) that the stated property holds for infinitely many $n \in \mathbb{N}$ (cf. [15]).

Proof. If (T_n) satisfies the Hypercyclicity Criterion then by Theorem 2.2 the sequence $(T_n \oplus T_n)$ is densely hypercyclic, which by Lemma 2.1 implies (GS) when we choose $U \times W$ and $W \times V$ as open subsets of $X \times X$ in condition (ii) there.

Conversely, suppose that (T_n) satisfies (GS). It suffices to show that then also condition (D) of Theorem 3.3 holds. First, by the proof of [15, Corollary 1.3], (T_n) has a dense G_δ-set of hypercyclic vectors. Since the intersection of two dense G_δ-sets in an F-space is non-empty, condition (D) will follow once we have shown that the following property is satisfied by a dense G_δ-set of points $x \in X$: *there exists a sequence* (u_k) *in* X *with* $u_k \to 0$ *and an increasing sequence* (n_k) *of positive integers such that* $T_{n_k}x \to 0$ *and* $T_{n_k}u_k \to x$. Let P be the set of such points x and let (U_k) be a base of open neighbourhoods of zero in X. Then we clearly have $P = \bigcap_{k=1}^{\infty} O_k$ with

$$O_k = \bigcup_{n=1}^{\infty} (T_n^{-1}(U_k) \cap \{x \in X : T_n(U_k) \cap (x + U_k) \neq \emptyset\}).$$

Since each T_n is continuous and $x + U_k$ is open it follows easily that each set O_k is open. Now, condition (GS) implies that each O_k is also dense. To see this, fix $k \in \mathbb{N}$ and let V be a non-empty open subset of X. Choose an open non-empty subset $\widetilde{V}$ of V with $\widetilde{V} - \widetilde{V} \subset U_k$ (for example, let $\widetilde{V} = v + W$ with $v \in V$ and an open neighbourhood W of zero such that $v + W \subset V$ and $W - W \subset U_k$). Then by (GS) there is some $n \in \mathbb{N}$ and points $x, y \in \widetilde{V}$ such that $T_n x \in U_k$ and $y \in T_n(U_k)$. Since $y - x \in \widetilde{V} - \widetilde{V} \subset U_k$ we deduce that $y \in x + U_k$. Therefore $x \in T_n^{-1}(U_k)$ and $T_n(U_k) \cap (x + U_k) \neq \emptyset$. Since $x \in \widetilde{V} \subset V$ we see that O_k contains a point from V. Thus we have shown that each set O_k is dense and open, which makes P a dense G_δ-subset of X by the Baire Category Theorem. ∎

Remark 3.5. Even the special case of this theorem for single operators T gives a new result. Namely, an operator T on X satisfies the Hypercyclicity Criterion if and only if the following condition holds:

(GS) *For every pair* U, V *of non-empty open subsets of* X *and every neighbourhood* W *of zero there is some* $n \in \mathbb{N}$ *with*

$$T^n(U) \cap W \neq \emptyset \quad \text{and} \quad T^n(W) \cap V \neq \emptyset.$$

During the preparation of this paper we were kindly informed by J. Bès [10], F. León [24] and L. Saldivia [30] that they have independently obtained this result. We are grateful to all these people.

In view of the second part of the proof of Theorem 3.4 it is natural to pose the following problem: *Is* $U = V$ *in condition* (GS) *sufficient for hypercyclicity of a single operator* T*?* If this is the case then the proof shows that the weaker condition is also equivalent to the Hypercyclicity Criterion for T. We are grateful to J. Bès and L. Saldivia for interesting discussions about this matter.

4. Chaos for sequences of operators. In this section we show that many hypercyclic weakly commuting sequences (T_n), including all chaotic weakly commuting sequences, satisfy the Hypercyclicity Criterion, which we may take as further evidence for the importance of this criterion. Recall that a subset A of a metric space is called *precompact* (or *totally bounded*) if for every $\varepsilon > 0$ the set A can be covered by a finite union of ε-balls in the space.

THEOREM 4.1. *Let (T_n) be a hypercyclic weakly commuting sequence of operators on X. If there is a dense set of points $x \in X$ with precompact orbit $\{T_n x : n \in \mathbb{N}\}$ then (T_n) satisfies the Hypercyclicity Criterion.*

Proof. By Theorem 2.2 it suffices to show that (T_n) has a densely hereditarily hypercyclic subsequence (T_{n_k}). To see this, let x_0 be a fixed hypercyclic vector for (T_n). Since also each vector $u_k := k^{-1}x_0$ is hypercyclic we can find an increasing sequence (n_k) of positive integers such that $T_{n_k}u_k \to x_0$ as $k \to \infty$, and we find that $u_k \to 0$.

Let

$$u_k^{(m)} := T_m u_k \quad (k, m \in \mathbb{N}).$$

Then Lemma 3.2 implies that

$$T_{n_k} u_k^{(m)} \to T_m x_0$$

as $k \to \infty$ for all $m \in \mathbb{N}$, and by continuity of T_m we have $u_k^{(m)} \to 0$ for all $m \in \mathbb{N}$. Hence condition (ii) in the Hypercyclicity Criterion is satisfied when we set $Y_0 = \{T_n x_0 : n \in \mathbb{N}\}$, which is dense in X by hypercyclicity of x_0.

We can now show that (T_{n_k}) is densely hereditarily hypercyclic. Let (m_k) be a subsequence of (n_k). Then also the sequence (T_{m_k}) satisfies condition (ii) in the Hypercyclicity Criterion for the same dense subset Y_0. On the other hand, let X_0 be a dense set of points in X so that each $x \in X_0$ has a precompact orbit. Then, for each $x \in X_0$, $(T_{m_k}x)$ has a convergent subsequence. Hence (T_{m_k}) also satisfies condition (i′) in Remark 1.2, which implies that it is densely hypercyclic. This had to be shown. ■

Bermúdez, Bonilla and Peris [4, Corollary 2.1] have obtained Theorem 4.1 for commuting operators in an F-space using a different argument.

The assumption of a dense set of points with precompact orbits includes several interesting special cases. We refer to

$$\liminf_{n\to\infty} \operatorname{Ker}(T_n) = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \operatorname{Ker}(T_j)$$

as the *generalized kernel* of the sequence (T_n). Observe that it is always a linear submanifold of X. This notion extends the known concept of a generalized kernel of a single operator (cf. [22] or [11]). And we call x an *almost-periodic point* for (T_n) if for every neighbourhood U of zero there is some $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there is some k with $m \le k \le m + N$ such that $T_{n+k}x - T_n x \in U$ for all $n \in \mathbb{N}$ (cf. [31, p. 93]). An almost-periodic point for an operator T is one for the sequence (T^n).

Corollary 4.2. *Let* (T_n) *be a hypercyclic weakly commuting sequence of operators on* X. *Suppose that at least one of the following conditions is satisfied*:

(a) (T_n) *has dense generalized kernel.*
(b) (T_n) *converges pointwise on a dense subset of* X.
(c) (T_n) *is chaotic.*
(d) (T_n) *has a dense set of almost-periodic points.*

Then (T_n) *satisfies the Hypercyclicity Criterion.*

Proof. Parts (a) to (c) are straightforward, but (d) needs some explanation. In view of Theorem 4.1, we should show that every almost-periodic point x for (T_n) has precompact orbit. For this, fix such an x and a neighbourhood U of zero. Our goal is to find finitely many vectors y_j, $j = 1, \dots, p$, with $T_n x \in \bigcup_{j=1}^{p} (y_j + U)$ for all $n \in \mathbb{N}$. Now, if N is as in the definition of almost-periodicity and $n > N$ then there is $k = k(n)$ with $n - N \le k \le n$ such that $T_{m+k}x - T_m x \in U$ for all $m \in \mathbb{N}$. If we take in particular $m = n - k + 1$ then $T_{n+1}x \in T_{n-k+1}x + U \subset \bigcup_{j=1}^{N+1}(T_j x + U)$. Consequently, $T_n x \in \bigcup_{j=1}^{p}(y_j + U)$ for all $n \in \mathbb{N}$, where $p := N + 1$ and $y_j := T_j x$. ∎

Even in the special case of sequences (T^n) of iterates of a single operator Theorem 4.1 and Corollary 4.2 give new results.

Corollary 4.3. (1) *Let* T *be a hypercyclic operator on* X. *If there is a dense set of points* $x \in X$ *with precompact orbit* $\{T^n x : n \in \mathbb{N}\}$ *then* T *satisfies the Hypercyclicity Criterion.*

(2) *In particular*, *the Hypercyclicity Criterion is satisfied for a hypercyclic operator* T *on* X *if at least one of the following conditions holds*:

(a) T *has dense generalized kernel.*
(b) (T^n) *converges pointwise on a dense subset of* X.
(c) T *is chaotic.*
(d) T *has a dense set of almost-periodic points.*

F. León [24] has independently proved part (1) of Corollary 4.3 on Hilbert spaces. Moreover, part (2) of this corollary includes two results of Bès and Peris [11, Propositions 2.11 and 2.14]; but see also [4, Remark 2.1(1)]. Note, in particular, that for chaotic operators our proof is considerably simpler than the one by Bès and Peris who use a deep theorem of Ansari [1], which forced these authors to assume that X is a Fréchet space, that is, a locally convex F-space; see also [2, Theorem 1.1].

Acknowledgments. The authors are grateful to the referee for helpful comments and suggestions.

References

[1] S. I. Ansari, *Hypercyclic and cyclic vectors*, J. Funct. Anal. 128 (1995), 374–383.

[2] J. Banks, *Regular periodic decompositions for topologically transitive maps*, Ergodic Theory Dynam. Systems 17 (1997), 505–529.

[3] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly 99 (1992), 332–334.

[4] T. Bermúdez, A. Bonilla and A. Peris, *On hypercyclicity and supercyclicity criteria*, preprint.

[5] L. Bernal-González, *Densely hereditarily hypercyclic sequences and large hypercyclic manifolds*, Proc. Amer. Math. Soc. 127 (1999), 3279–3285.

[6] —, *Universal images of universal elements*, Studia Math. 138 (2000), 241–250.

[7] L. Bernal-González and M. C. Calderón-Moreno, *Dense linear manifolds of monsters*, J. Approx. Theory 119 (2002), 156–180.

[8] L. Bernal-González and A. Montes-Rodríguez, *Universal functions for composition operators*, Complex Variables Theory Appl. 27 (1995), 47–56.

[9] J. Bès, *Three Problems on Hypercyclic Operators*, dissertation, Kent State Univ., Kent, OH, 1998.

[10] J. Bès, personal communication.

[11] J. Bès and A. Peris, *Hereditarily hypercyclic operators*, J. Funct. Anal. 167 (1999), 94–112.

[12] J. Bonet, *Hypercyclic and chaotic convolution operators*, J. London Math. Soc. 62 (2000), 253–262.

[13] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, Redwood City, CA, 1989.

[14] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. 100 (1987), 281–288.

[15] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. 98 (1991), 229–269.

[16] S. Grivaux, *Construction of operators with prescribed behaviour*, Arch. Math. (Basel), to appear.

[17] K.-G. Grosse-Erdmann, *Holomorphe Monster und universelle Funktionen*, Mitt. Math. Sem. Giessen 176 (1987).

[18] —, *On the universal functions of G. R. MacLane*, Complex Variables Theory Appl. 15 (1990), 193–196.

[19] —, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), 345–381.

[20] D. Herrero, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. 99 (1991), 179–190.

[21] —, *Hypercyclic operators and chaos*, J. Operator Theory 28 (1992), 93–103.

[22] G. Herzog and C. Schmoeger, *On operators T such that $f(T)$ is hypercyclic*, Studia Math. 108 (1994), 209–216.

[23] C. Kitai, *Invariant closed sets for linear operators*, dissertation, Univ. of Toronto, Toronto, 1982.

[24] F. León-Saavedra, *Notes about the hypercyclicity criterion*, Bull. London Math. Soc., to appear.

[25] F. León-Saavedra and A. Montes-Rodríguez, *Spectral theory and hypercyclic subspaces*, Trans. Amer. Math. Soc. 353 (2000), 247–267.

[26] A. Montes-Rodríguez, *A note on Birkhoff open sets*, Complex Variables Theory Appl. 30 (1996), 193–198.

[27] S. Rolewicz, *On orbits of elements*, Studia Math. 32 (1969), 17–22.
[28] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.
[29] H. Salas, *A hypercyclic operator whose adjoint is also hypercyclic*, Proc. Amer. Math. Soc. 112 (1991), 765–770.
[30] L. Saldivia, personal communication.
[31] K. S. Sibirsky, *Introduction to Topological Dynamics*, Noordhoff International, Leyden, 1975.

Departamento de Análisis Matemático
Facultad de Matemáticas, Apdo. 1160
Universidad de Sevilla
Avda. Reina Mercedes
41080 Sevilla, Spain
E-mail: lbernal@us.es

Fachbereich Mathematik
Fern Universität Hagen
D-58084 Hagen, Germany
E-mail: kg.grosse-erdmann@fernuni-hagen.de

Received September 26, 2001
Revised version November 4, 2002 (4812)