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# Geometric characterization for affine mappings and Teichmüller mappings

by

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**Abstract.** We characterize affine mappings on the unit disk and on rectangles by module conditions. The main result generalizes the classic Schwarz lemma. As an application, we give a sufficient condition for a $K$-quasiconformal mapping on a Riemann surface to be a Teichmüller mapping.

**1. Preliminaries.** Let $\Gamma$ be a family of curves in the plane. Each $\gamma \in \Gamma$ is a countable union of open arcs, closed arcs or closed curves, and every closed subarc is rectifiable. We shall define the *extremal length* $\lambda(\Gamma)$ of $\Gamma$.

A function $\varrho$, defined on the whole plane, is called *admissible* if the following conditions are satisfied:

(i) $\varrho$ is a non-negative Borel function,
(ii) $A(\varrho) = \iint_{\mathbb{C}} \varrho^2 \, dx \, dy \neq 0, \infty$.

If such a $\varrho$ is measurable as a function of arc length on $\gamma$, set

$$L_\gamma(\varrho) = \int_\gamma \varrho \, |dz|. \tag{1.1}$$

Otherwise, set $L_\gamma(\varrho) = \infty$. Define

$$L(\varrho) = \inf_{\gamma \in \Gamma} L_\gamma(\varrho), \tag{1.2}$$

$$\lambda(\Gamma) = \sup_{\varrho} \frac{L(\varrho)^2}{A(\varrho)}, \tag{1.3}$$

where the supremum is taken over all admissible $\varrho$. We write $\Gamma_1 < \Gamma_2$ if for every $\gamma_2 \in \Gamma_2$ there is a $\gamma_1 \in \Gamma_1$ which is a subarc of $\Gamma_2$. By the above definitions, the extremal length is monotonic:

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Proposition 1.1. *If* $\mathit{\Gamma}_1 < \mathit{\Gamma}_2$, *then* $\lambda(\mathit{\Gamma}_1) \le \lambda(\mathit{\Gamma}_2)$.

Let $\mathit{\Gamma}_1, \mathit{\Gamma}_2$ be two families of curves. Set

$$\mathit{\Gamma}_1 + \mathit{\Gamma}_2 = \{\gamma_1 + \gamma_2 \mid \gamma_1 \in \mathit{\Gamma}_1,\ \gamma_2 \in \mathit{\Gamma}_2\}.$$

The extremal length of curve families has the following subadditivity property:

Proposition 1.2. *If* $\mathit{\Gamma}_1$ *and* $\mathit{\Gamma}_2$ *lie in mutually disjoint sets*, *then*

$$\lambda(\mathit{\Gamma}_1 + \mathit{\Gamma}_2) \ge \lambda(\mathit{\Gamma}_1) + \lambda(\mathit{\Gamma}_2).$$

For details about the properties of extremal length we refer to [1].

A *quadrilateral* consists of a Jordan domain $Q$ and a sequence $z_1, z_2, z_3, z_4$ of boundary points of $Q$. The points $z_i$ are called the *vertices* of the quadrilateral, and divide its boundary into four Jordan arcs, called the *sides* of the quadrilateral. The arcs $z_1z_2$ and $z_3z_4$ are called the *a-sides* and the other two the *b-sides* of $Q$. Let $\mathit{\Gamma}_a$ be the family of curves that connect the $a$-sides in $Q$, and $\mathit{\Gamma}_b$ the family of curves that connect the $b$-sides in $Q$. Define the *module* of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ to be $\lambda(\mathit{\Gamma}_a)$, the extremal length of $\mathit{\Gamma}_a$. That is,

$$\operatorname{mod} Q(z_1, z_2, z_3, z_4) = \sup_{\varrho} \frac{(\inf_{\gamma\in\mathit{\Gamma}_a} L_\gamma(\varrho))^2}{A(\varrho)}.$$

For example, for a rectangle with width $a$ and height $b$, its module is $a/b$.

If $\varrho = 1$ in (1.1), then $L_\gamma(\varrho)$ is the euclidean length of $\gamma$, simply denoted by $|\gamma|$. We call

$$s_a = s_a(Q) = \inf_{\gamma\in\mathit{\Gamma}_b} |\gamma|$$

the *distance between the a-sides* of $Q$. The *distance* $s_b$ *between the b-sides* is defined analogously. Let $m(Q)$ be the euclidean area of $Q$. The following Rengel inequality plays an important role in this paper.

Proposition 1.3 ([7]). *The module of a quadrilateral* $Q$ *satisfies the double inequality*

$$\frac{(s_b(Q))^2}{m(Q)} \le \operatorname{mod} Q \le \frac{m(Q)}{(s_a(Q))^2}. \tag{1.4}$$

Remark 1.4. This inequality is usually called Rengel's inequality in the literature. However, it was first given by H. Grötzsch (see [6]).

The theory of quasiconformal mappings is closely related with the study of extremal length. In fact, the geometric definition of a quasiconformal mapping is based on moduli of quadrilaterals, which are represented by extremal length. Precisely, a function $f(z)$, which is a sense-preserving homeomorphism of $\mathit{\Omega}$ onto $\mathit{\Omega}'$, is $K$-quasiconformal if for every quadrilateral

$Q = Q(z_1, z_2, z_3, z_4) \subset \Omega$,

$$\frac{1}{K} \operatorname{mod} Q \le \operatorname{mod} f(Q) \le K \operatorname{mod} Q. \tag{1.5}$$

Denote the unit disk by $\Delta$. It follows from the Riemann mapping theorem that every quadrilateral $Q(z_1, z_2, z_3, z_4)$ can be mapped onto a quadrilateral $\Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ with domain $\Delta$ and vertices $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ on $\partial\Delta$. By Schwarz–Christoffel,

$$\Phi(\zeta) = \int^{\zeta} \frac{d\zeta}{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}$$

conformally maps $\Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ onto a rectangle. By combining the above mappings, we can map an arbitrary quadrilateral conformally onto a rectangle. Therefore, from conformal invariance of extremal length, the module of any quadrilateral can always be represented by that of its conformally equivalent rectangle.

Define the *module of an annulus*

$$A(r_1, r_2) = \{z \mid r_1 \le |z| \le r_2\} \tag{1.6}$$

to be the extremal length of the family $\Gamma$ of curves that connect $\{z \mid |z| = r_1\}$ and $\{z \mid |z| = r_2\}$ in $A(r_1, r_2)$. Since $\lambda(\Gamma)$ is conformally invariant and every ring domain can be mapped conformally onto an annulus, the number

$$\operatorname{mod} A(r_1, r_2) = \frac{1}{2\pi} \log \frac{r_2}{r_1} \tag{1.7}$$

represents the moduli of all ring domains which are conformally equivalent to $A$.

Define the *module of a sector*

$$A(r_1, r_2; \theta_1, \theta_2) = \{z \mid r_1 \le |z| \le r_2,\ \theta_1 \le \arg z \le \theta_2\} \tag{1.8}$$

to be the extremal length of the family $\Gamma$ of curves that connect $\{z \mid |z| = r_1\}$ and $\{z \mid |z| = r_2\}$ in $A(r_1, r_2; \theta_1, \theta_2)$. Then

$$\operatorname{mod} A(r_1, r_2; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \log \frac{r_2}{r_1}. \tag{1.9}$$

**2. Affine mappings on the unit disk $\Delta$.** Quasiconformality of a domain is characterized not only by the module of a quadrilateral $Q$ as in inequality (1.5), but also by that of horizontal rectangles. A *horizontal rectangle* is a quadrilateral whose $a$-sides are parallel to the $x$-axis and $b$-sides are parallel to the $y$-axis. For the relevant results, we refer to [3] and [4]. In this section, we shall prove

THEOREM 2.1. *Let $f(z)$ be a sense-preserving homeomorphism of the unit disk $\Delta$ onto itself, with normalization $f(0) = 0$. If*

(a) $$\frac{1}{K}\operatorname{mod} A \le \operatorname{mod} f(A),$$

*where* $A$ *stands for all* $A(r_1, r_2)$ *and* $A(r_1, r_2; \theta_1, \theta_2)$ *as in* (1.6) *and* (1.8), *and*

(b) $$\lim_{z\to 0}\frac{|f(z)|}{|z|^{1/K}} = 1,$$

*then*

$$f(z) = \lambda z|z|^{1/K-1} \qquad (|\lambda| = 1).$$

In view of Theorem 2.1, we immediately generalize the Schwarz lemma to sense-preserving homeomorphisms.

COROLLARY 2.2. *Assume that* $f(z)$ *is a sense-preserving homeomorphism of the unit disk* $\Delta$ *onto itself*, *with normalization* $f(0) = 0$. *If* $f$ *satisfies conditions* (a) *and* (b), *where* $K = 1$ *in* (a), *then*

$$f(z) = \lambda z \qquad (|\lambda| = 1).$$

For the proof of the theorem, we need three lemmas.

LEMMA 2.3. *If* $f$ *satisfies condition* (a) *of Theorem* 2.1, *then it is absolutely continuous on* $\{|z| = r\}$.

*Proof.* Let $A = \{z = re^{i\theta} \mid r_1 < r < r_2,\ 0 \le \theta < 2\pi\}$ be an annulus in $\Delta$. Set

$$q(t) = m(f(A_t)),$$

where $A_t = \{z = re^{i\theta} \mid r_1 < r < t,\ 0 \le \theta < 2\pi\}$ and $m(f(A_t))$ is the euclidean area of $f(A_t)$. Obviously, $q(t)$ is an increasing function of $t$ and thus has a finite derivative $q'(t)$ for all $t$, $r_1 < t < r_2$, except for a set of zero linear measure. Assume that $q'(t_0)$ exists and is finite. We shall prove that $f(t_0e^{i\theta})$ is absolutely continuous on $[0, 2\pi]$.

We first choose a positive $\delta$ such that $t_0+\delta < r_2$. Let $(\theta_k, \theta_k^*)$, $k = 1, \dots, n$, be an arbitrary system of non-intersecting open subintervals of $[0, 2\pi]$. Define a sector (a special quadrilateral) by

$$G_k^\delta = \{z = re^{i\theta} \mid t_0 < r < t_0 + \delta,\ \theta_k < \theta < \theta_k^*\}.$$

Then by (1.9),

$$\operatorname{mod} G_k^\delta = \frac{1}{\theta_k^* - \theta_k}\log\frac{t_0+\delta}{t_0}. \tag{2.1}$$

For the module of the image of $G_k^\delta$, from the right hand side of Rengel's inequality we have

$$\operatorname{mod} f(G_k^\delta) \le \frac{m(f(G_k^\delta))}{(d_k^\delta)^2}, \tag{2.2}$$

where $d_k^\delta$ denotes the euclidean distance between the $a$-sides of $f(G_k^\delta)$. These sides converge to the points $f(t_0e^{i\theta_k})$ and $f(t_0e^{i\theta_k^*})$ as $\delta \to 0$, and so

$$\lim_{\delta\to 0} d_k^\delta = |f(t_0e^{i\theta_k^*}) - f(t_0e^{i\theta_k})|. \tag{2.3}$$

According to condition (a), we have

$$\frac{1}{K}\operatorname{mod} G_k^\delta \le \operatorname{mod} f(G_k^\delta). \tag{2.4}$$

From (2.2), (2.4) and (2.1), we get

$$\frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \le \frac{K(\theta_k^* - \theta_k)}{\log(1+\delta/t_0)}, \qquad k = 1, \dots, n.$$

Noticing that $\log(1 + \delta/t_0) \ge \delta/(2t_0)$ for $\delta < t_0$ and adding the above inequalities over $k$ we have

$$\sum_{k=1}^n \frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \le 2Kt_0 \sum_{k=1}^n \frac{\theta_k^* - \theta_k}{\delta}, \tag{2.5}$$

By the Schwarz inequality,

$$\Big(\sum_{k=1}^n d_k^\delta\Big)^2 \le \sum_{k=1}^n \frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \sum_{k=1}^n m(f(G_k^\delta)).$$

Observing that

$$\sum_{k=1}^n m(f(G_k^\delta)) \le q(t_0+\delta) - q(t_0),$$

we obtain

$$\Big(\sum_{k=1}^n d_k^\delta\Big)^2 \le 2Kt_0\,\frac{q(t_0+\delta)-q(t_0)}{\delta} \sum_{k=1}^n |\theta_k^* - \theta_k|. \tag{2.6}$$

Letting $\delta \to 0$, from (2.3) and (2.6) we obtain

$$\Big(\sum_{k=1}^n |f(t_0e^{i\theta_k^*}) - f(t_0e^{i\theta_k})|\Big)^2 \le 2Kt_0q'(t_0) \sum_{k=1}^n |\theta_k^* - \theta_k|.$$

This completes the proof of the lemma. ∎

Define

$$R(r) = \{z \mid r \le |z| \le 1\}.$$

There exists a conformal maping $\Phi_r$ such that

$$\Phi_r \circ f(R(r)) = \{\zeta \mid \phi(r) \le |\zeta| \le 1\}.$$

Set $g_r = \Phi_r \circ f$. We shall give the properties of $\phi(r)$, which was first considered by H. Grötzsch for $f$ $K$-quasiconformal (see [5] and [2]). However, it is not obvious from conditions (a) and (b) that the sense-preserving mapping

$f$ in Theorem 2.1 is $K$-quasiconformal. Following Grötzsch's idea, we prove Lemma 2.4 directly by extremal length methods.

LEMMA 2.4. *If $f$ satisfies condition* (a), *then $\phi(r)/r^{1/K}$ is an increasing function on* $(0,1]$, *and* $\phi(r)/r^{1/K} \le 1$.

*Proof.* Assume that $0 < r_1 < r_2 \le 1$. Then $g_{r_1}(z)$ maps $\{z \mid r_1 \le |z| \le 1\}$ onto $\{\zeta \mid \phi(r_1) \le |\zeta| \le 1\}$. The definition of moduli of ring domains together with Proposition 1.2 yields

$$\begin{aligned} \text{(2.7)} \qquad &\operatorname{mod} g_{r_1}\{z \mid r_1 \le |z| \le r_2\} + \operatorname{mod} g_{r_1}\{z \mid r_2 \le |z| \le 1\} \\ &\qquad \le \operatorname{mod} g_{r_1}\{z \mid r_1 \le |z| \le 1\} = \frac{1}{2\pi}\log\frac{1}{\phi(r_1)}. \end{aligned}$$

From conformal invariance of extremal length, it follows that

$$\begin{aligned} \text{(2.8)} \qquad &\operatorname{mod} g_{r_1}\{z \mid r_2 \le |z| \le 1\} = \operatorname{mod} f\{z \mid r_2 \le |z| \le 1\} \\ &\qquad = \operatorname{mod} g_{r_2}\{z \mid r_2 \le |z| \le 1\} = \frac{1}{2\pi}\log\frac{1}{\phi(r_2)}. \end{aligned}$$

Substituting (2.8) into (2.7), we have

$$\text{(2.9)} \qquad \operatorname{mod} g_{r_1}\{z \mid r_1 \le |z| \le r_2\} \le \frac{1}{2\pi}\log\frac{\phi(r_2)}{\phi(r_1)}.$$

On the other hand, from condition (a) it follows that

$$\text{(2.10)} \qquad \operatorname{mod} g_{r_1}\{z \mid r_1 \le |z| \le r_2\} = \operatorname{mod} f\{z \mid r_1 \le |z| \le r_2\} \ge \frac{1}{2\pi K}\log\frac{r_2}{r_1}.$$

Combining (2.9) with (2.10), we obtain

$$\left(\frac{r_2}{r_1}\right)^{1/K} \le \frac{\phi(r_2)}{\phi(r_1)}, \quad \text{that is,} \quad \frac{\phi(r_1)}{r_1^{1/K}} \le \frac{\phi(r_2)}{r_2^{1/K}}.$$

So, the monotonicity of $\phi(r)/r^{1/K}$ is proved. From the above inequality, by letting $r_2 = 1$, we get $\phi(r)/r^{1/K} \le 1$ for $r \in (0,1]$. This completes the proof of the lemma. ■

LEMMA 2.5. *If $f$ satisfies condition* (b), *then*

$$\lim_{r\to 0}\frac{\phi(r)}{r^{1/K}} = 1.$$

*Proof.* For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $r < \delta$,

$$1 - \varepsilon < \frac{|f(re^{i\theta})|}{r^{1/K}} < 1 + \varepsilon.$$

Therefore,

$$\{w \mid (1+\varepsilon)r^{1/K} < |w| \le 1\} \subset f(R(r)) \subset \{w \mid (1-\varepsilon)r^{1/K} < |w| \le 1\}.$$

By Proposition 1.1,

$$\frac{1}{2\pi}\log\frac{1}{(1+\varepsilon)r^{1/K}} \le \operatorname{mod} f(R(r)) \le \frac{1}{2\pi}\log\frac{1}{(1-\varepsilon)r^{1/K}}.$$

Therefore,

$$\frac{1}{2\pi}\log\frac{1}{(1+\varepsilon)r^{1/K}} \le \frac{1}{2\pi}\log\frac{1}{\phi(r)} \le \frac{1}{2\pi}\log\frac{1}{(1-\varepsilon)r^{1/K}}.$$

That is,

$$1-\varepsilon \le \frac{\phi(r)}{r^{1/K}} \le 1+\varepsilon.$$

This shows $\lim_{r\to 0}\phi(r)/r^{1/K} = 1$. ■

*Proof of Theorem 2.1.* From Lemmas 2.4 and 2.5, we obtain

$$\phi(r) = r^{1/K}, \quad 0 \le r \le 1.$$

This means

$$|g_r(z)| = |z|^{1/K} \quad (r \le |z| \le 1).$$

Taking $r = 1/n$, we write

$$g_n(z) = |z|^{1/K} e^{ih_n(\theta)} \quad (1/n \le |z| \le 1), \tag{2.11}$$

where $\theta = \arg z$ and $h_n$ is a homeomorphism from $[0, 2\pi]$ onto $[\theta_n, \theta_n + 2\pi]$ $(0 \le \theta_n < 2\pi)$. According to Lemma 2.3, $f$ is absolutely continuous on $\{|z| = r\}$, and so is $g_n$. Therefore, $h_n'(\theta)$ exists almost everywhere and is integrable on $[0, 2\pi]$, and

$$\int_0^{2\pi} h_n'(\theta)\, d\theta = 2\pi. \tag{2.12}$$

Direct computation on generalized derivatives of $g_n(z)$ in (2.11) shows

$$\begin{aligned} \partial_z g_n(z) &= \frac{z}{2|z|^{2+1/K}}\, e^{-ih_n(\theta)}\left(\frac{1}{K} + h_n'(\theta)\right),\\ \partial_{\bar z} g_n(z) &= \frac{\bar z}{2|z|^{2+1/K}}\, e^{-ih_n(\theta)}\left(\frac{1}{K} - h_n'(\theta)\right). \end{aligned} \tag{2.13}$$

Since $g_n$ is sense-preserving,

$$|\partial_z g_n(z)| > |\partial_{\bar z} g_n(z)|. \tag{2.14}$$

Therefore, from (2.13) and (2.14) we infer that $h_n'(\theta) > 0$. Set

$$P = \{z \mid r \le |z| \le r + dr,\ \theta \le \arg z \le \theta + d\theta\}.$$

Then

$$g_n(P) = \{\zeta \mid r^{1/K} \le |\zeta| \le (r+dr)^{1/K},\ h_n(\theta) \le \arg\zeta \le h_n(\theta) + h_n'(\theta)d\theta\}.$$

By conformal invariance of extremal length and condition (a), we have

$$\operatorname{mod} g_n(P) \geq \frac{1}{K} \operatorname{mod} P.$$

From the above inequality, it follows that $h_n'(\theta) \leq 1$. Thus,

$$0 < h_n'(\theta) \leq 1. \tag{2.15}$$

From (2.12) and (2.15), we obtain $h_n'(\theta) = 1$ a.e. on $[0, 2\pi)$. Therefore, $h_n(\theta) = \theta_n + \theta$, and we can rewrite (2.11) as follows:

$$g_n(z) = z|z|^{1/K-1} e^{i\theta_n} \qquad (1/n \leq |z| \leq 1). \tag{2.16}$$

Since $\theta_n \in [0, 2\pi)$, there exists a subsequence $\{\theta_{n_k}\}$ such that $\lim_{k\to\infty} \theta_{n_k} = \theta_0$. Therefore,

$$\lim_{k\to\infty} g_{n_k}(z) = g(z) = z|z|^{1/K-1} e^{i\theta_0} \qquad (\text{locally uniformly in } \Delta - \{0\}).$$

So,

$$\begin{aligned} \lim_{k\to\infty} \Phi_{n_k}(w) &= \lim_{k\to\infty} g_{n_k} \circ f^{-1}(w) \\ &= g \circ f^{-1}(w) := \Phi(w) \qquad (\text{locally uniformly in } \Delta - \{0\}). \end{aligned}$$

From compactness of the family of analytic functions, $\Phi$ is analytic in $\Delta - \{0\}$. Observing that $\lim_{w\to 0} \Phi(w) = 0$, we conclude that the origin is a removable singular point and hence $\Phi$ is analytic in the unit disk. Since

$$\lim_{w\to 0} \frac{|\Phi(w)|}{|w|} = \lim_{z\to 0} \frac{\Phi \circ f(z)}{f(z)} = \lim_{z\to 0} \frac{|z|^{1/K}}{|f(z)|} = 1,$$

by the Schwarz lemma we have $\Phi(w) = \lambda w$. Thus, $f(z) = \lambda z|z|^{1/K-1}$ where $|\lambda| = 1$. ∎

Corollary 2.6. *If $f$ is $K$-quasiconformal mapping of $\Delta$ onto itself which satisfies $f(0) = 0$ and*

$$\lim_{z\to 0} \frac{|f(z)|}{|z|^K} = 1,$$

*then $f(z) = \lambda z|z|^{K-1}$.*

Problem 2.7. If we replace conditions (a) and (b) by

$$\operatorname{mod} f(A) \leq K \operatorname{mod} A \qquad \text{for all } A \subset \Delta, \tag{a$^*$}$$

$$\lim_{z\to 0} \frac{|f(z)|}{|z|^K} = 1, \tag{b$^*$}$$

is it true that $f(z) = \lambda z|z|^{K-1}$?

## 3. Affine mappings on rectangles

LEMMA 3.1 ([1]). *Let $R = \{z \mid 0 \le x \le a,\ 0 \le y \le 1\}$. Let $C$ be a curve in $R$ that connects the $a$-sides of $R$ and hence divides $R$ into two parts $R_1$ and $R_2$. Then $R_1$ and $R_2$ are rectangles if and only if*

$$\operatorname{mod} R_1 + \operatorname{mod} R_2 = \operatorname{mod} R = a.$$

LEMMA 3.2. *Let $f$ be a sense-preserving homeomorphism which maps $\{z \mid r_0 \le |z| \le 1\}$ onto $\{w \mid r_0^{1/K} \le |w| \le 1\}$. If*

$$\frac{1}{K} \operatorname{mod} A \le \operatorname{mod} f(A) \tag{3.0}$$

*for all $A \subset \{z \mid r_0 \le |z| \le 1\}$ appearing in* (1.6) *and* (1.8), *then*

$$f(z) = \lambda z|z|^{1/K-1} \qquad (|\lambda| = 1).$$

*Proof.* Extend $f$ by setting

$$f(z) = r_0^{2n/K}/f(r_0^{2n}/\bar{z}), \qquad r_0^{n+1} \le |z| \le r_0^n,\ n = 1, 2, \ldots$$

Then the extended $f$ is a sense-preserving homeomorphism of $\Delta$ onto itself with $f(0) = 0$. Furthermore, it is not difficult to prove that the extended $f$ satisfies (a) and (b). Hence, by Theorem 2.1,

$$f(z) = \lambda z|z|^{1/K-1}. \ \blacksquare$$

THEOREM 3.3. *Assume that $f$ is a sense-preserving homeomorphism which maps the rectangle $R = \{z \mid 0 \le x \le a,\ 0 \le y \le 1\}$ onto $f(R) = \{w \mid 0 \le u \le a',\ 0 \le v \le 1\}$. If for every horizontal subrectangle $R_h$ ($\subset R$),*

$$\frac{1}{K} \operatorname{mod} R_h \le \operatorname{mod} f(R_h), \tag{3.1}$$

*then*

$$f(z) = \frac{1}{K} x + iy, \tag{3.2}$$

*where $K = \operatorname{mod} R/\operatorname{mod} f(R) = a/a'$.*

*Proof.* Set $R_1 = \{z \mid 0 \le x \le a_1,\ 0 \le y \le 1\}$, $R_2 = \{z \mid a_1 \le x \le a,\ 0 \le y \le 1\}$. By (3.1), we have

$$\frac{1}{K} \operatorname{mod} R_1 \le \operatorname{mod} f(R_1), \qquad \frac{1}{K} \operatorname{mod} R_2 \le \operatorname{mod} f(R_2).$$

Therefore,

$$\frac{1}{K} \operatorname{mod} R = \frac{1}{K}(\operatorname{mod} R_1 + \operatorname{mod} R_2) \le \operatorname{mod} f(R_1) + \operatorname{mod} f(R_2). \tag{3.3}$$

On the other hand, by Proposition 1.2,

$$\operatorname{mod} f(R_1) + \operatorname{mod} f(R_2) \le \operatorname{mod} f(R) = \frac{1}{K} \operatorname{mod} R. \tag{3.4}$$

Combining (3.3) with (3.4), we have

$$\operatorname{mod} f(R_1) + \operatorname{mod} f(R_2) = \operatorname{mod} f(R).$$

By Lemma 3.1, $f(R_1)$ and $f(R_2)$ are rectangles. That is,

$$f(x) = f(x+i), \qquad 0 \le x \le a. \tag{3.5}$$

It follows from (3.5) that $f$ can be extended to the strip

$$S = \{z \mid 0 \le x \le a\}.$$

By the extremal length method, it is not difficult to prove that for the extended function $f$, which maps $S$ onto another strip

$$S' = \{w \mid 0 \le u \le a'\},$$

inequality (3.1) still holds for any $R_h \subset S$. Set

$$G = \{\zeta \mid e^{-a} \le |\zeta| \le 1\}, \qquad G' = \{\eta \mid e^{-a'} \le |\eta| \le 1\}.$$

Then $\{S, e^{-a+z}\}$, $\{S', e^{-a'+w}\}$ are universal covering surfaces of $G$ and $G'$ respectively.

The projective mapping of the function $f$,

$$g(\zeta) = e^{-a'+f(a+\log\zeta)}, \tag{3.6}$$

is a sense-preserving homeomorphism and satisfies inequality (3.0). In view of Lemma 3.2, we have

$$g(\zeta) = \lambda\zeta|\zeta|^{1/K-1} \qquad (|\lambda| = 1). \tag{3.7}$$

Combining (3.6) with (3.7) and observing that $f(0) = 0$, we obtain

$$f(z) = \frac{1}{K}x + iy.$$

This ends the proof of the theorem. ■

The above theorem yields immediately

Corollary 3.4. *Let $f$ be a sense-preserving homeomorphism which maps a rectangle $R$ onto a quadrilateral $Q$ and satisfies the following two conditions*:

(c) $K^{-1} \operatorname{mod} R_h \le \operatorname{mod} f(R_h)$ *for every horizontal subrectangle* $R_h \subset R$,

(d) $K^{-1} \operatorname{mod} R = \operatorname{mod} Q$.

*Then $f$ is $K$-quasiconformal and is a Teichmüller mapping.*

Remark 3.5. It might seem that Corollary 3.4 should have a symmetric form, with (c) and (d) replaced respectively by

$$\operatorname{mod} R_h \le K \operatorname{mod} f(R_h), \tag{c*}$$

$$\operatorname{mod} R = K \operatorname{mod} f(R_h). \tag{d*}$$

But unfortunately, this is not the case. In fact, we can find a counterexample in [3], where $f$ is $(K + \sqrt{K^2 - 1})$-quasiconformal.

**4. Geometric characterization for Teichmüller mappings.** Let $S_1$ and $S_2$ be two Riemann surfaces. Let $\omega$ be a holomorphic quadratic differential on $S_1$. A quasiconformal mapping $f$ from $S_1$ onto $S_2$ is called a *Teichmüller mapping* if its Beltrami differential $\mu_f d\bar{z}/dz$ equals $k\overline{\omega}/\omega$. In this section, we want to characterize Teichmüller mappings by a local property.

Let $D_1$ and $D_2$ be simply connected domains of hyperbolic type. From the Riemann mapping theorem, there exist conformal mappings $\Phi_1$ and $\Phi_2$ which map $D_1$ and $D_2$ respectively onto $\Delta$.

THEOREM 4.1. *Suppose that $f$ is a $K$-quasiconformal mapping from $D_1$ onto $D_2$. To $z_0 \in D_1$, there correspond conformal mappings $\Phi_1$ and $\Phi_2$ as above with $\Phi_1(z_0) = 0$ and $\Phi_2(f(z_0)) = 0$. If either*

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} = \frac{|\Phi_1'(z_0)|^{1/K}}{|\Phi_2'(f(z_0))|}, \tag{4.1}$$

*or*

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{K}} = \frac{|\Phi_1'(z_0)|^{K}}{|\Phi_2'(f(z_0))|}, \tag{4.2}$$

*then $f$ is a Teichmüller mapping.*

*Proof.* Let $w = f(z)$, $\zeta = \Phi_1(z)$, $\eta = \Phi_2(w)$. Set $g(\zeta) = \Phi_2 \circ f \circ \Phi_1^{-1}(\zeta)$. From (4.1) and (4.2), it is not difficult to prove that

$$\text{either} \quad \lim_{\zeta\to 0} \frac{|g(\zeta)|}{|\zeta|^{1/K}} = 1 \quad \text{or} \quad \lim_{\zeta\to 0} \frac{|g(\zeta)|}{|\zeta|^{K}} = 1.$$

Therefore, by Theorem 2.1 and Corollary 2.6, $g(\zeta)$ is an affine mapping. Thus, $f$ is a Teichmüller mapping. ∎

The above theorem yields immediately

COROLLARY 4.2. *Suppose that $f$ is a $K$-quasiconformal mapping of $\Delta$ onto itself. If either*

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} = \frac{1 - |f(z_0)|^2}{(1 - |z_0|^2)^{1/K}} \tag{4.3}$$

*or*

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{K}} = \frac{1 - |f(z_0)|^2}{(1 - |z_0|^2)^{K}}, \tag{4.4}$$

*then $f$ is a Teichmüller mapping.*

THEOREM 4.3. *Let $(\Delta, \pi_1)$ and $(\Delta, \pi_2)$ be universal covering surfaces of $S_1$ and $S_2$ respectively. Let $\widetilde{f} : \Delta \to \Delta$ be a lift of a $K$-quasiconformal* *mapping $f$ of $S_1$ onto $S_2$. If for a neighbourhood $V$ of a lift of $f(p_0)$, either*

$$\lim_{p\to p_0} \frac{|(\pi_2|_V)^{-1}\circ f(p)-(\pi_2|_V)^{-1}\circ f(p_0)|}{|(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p)-(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p_0)|^{1/K}} = \frac{1-|(\pi_2|_V)^{-1}\circ f(p_0)|^2}{(1-|(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p_0)|^2)^{1/K}}$$

*or*

$$\lim_{p\to p_0} \frac{|(\pi_2|_V)^{-1}\circ f(p)-(\pi_2|_V)^{-1}\circ f(p_0)|}{|(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p)-(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p_0)|^{K}} = \frac{1-|(\pi_2|_V)^{-1}\circ f(p_0)|^2}{(1-|(\pi_1|_{\widetilde{f}^{-1}(V)})^{-1}(p_0)|^2)^{K}}$$

*then $f$ is a Teichmüller mapping.*

*Proof.* By hypothesis, $\widetilde{f}$ is a $K$-quasiconformal mapping of $\Delta$ onto itself and satisfies (4.3) or (4.4). In view of Corollary 4.2, $\widetilde{f}$ is a Teichmüller mapping. Hence so is $f$. ■

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