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Boundedness of commutators of strongly singular convolution operators on Herz-type spaces

by

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Abstract. The author investigates the boundedness of the higher order commutator of strongly singular convolution operator, T_b^m, on Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$, and on a new class of Herz-type Hardy spaces $H\dot{K}_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)$ and $HK_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)$, where $0 < p \le 1 < q < \infty$, $\alpha = n(1-1/q)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$.

1. Introduction. Let v be a smooth radial cut-off function with $\operatorname{supp} v \subset \{x \in \mathbb{R}^n : |x| \le 2\}$. We consider the strongly singular convolution kernel

$$K(x) = \frac{e^{i|x|^{-s'}}}{|x|^n}\, v(x),$$

where $0 < s < 1$ and $s' = s/(1-s)$. It is well known that, when $|x| > 2|y|$,

$$|K(x-y) - K(x)| \le \frac{C|y|}{|x|^{n+s'+1}}.$$

Denote by Tf the corresponding strongly singular convolution operator:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy.$$

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and m be a positive integer. The commutator of order m of T with b is defined by

$$T_b^m f(x) = \text{p.v.} \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x-y) f(y)\, dy.$$

The study of the strongly singular convolution operator T in the context of $L^q(\mathbb{R}^n)$ spaces was carried out by I. I. Hirschman and S. Wainger [11]. The sharp endpoint estimates for T were obtained by C. Fefferman and E. M. Stein [3] using the duality of H^1 and BMO. The weighted norm

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estimates (L^q and weak $(1,1)$) for T were established by S. Chanillo [1]. The properties of T on weighted Herz-type Hardy spaces were investigated by Li Xiaochun and Lu Shanzhen [6]. Recently, the boundedness of the commutator T_b^m on weighted $L^q(\mathbb{R}^n)$ was obtained by J. García-Cuerva, E. Harboure, C. Segovia and J. L. Torrea [4]. Its boundedness on Herz spaces can be viewed as a special case of a result belonging to Lu Shanzhen, Tanglin and Yang Dachun [8] for $m=1$.

The main purpose of this paper is to investigate the commutator T_b^m on Herz spaces and on a new class of Herz-type Hardy spaces with critical index $\alpha = n(1-1/q)$. First, let us introduce some definitions and notations.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$, $\widetilde{A}_k = A_k$ for $k \in \mathbb{N}$ and $\widetilde{A}_0 = B_0$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$ and $\widetilde{\chi}_k = \chi_{\widetilde{A}_k}$ for $k \in \mathbb{N} \cup \{0\}$, where χ_E is the characteristic function of E.

Definition 1.1 ([9]). Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$.

(a) The *homogeneous Herz space* is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}.$$

(b) The *nonhomogeneous Herz space* is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|^p_{L^q(\mathbb{R}^n)} \Big\}^{1/p}.$$

Definition 1.2 ([5]). Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$.

(1) A measurable function f on $\mathbb{R}^n$ is said to belong to the *homogeneous weak Herz space* $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} |\{x \in A_k : |f(x)| > \lambda\}|^{p/q} \Big\}^{1/p} < \infty.$$

(2) A measurable function f on $\mathbb{R}^n$ is said to belong to the *nonhomogeneous weak Herz space* $WK_q^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} |\{x \in \widetilde{A}_k : |f(x)| > \lambda\}|^{p/q} \Big\}^{1/p} < \infty.$$

Definition 1.3. Let $\alpha \in \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$, $1 < q < \infty$, $1/q + 1/q' = 1$ and $b \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^n)$.

(1) A function a on $\mathbb{R}^n$ is said to be a *central* $(\alpha, q, l; b, m)$*-atom* if it satisfies:

(a) $\operatorname{supp} a \subset B(0,r) := \{x \in \mathbb{R}^n : |x| < r\}$ for some $r > 0$,
(b) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n} = Cr^{-\alpha}$,
(c) $\int_{\mathbb{R}^n} x^\beta a(x) b(x)^t \, dx = 0$ for $|\beta| \le l$ and $t = 0, 1, \ldots, m$.

(2) A function a on $\mathbb{R}^n$ is said to be a *central* $(\alpha, q, l; b, m)$*-atom of restricted type* if it satisfies (b), (c) and

(a′) $\operatorname{supp} a \subset B(0,r)$ for some $r > 1$.

Definition 1.4. Let $\alpha \in \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$, $0 < p < \infty$, $1 < q < \infty$, $1/q + 1/q' = 1$ and $b \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^n)$. A tempered distribution f is said to belong to $H\dot{K}^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)$ (resp. $HK^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)$) if, in the $\mathcal{S}'(\mathbb{R}^n)$ sense, it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (resp. $f = \sum_{j=0}^{\infty} \lambda_j a_j$), where each a_j is a central $(\alpha, q, l; b, m)$-atom (resp. central $(\alpha, q, l; b, m)$-atom of restricted type) with $\operatorname{supp} a_j \subset B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (resp. $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). We define the quasinorms on $H\dot{K}^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)$ and $HK^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)$ by

$$\|f\|_{H\dot{K}^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)} = \inf\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{1/p},$$

$$\|f\|_{HK^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n)} = \inf\Big(\sum_{j=0}^{\infty} |\lambda_j|^p\Big)^{1/p},$$

where the infimum is taken over all central atomic decompositions (resp. central atomic decompositions of restricted type) of f.

Obviously, the Herz-type Hardy spaces in Definition 1.4 are subspaces of those introduced by Lu and Yang [10] when $m > 1$, and they coincide with them when $m = 1$. In particular, if $0 < p < \infty$, $1 < q < \infty$, $\alpha \ge n(1 - 1/q)$, $l \ge [\alpha + n(1 - 1/q)]$, $m = 0$ and $b \equiv 1$, then $H\dot{K}^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n) = H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $HK^{\alpha,p,l}_{q,b,m}(\mathbb{R}^n) = HK^{\alpha,p}_q(\mathbb{R}^n)$, the spaces studied by Lu and Yang [9].

To state our results, we need the following basic lemmas. The first lemma is the unweighted case of Theorem 2.2 of [4].

Lemma 1.1. *Let m be a positive integer, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $1 < q < \infty$. Then the commutator T^m_b is bounded on $L^q(\mathbb{R}^n)$.*

Lemma 1.2 ([1]). *Let $0 < s < 1$, $s' = s/(1-s)$, $t > 1$, $1/t + 1/t' = 1$ and $(s'+2)/t < 1$. Define $\widetilde{K}_{s',t}(x) = e^{i|x|^{-s'}}/|x|^{n(s'+2)/t}$. Then there exists a constant $C > 0$ such that for any $f \in L^{t'}(\mathbb{R}^n)$,*

$$\|\widetilde{K}_{s',t} * f\|_{L^t(\mathbb{R}^n)} \le C\|f\|_{L^{t'}(\mathbb{R}^n)}.$$

Lemma 1.3. *Let m be a positive integer. If $u > 1$ is such that $1 < 2^x/x^m \le u$ for some $x \ge N_m$, then $2^x \le Cu(\log_2^+ u)^m$, where C only depends on m.*

Proof. It is easy to see that, for a given integer m, there exists a positive integer N_m such that $\log_2^+ x < x/(2m)$ for any $x \ge N_m$. If $1 < 2^x/x^m \le u$, then $2^x \le ux^m$. This implies

$$x \le \log_2^+ u + m\log_2^+ x \le \log_2^+ u + x/2.$$

Thus we have $x \le 2\log_2^+ u$. This gives $2^x \le Cu(\log_2^+ u)^m$, where $C = 2^m$.

2. Commutators on Herz spaces. In this section, we first obtain the weak type $L(\log^+ L)^m$ estimate for the commutator T_b^m on nonhomogeneous Herz spaces. Then we show that T_b^m is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (resp. $K_q^{\alpha,p}(\mathbb{R}^n)$) to $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (resp. $WK_q^{\alpha,p}(\mathbb{R}^n)$) when $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies the so-called condition $\mathfrak{L}$, defined below. However, I do not know whether condition $\mathfrak{L}$ is optimal for the validity of Theorems 2.2 and 3.2 of this paper.

Definition 2.1. We say that $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies *condition* $\mathfrak{L}$ if there exists a constant $C > 0$ only depending n such that for any $k, j \in \mathbb{Z}$ with $j \le k-2$ and any $x \in A_k$,

$$|b(x) - b_j| \le C|b(x) - b_k|,$$

where b_j is the mean value of b over the ball B_j, i.e. $b_j = |B_j|^{-1}\int_{B_j} b(x)\,dx$.

Remark. Condition $\mathfrak{L}$ was first introduced by the author [7]. In that paper, two examples are presented to show that this condition is non-trivial. One of these examples is any odd $\mathrm{BMO}(\mathbb{R}^1)$ function b; it satisfies condition $\mathfrak{L}$ because $b_j = 0$ for any $j \in \mathbb{Z}$. By [2], the set of unbounded odd $\mathrm{BMO}(\mathbb{R}^1)$ functions is a non-trivial subspace of $\mathrm{BMO}(\mathbb{R}^1)$. Another example is a π-periodic function b on $\mathbb{R}^1$ with $b(x) = x$ for $x \in (0, \pi]$; it belongs to $\mathrm{BMO}(\mathbb{R}^1)$ and satisfies condition $\mathfrak{L}$ because it is a bounded function on $\mathbb{R}^1$ and $b_j = \pi/2$ for any $j \in \mathbb{Z}$.

Theorem 2.1. *Let $0 < p \le 1 < q < \infty$, $\alpha = n(1 - 1/q)$, m a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $f \in K_q^{\alpha,p}(\mathbb{R}^n)$ and any $\lambda > 0$,*

$$\Big\{\sum_{k=0}^{\infty} 2^{k\alpha p}|\{x \in \widetilde{A}_k : |T_b^m f(x)| > \lambda\}|^{p/q}\Big\}^{1/p} \le \frac{C\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg(1 + \bigg(\log^+ \frac{\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg)^m\bigg).$$

Proof. We write

$$\begin{aligned}\Big\{\sum_{k=0}^{\infty}2^{k\alpha p}|\{x\in\widetilde{A}_k:|T_b^m f(x)|>\lambda\}|^{p/q}\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=0}^{N_m-1}2^{k\alpha p}|\{x\in\widetilde{A}_k:|T_b^m f(x)|>\lambda\}|^{p/q}\Big\}^{1/p}\\
&\quad+C\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}|\{x\in A_k:|T_b^m f(x)|>\lambda\}|^{p/q}\Big\}^{1/p}\\
&=\mathcal{A}_1+\mathcal{A}_2,\end{aligned}$$

where N_m is the same constant as in Lemma 1.3. Since $0<p\le 1<q<\infty$, $\alpha=n(1-1/q)$ and T_b^m is bounded on $L^q(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathcal{A}_1&\le\frac{C}{\lambda}\|f\|_{L^q(\mathbb{R}^n)}\le\frac{C}{\lambda}\sum_{j=0}^{\infty}\|f\widetilde{\chi}_j\|_{L^q(\mathbb{R}^n)}\le\frac{C}{\lambda}\sum_{j=0}^{\infty}2^{j\alpha}\|f\widetilde{\chi}_j\|_{L^q(\mathbb{R}^n)}\\
&\le\frac{C}{\lambda}\Big(\sum_{j=0}^{\infty}2^{j\alpha p}\|f\widetilde{\chi}_j\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}=\frac{C}{\lambda}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

For any integer $k\ge N_m$, we decompose $f(x)$ as follows:

$$\begin{aligned}f(x)&=f(x)\chi_{\{|x|\le 2^{k-2}\}}(x)+f(x)\chi_{\{2^{k-2}<|x|<2^{k+2}\}}(x)+f(x)\chi_{\{|x|\ge 2^{k+2}\}}(x)\\
&=f_1(x)+f_2(x)+f_3(x).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{A}_2&\le C\sum_{i=1}^{3}\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}|\{x\in A_k:|T_b^m f_i(x)|>\lambda/3\}|^{p/q}\Big\}^{1/p}\\
&=\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3.\end{aligned}$$

The boundedness of T_b^m on $L^q(\mathbb{R}^n)$ implies that

$$\begin{aligned}\mathcal{B}_2&\le\frac{C}{\lambda}\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}\|f_2\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}\\
&\le\frac{C}{\lambda}\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}\Big(\sum_{j=k-1}^{k+1}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p}\\
&\le\frac{C}{\lambda}\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}\\
&\le\frac{C}{\lambda}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)},\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_3 &\le \frac{C}{\lambda}\Big\{\sum_{k=N_m}^{\infty} 2^{k\alpha p}\|f_3\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}\\
&\le \frac{C}{\lambda}\Big\{\sum_{k=N_m}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p}\\
&\le \frac{C}{\lambda}\Big\{\sum_{j=N_m+2}^{\infty}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big(\sum_{k=0}^{j-2}2^{k\alpha p}\Big)\Big\}^{1/p}\\
&\le \frac{C}{\lambda}\Big\{\sum_{j=0}^{\infty}2^{j\alpha p}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p}\\
&\le \frac{C}{\lambda}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate $\mathcal{B}_1$, we need a pointwise estimate for $T_b^m f_1(x)$ when $x \in A_k$, $y \in \widetilde{A}_j$ with $j \le k-2$ and $k \ge N_m$. In this case, we can easily see that $|x-y| \ge 2^{k-2}$. We now apply the Hölder inequality, the binomial theorem and the fact that each $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies $|b_k - b_j| \le (k-j)\|b\|_*$ for any integers k, j with $k \ge j$ and $(|B_j|^{-1}\int_{B_j}|b(y)-b_j|^r\,dy)^{1/r} \le C\|b\|_*$ for some $r > 1$, where $\|\cdot\|_*$ is the $\mathrm{BMO}(\mathbb{R}^n)$ norm and b_j is the mean value of b over B_j, to obtain

$$\begin{aligned}
|T_b^m f_1(x)| &\le \sum_{j=0}^{k-2}|T_b^m(f\widetilde{\chi}_j)(x)| \le \sum_{j=0}^{k-2}\int_{\widetilde{A}_j}\frac{|b(x)-b(y)|^m}{|x-y|^n}|f(y)|\,dy\\
&\le C2^{-kn}\sum_{i=0}^{m}\sum_{j=0}^{k-2}|b(x)-b_j|^i\int_{\widetilde{A}_j}|b(y)-b_j|^{m-i}|f(y)|\,dy\\
&\le C2^{-kn}\sum_{i=0}^{m}\sum_{j=0}^{k-2}|b(x)-b_j|^i\Big(\int_{\widetilde{A}_j}|b(y)-b_j|^{(m-i)q'}\,dy\Big)^{1/q'}\|f\widetilde{\chi}_j\|_{L^q(\mathbb{R}^n)}\\
&\le C2^{-kn}\sum_{i=0}^{m}\sum_{j=0}^{k-2}(|b(x)-b_k|^i+|b_k-b_j|^i)2^{jn/q'}\|b\|_*^{m-i}\|f\widetilde{\chi}_j\|_{L^q(\mathbb{R}^n)}\\
&\le C2^{-kn}\sum_{i=0}^{m}\|b\|_*^{m-i}|b(x)-b_k|^i\Big(\sum_{j=0}^{k-2}2^{j\alpha p}\|f\widetilde{\chi}_j\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\\
&\quad + C2^{-kn}\sum_{i=0}^{m}k^i\|b\|_*^m\Big(\sum_{j=0}^{k-2}2^{j\alpha p}\|f\widetilde{\chi}_j\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}
\end{aligned}$$

$$\leq C2^{-kn}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}\Big(\sum_{i=1}^{m}\|b\|_*^{m-i}|b(x)-b_k|^i+k^m\|b\|_*^m\Big)$$
$$=\sum_{i=1}^{m}\mathcal{C}_i+\mathcal{C}_{m+1}.$$

This implies that

$$\mathcal{B}_1\leq C\sum_{i=1}^{m+1}\Big\{\sum_{k=N_m}^{\infty}2^{k\alpha p}|\{x\in A_k:\mathcal{C}_i>\lambda/(m+1)\}|^{p/q}\Big\}^{1/p}=C\sum_{i=1}^{m+1}\mathcal{D}_i.$$

For $i=1,\dots,m$, using the John–Nirenberg inequality, we obtain

$$\mathcal{D}_i\leq C\bigg\{\sum_{k=N_m}^{\infty}2^{k\alpha p+knp/q}\exp\bigg(-\bigg(\frac{c\lambda 2^{kn}}{\|b\|_*^{m-i}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}\bigg)^{1/i}\bigg)\bigg\}^{1/p}$$
$$=C\bigg\{\sum_{k=N_m}^{\infty}2^{knp}\exp\bigg(-\bigg(\frac{c\lambda 2^{kn}}{\|b\|_*^{m-i}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}\bigg)^{1/i}\bigg)\bigg\}^{1/p}$$
$$\leq C\bigg\{\int_0^{\infty}u^{p-1}\exp\bigg(-\frac{c\lambda u}{\|b\|_*^{m-i}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}\bigg)^{1/i}du\bigg\}^{1/p}$$
$$=\frac{C}{\lambda}\|b\|_*^{m-i}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}\Big\{\int_0^{\infty}v^{p-1}\exp(-v^{1/i})\,dv\Big\}^{1/p}$$
$$\leq\frac{C}{\lambda}\|b\|_*^{m-i}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

For $\mathcal{D}_{m+1}$, if

$$|\{x\in A_k:Ck^m2^{-kn}\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}\|b\|_*^m>\lambda/(m+1)\}|\neq 0,$$

then

$$1<\frac{2^{kn}}{(kn)^m}<\frac{C\|b\|_*^m\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\quad\text{for } k\geq N_m.$$

By Lemma 1.3, this implies

$$2^{kn}\leq\frac{C\|b\|_*^m\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg(\log^+\frac{C\|b\|_*^m\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg)^m.$$

Denoting by K_λ the maximal integer k satisfying the last inequality, we have

$$\mathcal{D}_{m+1}\leq C\Big\{\sum_{k=N_m}^{K_\lambda}2^{k\alpha p+knp/q}\Big\}^{1/p}\leq C\Big\{\sum_{k=N_m}^{K_\lambda}2^{knp}\Big\}^{1/p}\leq C2^{K_\lambda n}$$
$$\leq\frac{C\|b\|_*^m\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg(\log^+\frac{C\|b\|_*^m\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg)^m.$$

The estimates for $\mathcal{D}_1, \dots, \mathcal{D}_{m+1}$ give

$$\mathcal{B}_1 \le \frac{C\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg(1+\bigg(\log^+ \frac{\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)}}{\lambda}\bigg)^m\bigg),$$

where C is independent of f and λ. This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfy condition $\mathfrak{L}$. If the assumptions of Theorem 2.1 are satisfied, then the commutator T_b^m is a bounded operator from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (resp. $K_q^{\alpha,p}(\mathbb{R}^n)$) to $W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (resp. $WK_q^{\alpha,p}(\mathbb{R}^n)$).*

Proof. We only prove the homogeneous case. Let $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. For any integer k, we decompose $f(x) = f_1(x) + f_2(x) + f_3(x)$ as in the previous proof. Thus

$$\begin{aligned}\|f\|_{W\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\le \sum_{i=1}^{3} \sup_{\lambda>0} \lambda\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}|\{x \in A_k : |T_b^m f_i(x)| > \lambda/3\}|^{p/q}\Big\}^{1/p} \\ &= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.\end{aligned}$$

The boundedness of T_b^m on $L^q(\mathbb{R}^n)$, and $0 < p \le 1$, give

$$\begin{aligned}\mathcal{E}_2 &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_2\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{k+1}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p} \le C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_3 &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_3\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big)\Big\}^{1/p} \\ &\le C\Big\{\sum_{j=-\infty}^{\infty} \|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{j-2} 2^{k\alpha p}\Big)\Big\}^{1/p} \\ &\le C\Big\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f\chi_j\|^p_{L^q(\mathbb{R}^n)}\Big\}^{1/p} \le C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

Similarly to the estimate for $\mathcal{B}_1$ in Theorem 2.1, when $x \in A_k$ and $j \le k-2$, we have

$$|T_b^m f_1(x)| \le C 2^{-kn} \sum_{i=0}^{m} \sum_{j=-\infty}^{k-2} |b(x)-b_j|^i 2^{j\alpha} \|b\|_*^{m-i} \|f\widetilde{\chi}_j\|_{L^q(\mathbb{R}^n)}$$
$$\le C 2^{-kn} \sum_{i=0}^{m} |b(x)-b_k|^i \|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

In the last inequality, we have applied condition $\mathfrak{L}$. We thus have

$$\mathcal{E}_1 \le C \sum_{i=0}^{m} \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} |\{x \in A_k : U(x) > \lambda/(m+1)\}|^{p/q} \Big\}^{1/p}$$
$$= \sum_{i=0}^{m} \mathcal{F}_i,$$

where

$$U(x) = C 2^{-kn} \|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} |b(x)-b_k|^i.$$

For $\mathcal{F}_i$ $(i = 1, \dots, m)$, using the John–Nirenberg inequality and proceeding similarly to the case of $\mathcal{D}_i$ in Theorem 2.1, we have

$$\mathcal{F}_i \le C \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p + knp/q} \exp\Big(-\Big(\frac{C\lambda 2^{kn}}{\|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}} \Big)^{1/i} \Big) \Big\}^{1/p}$$
$$\le C \sup_{\lambda>0} \lambda \Big\{ \int_0^\infty u^{p-1} \exp\Big(-\Big(\frac{C\lambda u}{\|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}} \Big)^{1/i} \Big) du \Big\}^{1/p}$$
$$\le C \|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

For any fixed $\lambda > 0$, if

$$|\{x \in A_k : C 2^{-kn} \|b\|_*^m \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} > \lambda/(m+1)\}| \ne 0,$$

then

$$2^{kn} \le \frac{C}{\lambda} \|b\|_*^m \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Letting K_λ be the maximal integer k satisfying the last inequality, we thus get

$$\mathcal{F}_0 \le \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{K_\lambda} 2^{k\alpha p + knp/q} \Big\}^{1/p} \le C \sup_{\lambda>0} \lambda 2^{K_\lambda} \le C \|b\|_*^m \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Hence we obtain the following estimate:

$$\mathcal{E}_1 \le \sum_{i=0}^{m} \mathcal{F}_i \le C \sum_{i=0}^{m} \|b\|_*^{m-i} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le C \|f\|_{\dot{K}_q^{\alpha,p}(R^n)}.$$

This finishes the proof of Theorem 2.2.

3. Commutators on Herz-type Hardy spaces. In this section, we obtain, first, the boundedness of T_b^m from nonhomogeneous Herz-type Hardy spaces to corresponding Herz spaces with $b \in \mathrm{BMO}(\mathbb{R}^n)$. Next we show that if $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfies condition $\mathfrak{L}$, then T_b^m is a bounded operator from homogeneous Herz-type Hardy spaces to corresponding Herz spaces. The main results are the following theorems.

THEOREM 3.1. *Let $0 < p \le 1 < q < \infty$, $\alpha = n(1-1/q)$, m a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $f \in HK_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)$,*

$$\|T_b^m f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \le C\|f\|_{HK_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)}.$$

Proof. Since $0 < p \le 1$, by the definition of $HK_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)$ it is sufficient to show that for any central $(\alpha, q, 0; b, m)$-atom a of restricted type, we have $\|T_b^m a\|_{K_q^{\alpha,p}(\mathbb{R}^n)}^p \le C$, where $C > 0$ is independent of a.

Let $\operatorname{supp} a \subset B(0,r)$ and $r = 2^{k_0-1}$ for some $k_0 \in \mathbb{N}$. Then

$$\begin{aligned}\|T_b^m a\|_{K_q^{\alpha,p}(\mathbb{R}^n)}^p &= \sum_{k=0}^{k_0} 2^{k\alpha p}\|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p + \sum_{k=k_0+1}^{\infty} 2^{k\alpha p}\|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p \\ &= \mathcal{G}_1 + \mathcal{G}_2.\end{aligned}$$

The boundedness of T_b^m on $L^q(\mathbb{R}^n)$ gives

$$\mathcal{G}_1 \le C\sum_{k=0}^{k_0} 2^{k\alpha p}\|a\|_{L^q(\mathbb{R}^n)}^p \le C\sum_{k=0}^{k_0} 2^{k\alpha p} r^{-\alpha p} \le C\sum_{k=0}^{k_0} 2^{(k-k_0+1)\alpha p} \le C.$$

We proceed to estimate $\mathcal{G}_2$. It is easy to see that $|x| > 2|y|$ whenever $y \in B(0,r)$ and $x \in A_k$ with $k = k_0+1, k_0+2, \ldots$ By the Hölder inequality and the vanishing moments condition on a, we have

$$\begin{aligned}|T_b^m a(x)| &\le \int_{B(0,r)} |K(x-y) - K(x)|\,|b(x)-b(y)|^m |a(y)|\,dy \\ &\le \int_{B(0,r)} \frac{|y|}{|x|^{n+s'+1}}\,|b(x)-b(y)|^m |a(y)|\,dy \\ &\le Cr2^{-k(n+s'+1)} \sum_{i=0}^{m} |b(x)-b_r|^i \int_{B(0,r)} |b(y)-b_r|^{m-i}|a(y)|\,dy \\ &\le Cr2^{-k(n+s'+1)} \sum_{i=0}^{m} |b(y)-b_r|^i \Big(\int_{B(0,r)} |b(y)-b_r|^{(m-i)q'}\,dy\Big)^{1/q'} \|a\|_{L^q(\mathbb{R}^n)} \\ &\le Cr2^{-k(n+s'+1)} \sum_{i=0}^{m} \|b\|_*^{m-i}|b(x)-b_r|^i,\end{aligned}$$

where b_r is the mean value of b over $B(0,r)$.

Let $2^{j_0-1} < r^{1-s} \le 2^{j_0}$ for some $j_0 \in \mathbb{Z}$. It is easy to see that $j_0 < k_0$ because $r > 1$. Thus

$$\begin{aligned}\mathcal{G}_2 &\le C \sum_{k=k_0+1}^{\infty} 2^{k\alpha p - kp(n+s'+1)} r^p \Big(\sum_{i=0}^{m} \|b\|_*^{(m-i)p} \|(b-b_r)^i \chi_k\|_{L^q(\mathbb{R}^n)}^p \Big) \\ &\le C \sum_{k=k_0+1}^{\infty} 2^{k\alpha p - kp(n+s'+1)} r^p \Big(\sum_{i=0}^{m} \|b\|_*^{(m-i)p} (k-k_0)^{ip} 2^{knp/q} \|b\|_*^{ip} \Big) \\ &\le C r^p \|b\|_*^{mp} \sum_{k=j_0+1}^{\infty} (k-j_0)^{mp} 2^{-kp(s'+1)} \\ &\le c r^p \|b\|_*^{mp} 2^{-j_0 p(s'+1)} \le c r^p \|b\|_*^{mp} r^{-(1-s)(s'+1)p} \le c \|b\|_*^{mp}.\end{aligned}$$

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ satisfy condition $\mathfrak{L}$. If $0 < p \le 1 < q < \infty$, $\alpha = n(1-1/q)$ and m is a positive integer, then there exists a constant $C > 0$ such that for any $f \in H\dot{K}_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)$,*

$$\|T_b^m f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le C \|f\|_{H\dot{K}_{q,b,m}^{\alpha,p,0}(\mathbb{R}^n)}.$$

Proof. As in the proof of Theorem 3.1, it is sufficient to show that for any central $(\alpha, q, 0; b, m)$-atom a, we have $\|T_b^m a\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le C$, where $C > 0$ is independent of a. Let supp $a \subset B(0,r)$ and $r = 2^{k_0-1}$ for some $k_0 \in \mathbb{Z}$. Then

$$\begin{aligned}\|T_b^m a\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p + \sum_{k=k_0+1}^{\infty} 2^{k\alpha p} \|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p \\ &= \mathcal{H}_1 + \mathcal{H}_2.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\mathcal{H}_1 &\le C \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|a\|_{L^q(\mathbb{R}^n)}^p \le C \sum_{k=-\infty}^{k_0} 2^{k\alpha p} r^{-\alpha p} \\ &\le C \sum_{k=-\infty}^{k_0} 2^{(k-k_0+1)\alpha p} \le C.\end{aligned}$$

Let $2^{j_0-1} < r^{1-s} \le 2^{j_0}$ for some $j_0 \in \mathbb{Z}$. Since a is a central $(\alpha, q, 0; b, m)$-atom, we only need to estimate $\mathcal{H}_2$ with $k_0 < j_0$, because in the case of $j_0 \le k_0$, the estimate follows from Theorem 3.1. Thus

$$\mathcal{H}_2 \le \sum_{k=k_0+1}^{j_0} 2^{k\alpha p} \|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p + \sum_{k=j_0+1}^{\infty} 2^{k\alpha p} \|\chi_k T_b^m a\|_{L^q(\mathbb{R}^n)}^p = \mathcal{I}_1 + \mathcal{I}_2.$$

Similarly to the estimate for $\mathcal{G}_2$, we get

$$\mathcal{I}_2 \le C\|b\|_*^{mp}.$$

To estimate $\mathcal{I}_1$, we write

$$\begin{aligned} T_b^m a(x) &= \int_{B(0,r)} \frac{e^{i|x-y|^{-s'}}}{|x-y|^n} (b(x)-b(y))^m a(y)\, dy \\ &= \int_{B(0,r)} \frac{e^{i|x-y|^{-s'}}}{|x-y|^{n(s'+2)/t}} \left(\frac{1}{|x-y|^{n(1-(s'+2)/t)}} - \frac{1}{|x|^{n(1-(s'+2)/t)}} \right) \\ &\qquad\qquad \times (b(x)-b(y))^m a(y)\, dy \\ &\quad + \frac{1}{|x|^{n(1-(s'+2)/t)}} \int_{B(0,r)} \frac{e^{i|x-y|^{-s'}}}{|x-y|^{n(s'+2)/t}} (b(x)-b(y))^m a(y)\, dy \\ &= \mathcal{I}_{11}(x) + \mathcal{I}_{12}(x). \end{aligned}$$

Applying the mean value theorem to the term in brackets in the integrand of $\mathcal{I}_{11}(x)$, we obtain a pointwise estimate for $\mathcal{I}_{11}(x)$ on A_k as follows:

$$\begin{aligned} |\mathcal{I}_{11}(x)| &\le C \int_{B(0,r)} \frac{|y|}{|x|^{n+1}} |\, b(x)-b(y)|^m |a(y)|\, dy \\ &\le Cr2^{-k(n+1)} \sum_{i=0}^{m} |b(x)-b_r|^i \int_{B(0,r)} |b(y)-b_r|^{m-i} |a(y)|\, dy \\ &\le Cr2^{-k(n+1)} \sum_{i=0}^{m} |b(x)-b_r|^i \left(\int_{B(0,r)} |b(y)-b_r|^{(m-i)q'}\, dy \right)^{1/q'} \|a\|_{L^q(\mathbb{R}^n)} \\ &\le Cr2^{-k(n+1)} \sum_{i=0}^{m} \|b\|_*^{m-i} |b(x)-b_r|^i. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=k_0+1}^{j_0} 2^{k\alpha p} \|\chi_k \mathcal{I}_{11}\|_{L^q(\mathbb{R}^n)}^p \\ &\qquad \le C \sum_{k=k_0+1}^{j_0} r^p 2^{k(\alpha-n-1)p} \left(\sum_{i=0}^{m} \|b\|_*^{(m-i)p} \left(\int_{C_k} |b(x)-b_r|^{iq}\, dx \right)^{p/q} \right) \end{aligned}$$

$$\leq C \sum_{k=k_0+1}^{j_0} r^p 2^{k(\alpha-n-1)p}\Big(\sum_{i=0}^{m} \|b\|_*^{mp}(k-k_0)^{ip}2^{knp/q}\Big)$$

$$\leq Cr^p\|b\|_*^{mp} \sum_{k=k_0+1}^{j_0} (k-k_0)^{mp}2^{-kp} \leq Cr^p\|b\|_*^{mp}2^{-k_0p} \leq C\|b\|_*^{mp}.$$

On the other hand, we have

$$|\mathcal{I}_{12}(x)| \leq C\sum_{i=0}^{m} \frac{|b(x)-b_r|^i}{|x|^{n(1-(s'+2)/t)}}\,|\widetilde{K}_{s',t} * ((b-b_r)^{m-i}a)(x)|.$$

Noticing that t may be chosen large enough to satisfy $t > \max(q, q')$ and $(s'+2)/t < 1$, we apply the Hölder inequality, Lemma 1.2 and condition $\mathfrak{L}$ to obtain

$$\|\chi_k\mathcal{I}_{12}\|_{L^q(\mathbb{R}^n)} \leq C\sum_{i=0}^{m}\bigg\{\int_{A_k} \frac{|b(x)-b_r|^{iq}}{|x|^{nq(1-(s'+2)/t)}}|\widetilde{K}_{s',t} * ((b-b_r)^{m-i}a)(x)|^q\,dx\bigg\}^{1/q},$$

where

$$\bigg\{\int_{A_k} |b(x)-b_r|^{iq}|\widetilde{K}_{s',t} * ((b-b_r)^{m-i}a)(x)|^q\,dx\bigg\}^{1/q}$$

$$\leq \bigg(\int_{A_k} |b(x)-b_r|^{itq/(t-q)}\,dx\bigg)^{(t-q)/tq}\|\widetilde{K}_{s',t} * ((b-b_r)^{m-i}a)\|_{L^t(\mathbb{R}^n)}$$

$$\leq \bigg(\int_{A_k} |b(x)-b_k|^{itq/(t-q)}\,dx\bigg)^{(t-q)/tq}\|(b-b_r)^{m-i}a\|_{L^{t'}(\mathbb{R}^n)}$$

$$\leq 2^{kn(1/q-1/t)}\sum_{i=0}^{m}\|b\|_*^i\bigg(\int_{B(0,r)} |b(x)-b_r|^{(m-i)t'q/(q-t')}\,dx\bigg)^{1/t'-1/q}\|a\|_{L^q(\mathbb{R}^n)}.$$

This implies that

$$\|\chi_k\mathcal{I}_{12}\|_{L^q(\mathbb{R}^n)} \leq C2^{-k\alpha+kn(s'+1)/t}\sum_{i=0}^{m}\|b\|_*^i r^{n(1/t'-1/q)-\alpha}\|b\|_*^{m-i}$$
$$\leq C2^{-k\alpha+kn(s'+1)/t}r^{-n/t}\|b\|_*^m.$$

This gives

$$\sum_{k=k_0+1}^{j_0} 2^{k\alpha p}\|\chi_k\mathcal{I}_{12}\|_{L^q(\mathbb{R}^n)}^p \leq C\sum_{k=k_0+1}^{j_0} 2^{kn(s'+1)p/t}r^{-np/t}\|b\|_*^{mp}$$
$$\leq C2^{j_0n(s'+1)p/t}r^{-np/t}\|b\|_*^{mp}$$
$$\leq Cr^{(1-s)(s'+1)np/t}r^{-np/t}\|b\|_*^{mp} \leq C\|b\|_*^{mp}.$$

Thus, we have $\mathcal{I}_1 \leq C\|b\|_*^{mp}$. The proof of Theorem 3.2 is complete.

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