Geometric characterization for affine mappings and Teichmüller mappings

by

ZHIGUO CHEN (Hangzhou)

Abstract. We characterize affine mappings on the unit disk and on rectangles by module conditions. The main result generalizes the classic Schwarz lemma. As an application, we give a sufficient condition for a $K$-quasiconformal mapping on a Riemann surface to be a Teichmüller mapping.

1. Preliminaries. Let $\Gamma$ be a family of curves in the plane. Each $\gamma \in \Gamma$ is a countable union of open arcs, closed arcs or closed curves, and every closed subarc is rectifiable. We shall define the extremal length $\lambda(\Gamma)$ of $\Gamma$.

A function $\varrho$, defined on the whole plane, is called admissible if the following conditions are satisfied:

(i) $\varrho$ is a non-negative Borel function,
(ii) $A(\varrho) = \iint \varrho^2 \, dx \, dy \neq 0, \infty$.

If such a $\varrho$ is measurable as a function of arc length on $\gamma$, set

$$L_\gamma(\varrho) = \int \varrho |dz|.$$  \hfill (1.1)

Otherwise, set $L_\gamma(\varrho) = \infty$. Define

$$L(\varrho) = \inf_{\gamma \in \Gamma} L_\gamma(\varrho),$$ \hfill (1.2)

$$\lambda(\Gamma) = \sup_{\varrho} \frac{L(\varrho)^2}{A(\varrho)},$$ \hfill (1.3)

where the supremum is taken over all admissible $\varrho$. We write $\Gamma_1 < \Gamma_2$ if for every $\gamma_2 \in \Gamma_2$ there is a $\gamma_1 \in \Gamma_1$ which is a subarc of $\Gamma_2$. By the above definitions, the extremal length is monotonic:

2000 Mathematics Subject Classification: 30C62, 30C55, 30F60.

Key words and phrases: quasiconformal mapping, extremal length, Teichmüller mapping.

Research supported by National Science Foundation of China (Grant No. 10101023).
Proposition 1.1. If $\Gamma_1 < \Gamma_2$, then $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.

Let $\Gamma_1, \Gamma_2$ be two families of curves. Set
\[ \Gamma_1 + \Gamma_2 = \{ \gamma_1 + \gamma_2 \mid \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \}. \]
The extremal length of curve families has the following subadditivity property:

Proposition 1.2. If $\Gamma_1$ and $\Gamma_2$ lie in mutually disjoint sets, then
\[ \lambda(\Gamma_1 + \Gamma_2) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2). \]

For details about the properties of extremal length we refer to [1].

A quadrilateral consists of a Jordan domain $Q$ and a sequence $z_1, z_2, z_3, z_4$ of boundary points of $Q$. The points $z_i$ are called the vertices of the quadrilateral, and divide its boundary into four Jordan arcs, called the sides of the quadrilateral. The arcs $z_1z_2$ and $z_3z_4$ are called the a-sides and the other two the b-sides of $Q$. Let $\Gamma_a$ be the family of curves that connect the a-sides in $Q$, and $\Gamma_b$ the family of curves that connect the b-sides in $Q$. Define the module of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ to be $\lambda(\Gamma_a)$, the extremal length of $\Gamma_a$. That is,
\[ \text{mod } Q(z_1, z_2, z_3, z_4) = \sup \frac{(\inf_{\gamma \in \Gamma_a} L_{\gamma}(\varrho))^2}{A(\varrho)}. \]

For example, for a rectangle with width $a$ and height $b$, its module is $a/b$.

If $\varrho = 1$ in (1.1), then $L_{\gamma}(\varrho)$ is the euclidean length of $\gamma$, simply denoted by $|\gamma|$. We call
\[ s_a = s_a(Q) = \inf_{\gamma \in \Gamma_b} |\gamma| \]
the distance between the a-sides of $Q$. The distance $s_b$ between the b-sides is defined analogously. Let $m(Q)$ be the euclidean area of $Q$. The following Rengel inequality plays an important role in this paper.

Proposition 1.3 ([7]). The module of a quadrilateral $Q$ satisfies the double inequality
\[ \frac{(s_b(Q))^2}{m(Q)} \leq \text{mod } Q \leq \frac{m(Q)}{(s_a(Q))^2}. \]

Remark 1.4. This inequality is usually called Rengel’s inequality in the literature. However, it was first given by H. Grötzsch (see [6]).

The theory of quasiconformal mappings is closely related with the study of extremal length. In fact, the geometric definition of a quasiconformal mapping is based on moduli of quadrilaterals, which are represented by extremal length. Precisely, a function $f(z)$, which is a sense-preserving homeomorphism of $\Omega$ onto $\Omega'$, is $K$-quasiconformal if for every quadrilateral...
\[ Q = Q(z_1, z_2, z_3, z_4) \subset \Omega, \]
\[
\frac{1}{K} \mod Q \leq \mod f(Q) \leq K \mod Q.
\]

Denote the unit disk by \( \Delta \). It follows from the Riemann mapping theorem that every quadrilateral \( Q(z_1, z_2, z_3, z_4) \) can be mapped onto a quadrilateral \( \Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) with domain \( \Delta \) and vertices \( \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) on \( \partial \Delta \). By Schwarz–Christoffel,
\[
\Phi(\zeta) = \int \frac{d\zeta}{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}
\]
conformally maps \( \Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) onto a rectangle. By combining the above mappings, we can map an arbitrary quadrilateral conformally onto a rectangle. Therefore, from conformal invariance of extremal length, the module of any quadrilateral can always be represented by that of its conformally equivalent rectangle.

Define the module of an annulus
\[
A(r_1, r_2) = \{ z \mid r_1 \leq |z| \leq r_2 \}
\]
to be the extremal length of the family \( \Gamma \) of curves that connect \( \{ z \mid |z| = r_1 \} \) and \( \{ z \mid |z| = r_2 \} \) in \( A(r_1, r_2) \). Since \( \lambda(\Gamma) \) is conformally invariant and every ring domain can be mapped conformally onto an annulus, the number
\[
\mod A(r_1, r_2) = \frac{1}{2\pi} \log \frac{r_2}{r_1}
\]
represents the moduli of all ring domains which are conformally equivalent to \( A \).

Define the module of a sector
\[
A(r_1, r_2; \theta_1, \theta_2) = \{ z \mid r_1 \leq |z| \leq r_2, \theta_1 \leq \arg z \leq \theta_2 \}
\]
to be the extremal length of the family \( \Gamma \) of curves that connect \( \{ z \mid |z| = r_1 \} \) and \( \{ z \mid |z| = r_2 \} \) in \( A(r_1, r_2; \theta_1, \theta_2) \). Then
\[
\mod A(r_1, r_2; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \log \frac{r_2}{r_1}.
\]

2. Affine mappings on the unit disk \( \Delta \). Quasiconformality of a domain is characterized not only by the module of a quadrilateral \( Q \) as in inequality (1.5), but also by that of horizontal rectangles. A horizontal rectangle is a quadrilateral whose \( a \)-sides are parallel to the \( x \)-axis and \( b \)-sides are parallel to the \( y \)-axis. For the relevant results, we refer to [3] and [4]. In this section, we shall prove

**Theorem 2.1.** Let \( f(z) \) be a sense-preserving homeomorphism of the unit disk \( \Delta \) onto itself, with normalization \( f(0) = 0 \). If
(a) \[ \frac{1}{K} \mod A \leq \mod f(A), \]

where \( A \) stands for all \( A(r_1, r_2) \) and \( A(r_1, r_2; \theta_1, \theta_2) \) as in (1.6) and (1.8), and

(b) \[ \lim_{z \to 0} \frac{|f(z)|}{|z|^{1/K}} = 1, \]

then

\[ f(z) = \lambda z |z|^{1/K-1} \quad (|\lambda| = 1). \]

In view of Theorem 2.1, we immediately generalize the Schwarz lemma to sense-preserving homeomorphisms.

Corollary 2.2. Assume that \( f(z) \) is a sense-preserving homeomorphism of the unit disk \( \Delta \) onto itself, with normalization \( f(0) = 0 \). If \( f \) satisfies conditions (a) and (b), where \( K = 1 \) in (a), then

\[ f(z) = \lambda z \quad (|\lambda| = 1). \]

For the proof of the theorem, we need three lemmas.

Lemma 2.3. If \( f \) satisfies condition (a) of Theorem 2.1, then it is absolutely continuous on \( \{ |z| = r \} \).

Proof. Let \( A = \{ z = re^{i\theta} \mid r_1 < r < r_2, 0 \leq \theta < 2\pi \} \) be an annulus in \( \Delta \). Set

\[ q(t) = m(f(A_t)), \]

where \( A_t = \{ z = re^{i\theta} \mid r_1 < r < t, 0 \leq \theta < 2\pi \} \) and \( m(f(A_t)) \) is the euclidean area of \( f(A_t) \). Obviously, \( q(t) \) is an increasing function of \( t \) and thus has a finite derivative \( q'(t) \) for all \( t, r_1 < t < r_2 \), except for a set of zero linear measure. Assume that \( q'(t_0) \) exists and is finite. We shall prove that \( f(t_0e^{i\theta}) \) is absolutely continuous on \([0, 2\pi]\).

We first choose a positive \( \delta \) such that \( t_0 + \delta < r_2 \). Let \( (\theta_k, \theta_k^*), k = 1, \ldots, n \), be an arbitrary system of non-intersecting open subintervals of \([0, 2\pi]\). Define a sector (a special quadrilateral) by

\[ G_k^\delta = \{ z = re^{i\theta} \mid t_0 < r < t_0 + \delta, \theta_k < \theta < \theta_k^* \}. \]

Then by (1.9),

\[ \mod G_k^\delta = \frac{1}{\theta_k^* - \theta_k} \log \frac{t_0 + \delta}{t_0}. \]

For the module of the image of \( G_k^\delta \), from the right hand side of Rengel’s inequality we have

\[ \mod f(G_k^\delta) \leq \frac{m(f(G_k^\delta))}{(d_k^\delta)^2}, \]

where \( d_k^\delta \) is
where $d_k^\delta$ denotes the euclidean distance between the $a$-sides of $f(G_k^\delta)$. These sides converge to the points $f(t_0e^{i\theta_k})$ and $f(t_0e^{i\theta^*_k})$ as $\delta \to 0$, and so

$$\lim_{\delta \to 0} d_k^\delta = |f(t_0e^{i\theta_k^*}) - f(t_0e^{i\theta_k})|.$$  

According to condition (a), we have

$$\frac{1}{K} \mod G_k^\delta \leq \mod f(G_k^\delta).$$  

From (2.2), (2.4) and (2.1), we get

$$\frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \leq \frac{K(\theta_k^* - \theta_k)}{\log(1 + \delta/t_0)}, \quad k = 1, \ldots, n.$$  

Noticing that $\log(1 + \delta/t_0) \geq \delta/(2t_0)$ for $\delta < t_0$ and adding the above inequalities over $k$ we have

$$\sum_{k=1}^n \frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \leq 2Kt_0 \sum_{k=1}^n \frac{\theta_k^* - \theta_k}{\delta},$$  

By the Schwarz inequality,

$$\left(\sum_{k=1}^n d_k^\delta\right)^2 \leq \sum_{k=1}^n \frac{(d_k^\delta)^2}{m(f(G_k^\delta))} \sum_{k=1}^n m(f(G_k^\delta)).$$  

Observing that

$$\sum_{k=1}^n m(f(G_k^\delta)) \leq q(t_0 + \delta) - q(t_0),$$

we obtain

$$\left(\sum_{k=1}^n d_k^\delta\right)^2 \leq 2Kt_0 \frac{q(t_0 + \delta) - q(t_0)}{\delta} \sum_{k=1}^n |\theta_k^* - \theta_k|.$$  

Letting $\delta \to 0$, from (2.3) and (2.6) we obtain

$$\left(\sum_{k=1}^n |f(t_0e^{i\theta_k^*}) - f(t_0e^{i\theta_k})|\right)^2 \leq 2Kt_0q^*(t_0) \sum_{k=1}^n |\theta_k^* - \theta_k|.$$  

This completes the proof of the lemma.  

Define

$$R(r) = \{z \mid r \leq |z| \leq 1\}.$$  

There exists a conformal mapping $\Phi_r$ such that

$$\Phi_r \circ f(R(r)) = \{\zeta \mid \phi(r) \leq |\zeta| \leq 1\}.$$  

Set $g_r = \Phi_r \circ f$. We shall give the properties of $\phi(r)$, which was first considered by H. Grötzsch for $f$ $K$-quasiconformal (see [5] and [2]). However, it is not obvious from conditions (a) and (b) that the sense-preserving mapping
f in Theorem 2.1 is $K$-quasiconformal. Following Grötzsch’s idea, we prove Lemma 2.4 directly by extremal length methods.

**Lemma 2.4.** If $f$ satisfies condition (a), then $\phi(r)/r^{1/K}$ is an increasing function on $(0, 1]$, and $\phi(r)/r^{1/K} \leq 1$.

**Proof.** Assume that $0 < r_1 < r_2 \leq 1$. Then $g_{r_1}(z)$ maps $\{z \mid r_1 \leq |z| \leq 1\}$ onto $\{\zeta \mid \phi(r_1) \leq |\zeta| \leq 1\}$. The definition of moduli of ring domains together with Proposition 1.2 yields

\begin{equation}
\text{mod} g_{r_1}\{z \mid r_1 \leq |z| \leq r_2\} + \text{mod} g_{r_1}\{z \mid r_2 \leq |z| \leq 1\} \\
\leq \text{mod} g_{r_1}\{z \mid r_1 \leq |z| \leq 1\} = \frac{1}{2\pi} \log \frac{1}{\phi(r_1)}.
\end{equation}

From conformal invariance of extremal length, it follows that

\begin{equation}
\text{mod} g_{r_1}\{z \mid r_2 \leq |z| \leq 1\} = \text{mod} f\{z \mid r_2 \leq |z| \leq 1\} \\
= \text{mod} g_{r_2}\{z \mid r_2 \leq |z| \leq 1\} = \frac{1}{2\pi} \log \frac{1}{\phi(r_2)}.
\end{equation}

Substituting (2.8) into (2.7), we have

\begin{equation}
\text{mod} g_{r_1}\{z \mid r_1 \leq |z| \leq r_2\} \leq \frac{1}{2\pi} \log \frac{\phi(r_2)}{\phi(r_1)}.
\end{equation}

On the other hand, from condition (a) it follows that

\begin{equation}
\text{mod} g_{r_1}\{z \mid r_1 \leq |z| \leq r_2\} = \text{mod} f\{z \mid r_1 \leq |z| \leq r_2\} \geq \frac{1}{2\pi K} \log \frac{r_2}{r_1}.
\end{equation}

Combining (2.9) with (2.10), we obtain

\[ \left( \frac{r_2}{r_1} \right)^{1/K} \leq \frac{\phi(r_2)}{\phi(r_1)}, \quad \text{that is,} \quad \frac{\phi(r_1)}{r_1^{1/K}} \leq \frac{\phi(r_2)}{r_2^{1/K}}. \]

So, the monotonicity of $\phi(r)/r^{1/K}$ is proved. From the above inequality, by letting $r_2 = 1$, we get $\phi(r)/r^{1/K} \leq 1$ for $r \in (0, 1]$. This completes the proof of the lemma. \hfill \blacksquare

**Lemma 2.5.** If $f$ satisfies condition (b), then

\[ \lim_{r \to 0} \frac{\phi(r)}{r^{1/K}} = 1. \]

**Proof.** For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $r < \delta$,

\[ 1 - \varepsilon < \frac{|f(re^{i\theta})|}{r^{1/K}} < 1 + \varepsilon. \]

Therefore,

\[ \{w \mid (1 + \varepsilon)r^{1/K} < |w| \leq 1\} \subset f(R(r)) \subset \{w \mid (1 - \varepsilon)r^{1/K} < |w| \leq 1\}. \]
By Proposition 1.1,
\[ \frac{1}{2\pi} \log \frac{1}{(1 + \varepsilon)r^{1/K}} \leq \mod f(R(r)) \leq \frac{1}{2\pi} \log \frac{1}{(1 - \varepsilon)r^{1/K}}. \]
Therefore,
\[ \frac{1}{2\pi} \log \frac{1}{(1 + \varepsilon)r^{1/K}} \leq \frac{1}{2\pi} \log \frac{1}{\phi(r)} \leq \frac{1}{2\pi} \log \frac{1}{(1 - \varepsilon)r^{1/K}}. \]
That is,
\[ 1 - \varepsilon \leq \frac{\phi(r)}{r^{1/K}} \leq 1 + \varepsilon. \]
This shows \( \lim_{r \to 0} \phi(r)/r^{1/K} = 1 \).

**Proof of Theorem 2.1.** From Lemmas 2.4 and 2.5, we obtain
\[ \phi(r) = r^{1/K}, \quad 0 \leq r \leq 1. \]
This means
\[ |g_r(z)| = |z|^{1/K} \quad (r \leq |z| \leq 1). \]
Taking \( r = 1/n \), we write
\[ g_n(z) = |z|^{1/K}e^{ih_n(\theta)} \quad (1/n \leq |z| \leq 1), \]
where \( \theta = \arg z \) and \( h_n \) is a homeomorphism from \([0, 2\pi]\) onto \([\theta_n, \theta_n + 2\pi]\) \((0 \leq \theta_n < 2\pi)\). According to Lemma 2.3, \( f \) is absolutely continuous on \(|z| = r\}, and so is \( g_n \). Therefore, \( h'_n(\theta) \) exists almost everywhere and is integrable on \([0, 2\pi]\), and
\[ \int_0^{2\pi} h'_n(\theta) d\theta = 2\pi. \]
Direct computation on generalized derivatives of \( g_n(z) \) in (2.11) shows
\[ \partial_z g_n(z) = \frac{z}{2|z|^{2+1/K}} e^{-ih_n(\theta)} \left( \frac{1}{K} + h'_n(\theta) \right), \]
\[ \partial z g_n(z) = \frac{\bar{z}}{2|z|^{2+1/K}} e^{-ih_n(\theta)} \left( \frac{1}{K} - h'_n(\theta) \right). \]
Since \( g_n \) is sense-preserving,
\[ |\partial_z g_n(z)| > |\partial z g_n(z)|. \]
Therefore, from (2.13) and (2.14) we infer that \( h'_n(\theta) > 0 \). Set
\[ P = \{ z \mid r \leq |z| \leq r + dr, \theta \leq \arg z \leq \theta + d\theta \}. \]
Then
\[ g_n(P) = \{ \zeta \mid r^{1/K} \leq |\zeta| \leq (r + dr)^{1/K}, h_n(\theta) \leq \arg \zeta \leq h_n(\theta) + h'_n(\theta)d\theta \}. \]
By conformal invariance of extremal length and condition (a), we have
\[ \text{mod } g_n(P) \geq \frac{1}{K} \text{mod } P. \]
From the above inequality, it follows that \( h_n' (\theta) \leq 1. \) Thus,

(2.15) \[ 0 < h_n' (\theta) \leq 1. \]
From (2.12) and (2.15), we obtain \( h_n' (\theta) = 1 \) a.e. on \( [0, 2\pi) \). Therefore, \( h_n(\theta) = \theta_n + \theta \), and we can rewrite (2.11) as follows:

(2.16) \[ g_n(z) = z|z|^{1/K-1} e^{i\theta_n} \quad (1/n \leq |z| \leq 1). \]
Since \( \theta_n \in [0, 2\pi) \), there exists a subsequence \( \{\theta_{n_k}\} \) such that \( \lim_{k \to \infty} \theta_{n_k} = \theta_0 \). Therefore,

\( \lim_{k \to \infty} g_{n_k}(z) = g(z) = z|z|^{1/K-1} e^{i\theta_0} \quad (\text{locally uniformly in } \Delta - \{0\}). \)
So,

\[ \lim_{k \to \infty} \Phi_{n_k}(w) = \lim_{k \to \infty} g_{n_k} \circ f^{-1}(w) = g \circ f^{-1}(w) := \Phi(w) \quad (\text{locally uniformly in } \Delta - \{0\}). \]
From compactness of the family of analytic functions, \( \Phi \) is analytic in \( \Delta - \{0\} \). Observing that \( \lim_{w \to 0} \Phi(w) = 0 \), we conclude that the origin is a removable singular point and hence \( \Phi \) is analytic in the unit disk. Since

\[ \lim_{w \to 0} \frac{|\Phi(w)|}{|w|} = \lim_{z \to 0} \frac{\Phi \circ f(z)}{f(z)} = \lim_{z \to 0} \frac{|z|^{1/K}}{|f(z)|} = 1, \]
by the Schwarz lemma we have \( \Phi(w) = \lambda w \). Thus, \( f(z) = \lambda z|z|^{1/K-1} \) where \( |\lambda| = 1. \)

**Corollary 2.6.** If \( f \) is \( K \)-quasiconformal mapping of \( \Delta \) onto itself which satisfies \( f(0) = 0 \) and
\[ \lim_{z \to 0} \frac{|f(z)|}{|z|^{K-1}} = 1, \]
then \( f(z) = \lambda z|z|^{K-1} \).

**Problem 2.7.** If we replace conditions (a) and (b) by

(a*) \[ \text{mod } f(A) \leq K \text{mod } A \quad \text{for all } A \subset \Delta, \]
(b*) \[ \lim_{z \to 0} \frac{|f(z)|}{|z|^K} = 1, \]
is it true that \( f(z) = \lambda z|z|^{K-1} ? \)
3. Affine mappings on rectangles

**Lemma 3.1** ([1]). Let \( R = \{ z \mid 0 \leq x \leq a, 0 \leq y \leq 1 \} \). Let \( C \) be a curve in \( R \) that connects the \( a \)-sides of \( R \) and hence divides \( R \) into two parts \( R_1 \) and \( R_2 \). Then \( R_1 \) and \( R_2 \) are rectangles if and only if
\[
\text{mod} R_1 + \text{mod} R_2 = \text{mod} R = a.
\]

**Lemma 3.2.** Let \( f \) be a sense-preserving homeomorphism which maps \( \{ z \mid r_0 \leq |z| \leq 1 \} \) onto \( \{ w \mid r_0^{1/K} \leq |w| \leq 1 \} \). If
\[
(3.0) \quad \frac{1}{K} \text{mod} A \leq \text{mod} f(A)
\]
for all \( A \subset \{ z \mid r_0 \leq |z| \leq 1 \} \) appearing in (1.6) and (1.8), then
\[
f(z) = \lambda z |z|^{1/K-1} \quad (|\lambda| = 1).
\]

**Proof.** Extend \( f \) by setting
\[
f(z) = r_0^{2n/K} / f(r_0^{2n}/z), \quad r_0^{n+1} \leq |z| \leq r_0^n, \quad n = 1, 2, \ldots
\]
Then the extended \( f \) is a sense-preserving homeomorphism of \( \Delta \) onto itself with \( f(0) = 0 \). Furthermore, it is not difficult to prove that the extended \( f \) satisfies (a) and (b). Hence, by Theorem 2.1,
\[
f(z) = \lambda z |z|^{1/K-1}.
\]

**Theorem 3.3.** Assume that \( f \) is a sense-preserving homeomorphism which maps the rectangle \( R = \{ z \mid 0 \leq x \leq a, 0 \leq y \leq 1 \} \) onto \( f(R) = \{ w \mid 0 \leq u \leq a', 0 \leq v \leq 1 \} \). If for every horizontal subrectangle \( R_h \subset R \),
\[
(3.1) \quad \frac{1}{K} \text{mod} R_h \leq \text{mod} f(R_h),
\]
then
\[
(3.2) \quad f(z) = \frac{1}{K} x + iy,
\]
where \( K = \text{mod} R / \text{mod} f(R) = a/a' \).

**Proof.** Set \( R_1 = \{ z \mid 0 \leq x \leq a_1, 0 \leq y \leq 1 \} \), \( R_2 = \{ z \mid a_1 \leq x \leq a, 0 \leq y \leq 1 \} \). By (3.1), we have
\[
\frac{1}{K} \text{mod} R_1 \leq \text{mod} f(R_1), \quad \frac{1}{K} \text{mod} R_2 \leq \text{mod} f(R_2).
\]
Therefore,
\[
(3.3) \quad \frac{1}{K} \text{mod} R = \frac{1}{K} (\text{mod} R_1 + \text{mod} R_2) \leq \text{mod} f(R_1) + \text{mod} f(R_2).
\]
On the other hand, by Proposition 1.2,
\[
(3.4) \quad \text{mod} f(R_1) + \text{mod} f(R_2) \leq \text{mod} f(R) = \frac{1}{K} \text{mod} R.
\]
Combining (3.3) with (3.4), we have
\[ \text{mod } f(R_1) + \text{mod } f(R_2) = \text{mod } f(R). \]
By Lemma 3.1, \( f(R_1) \) and \( f(R_2) \) are rectangles. That is,
\[ f(x) = f(x + i), \quad 0 \leq x \leq a. \]
It follows from (3.5) that \( f \) can be extended to the strip
\[ S = \{ z \mid 0 \leq x \leq a \}. \]
By the extremal length method, it is not difficult to prove that for the extended function \( f \), which maps \( S \) onto another strip
\[ S' = \{ w \mid 0 \leq u \leq a' \}, \]
inequality (3.1) still holds for any \( R_h \subset S \). Set
\[ G = \{ \zeta \mid e^{-a} \leq |\zeta| \leq 1 \}, \quad G' = \{ \eta \mid e^{-a'} \leq |\eta| \leq 1 \}. \]
Then \( \{ S, e^{-a+z} \}, \{ S', e^{-a'+w} \} \) are universal covering surfaces of \( G \) and \( G' \) respectively.

The projective mapping of the function \( f \),
\[ g(\zeta) = e^{-a' + f(a + \log \zeta)}, \tag{3.6} \]
is a sense-preserving homeomorphism and satisfies inequality (3.0). In view of Lemma 3.2, we have
\[ g(\zeta) = \lambda \zeta |\zeta|^{1/K-1} \quad (|\lambda| = 1). \tag{3.7} \]
Combining (3.6) with (3.7) and observing that \( f(0) = 0 \), we obtain
\[ f(z) = \frac{1}{K} x + iy. \]
This ends the proof of the theorem. □

The above theorem yields immediately

**Corollary 3.4.** Let \( f \) be a sense-preserving homeomorphism which maps a rectangle \( R \) onto a quadrilateral \( Q \) and satisfies the following two conditions:
\[ \begin{align*}
(c) & \quad K^{-1} \text{mod } R_h \leq \text{mod } f(R_h) \text{ for every horizontal subrectangle } R_h \subset R, \\
(d) & \quad K^{-1} \text{mod } R = \text{mod } Q.
\end{align*} \]
Then \( f \) is \( K \)-quasiconformal and is a Teichmüller mapping.

**Remark 3.5.** It might seem that Corollary 3.4 should have a symmetric form, with (c) and (d) replaced respectively by
\[ \begin{align*}
(c^*) & \quad \text{mod } R_h \leq K \text{mod } f(R_h), \\
(d^*) & \quad \text{mod } R = K \text{mod } f(R_h).
\end{align*} \]
But unfortunately, this is not the case. In fact, we can find a counterexample in [3], where $f$ is $(K + \sqrt{K^2 - 1})$-quasiconformal.

4. Geometric characterization for Teichmüller mappings. Let $S_1$ and $S_2$ be two Riemann surfaces. Let $\omega$ be a holomorphic quadratic differential on $S_1$. A quasiconformal mapping $f$ from $S_1$ onto $S_2$ is called a Teichmüller mapping if its Beltrami differential $\mu_f d\bar{z}/dz$ equals $k\bar{\omega}/\omega$. In this section, we want to characterize Teichmüller mappings by a local property.

Let $D_1$ and $D_2$ be simply connected domains of hyperbolic type. From the Riemann mapping theorem, there exist conformal mappings $\Phi_1$ and $\Phi_2$ which map $D_1$ and $D_2$ respectively onto $\Delta$.

**Theorem 4.1.** Suppose that $f$ is a $K$-quasiconformal mapping from $D_1$ onto $D_2$. To $z_0 \in D_1$, there correspond conformal mappings $\Phi_1$ and $\Phi_2$ as above with $\Phi_1(z_0) = 0$ and $\Phi_2(f(z_0)) = 0$. If either

\[ \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} = \frac{|\Phi_1'(z_0)|^{1/K}}{|\Phi_2'(f(z_0))|}, \tag{4.1} \]

or

\[ \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^K} = \frac{|\Phi_1'(z_0)|^K}{|\Phi_2'(f(z_0))|}, \tag{4.2} \]

then $f$ is a Teichmüller mapping.

**Proof.** Let $w = f(z)$, $\zeta = \Phi_1(z)$, $\eta = \Phi_2(w)$. Set $g(\zeta) = \Phi_2 \circ f \circ \Phi_1^{-1}(\zeta)$. From (4.1) and (4.2), it is not difficult to prove that

\[ \lim_{\zeta \to 0} \frac{|g(\zeta)|}{|\zeta|^{1/K}} = 1 \quad \text{or} \quad \lim_{\zeta \to 0} \frac{|g(\zeta)|}{|\zeta|^K} = 1. \]

Therefore, by Theorem 2.1 and Corollary 2.6, $g(\zeta)$ is an affine mapping. Thus, $f$ is a Teichmüller mapping. ■

The above theorem yields immediately

**Corollary 4.2.** Suppose that $f$ is a $K$-quasiconformal mapping of $\Delta$ onto itself. If either

\[ \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} = \frac{1 - |f(z_0)|^2}{(1 - |z_0|^2)^{1/K}}, \tag{4.3} \]

or

\[ \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^K} = \frac{1 - |f(z_0)|^2}{(1 - |z_0|^2)^K}, \tag{4.4} \]

then $f$ is a Teichmüller mapping.

**Theorem 4.3.** Let $(\Delta, \pi_1)$ and $(\Delta, \pi_2)$ be universal covering surfaces of $S_1$ and $S_2$ respectively. Let $\tilde{f} : \Delta \to \Delta$ be a lift of a $K$-quasiconformal
mapping $f$ of $S_1$ onto $S_2$. If for a neighbourhood $V$ of a lift of $f(p_0)$, either

$$\lim_{p \to p_0} \frac{|(\pi_2)_V^{-1} \circ f(p) - (\pi_2)_V^{-1} \circ f(p_0)|}{|\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p) - (\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p_0)|^{1/K}}$$

$$= \frac{1 - |(\pi_2)_V^{-1} \circ f(p_0)|^2}{(1 - |(\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p_0)|^2)^{1/K}}$$

or

$$\lim_{p \to p_0} \frac{|(\pi_2)_V^{-1} \circ f(p) - (\pi_2)_V^{-1} \circ f(p_0)|}{|\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p) - (\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p_0)|^{1/K}} = \frac{1 - |(\pi_2)_V^{-1} \circ f(p_0)|^2}{(1 - |(\pi_1|_{\tilde{f}^{-1}(V)}^{-1}(p_0)|^2)^{1/K}}$$

then $f$ is a Teichmüller mapping.

Proof. By hypothesis, $\tilde{f}$ is a $K$-quasiconformal mapping of $\Delta$ onto itself and satisfies (4.3) or (4.4). In view of Corollary 4.2, $\tilde{f}$ is a Teichmüller mapping. Hence so is $f$. ■

References


Department of Mathematics
XiXi campus
Zhejiang University
Hangzhou, Zhejiang, 310028
P.R. China
E-mail: zgchen@zju.edu.cn

Received April 18, 2002
Revised version October 18, 2002 (4931)