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Two-weight norm inequalities for potential type integral operators in the case $p > q > 0$ and $p > 1$

by

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Abstract. Sufficient conditions for a two-weight norm inequality for potential type integral operators to hold are given in the case $p > q > 0$ and $p > 1$ in terms of the Hedberg–Wolff potential.

1. Introduction. The purpose of this paper is to develop a theory of weights for potential type integral operators in the case $p > q > 0$ and $p > 1$. We first recall some standard notations.

We shall consider cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. For a cube $Q \in \mathcal{Q}$ we use $\ell(Q)$ to denote the side-length of Q, $|Q|$ to denote the volume of Q, and cQ to denote the cube with the same center as Q but with side-length $c\ell(Q)$. We denote by $\mathcal{D}$ the family of all dyadic cubes $Q = 2^{-i}(k + [0,1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$.

Let f be a locally integrable function on $\mathbb{R}^n$. The *Hardy–Littlewood maximal operator* M is defined by

$$Mf(x) = \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

and the *fractional integral operator* (or the *Riesz potential*) I_α, $0 < \alpha < n$, is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

In this paper a *weight* is simply a positive measurable function w on $\mathbb{R}^n$. For each $1 < p < \infty$, p' will denote the dual exponent of p, i.e., $p' = p/(p-1)$.

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As is well-known, for the Hardy–Littlewood maximal operator M and $p>1$, Muckenhoupt [7] showed that the one-weight strong type inequality

$$\|(Mf)w\|_{L^p(dx)} \le C\|fw\|_{L^p(dx)} \tag{1.1}$$

holds if and only if

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)^p\,dx\right)^{1/p} \left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,dx\right)^{1/p'} < \infty.$$

For $p>1$ one says that a weight w on $\mathbb{R}^n$ belongs to the class A_p when

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right) \left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)^{p-1} < \infty.$$

That is, (1.1) holds if and only if $w^p \in A_p$.

Muckenhoupt and Wheeden [8] showed that, for the fractional integral operator I_α and $q>p>1$ with $1/q = 1/p - \alpha/n$, the one-weight strong type inequality

$$\|(I_\alpha f)w\|_{L^q(dx)} \le C\|fw\|_{L^p(dx)}$$

holds if and only if

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q} \left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,dx\right)^{1/p'} < \infty.$$

Simple sufficient conditions are also known for two-weight strong type inequalities to hold.

Neugebauer [10] showed that, for the Hardy–Littlewood maximal operator M and $p>1$, the condition

$$\sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q u(x)^{ap}\,dx\right)^{1/ap} \left(\frac{1}{|Q|}\int_Q v(x)^{-ap'}\,dx\right)^{1/ap'} < \infty,$$

for some $a>1$, is sufficient for the two-weight strong type inequality

$$\|(Mf)u\|_{L^p(dx)} \le C\|fv\|_{L^p(dx)}$$

to hold. Sawyer and Wheeden [12] showed that, for the fractional integral operator I_α and $q \ge p > 1$, the condition

$$\sup_{Q\in\mathcal{Q}} |Q|^{\alpha/n+1/q-1/p} \left(\frac{1}{|Q|}\int_Q u(x)^{aq}\,dx\right)^{1/aq} \left(\frac{1}{|Q|}\int_Q v(x)^{-ap'}\,dx\right)^{1/ap'} < \infty,$$

for some $a>1$, is sufficient for the two-weight strong type inequality

$$\|(I_\alpha f)u\|_{L^q(dx)} \le C\|fv\|_{L^p(dx)} \tag{1.2}$$

to hold. For recent progress on this problem we refer to [3].

In the case $p > q > 0$ and $p > 1$, in terms of the Hedberg–Wolff potential, Cascante, Ortega and Verbitsky proved that the trace inequality

$$\|I_\alpha f\|_{L^q(d\mu)} \le C\|f\|_{L^p(dx)}$$

holds if and only if

$$\mathcal{W}_\alpha^{\mathcal{D}}[\mu]^{1/p'} \in L^r(d\mu), \qquad 1/q = 1/r + 1/p,$$

where μ is a nonnegative Borel measure on $\mathbb{R}^n$ and

$$\mathcal{W}_\alpha^{\mathcal{D}}[\mu](x) = \sum_{Q\in\mathcal{D}} \left(\frac{|Q|^{\alpha p/n}}{|Q|}\mu(Q)\right)^{p'-1} \chi_Q(x).$$

Here, χ_Q is the characteristic function of $Q \in \mathcal{D}$.

In this paper we shall investigate the two-weight strong type inequality (1.2) in the case $p > q > 0$ and $p > 1$.

In their significant paper [2], Cascante, Ortega and Verbitsky established the following: Let $K : \mathcal{D} \to \mathbb{R}^+$ be any map. Let σ be a nonnegative Borel measure on $\mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(d\sigma)$. The dyadic integral operator $T_K^{\mathcal{D}}[f\,d\sigma]$ is defined by

$$T_K^{\mathcal{D}}[f\,d\sigma](x) = \sum_{Q\in\mathcal{D}} K(Q) \int_Q f(y)\,d\sigma(y)\,\chi_Q(x).$$

We denote by $\overline{K}(Q)(x)$ the function

$$\overline{K}(Q)(x) = \frac{1}{\sigma(Q)} \sum_{Q'\subset Q} \sigma(Q')K(Q')\chi_{Q'}(x), \qquad x \in Q \in \mathcal{D},$$

and $\overline{K}(Q)(x) = 0$ when $\sigma(Q) = 0$. The pair (K, σ) is said to satisfy the *dyadic logarithmic bounded oscillation* (DLBO) condition if

$$\sup_{x\in Q} \overline{K}(Q)(x) \le A \inf_{x\in Q} \overline{K}(Q)(x),$$

where A does not depend on $Q \in \mathcal{D}$.

Proposition 1.1 ([2, Theorem A]). *Let $K : \mathcal{D} \to \mathbb{R}^+$, $0 < q < p < \infty$ and $1 < p < \infty$. Let μ and σ be nonnegative Borel measures on $\mathbb{R}^n$. Suppose that the pair (K, σ) satisfies the* DLBO *condition. Then the trace inequality*

$$\|T_K^{\mathcal{D}}[f\,d\sigma]\|_{L^q(d\mu)} \le C\|f\|_{L^p(d\sigma)} \tag{1.3}$$

holds if and only if

$$\mathcal{W}_{K,\sigma}^{\mathcal{D}}[\mu]^{1/p'} \in L^r(d\mu), \qquad 1/q = 1/r + 1/p,$$

where

$$\mathcal{W}_{K,\sigma}^{\mathcal{D}}[\mu](x) = \sum_{Q\in\mathcal{D}} \left(\int_Q \overline{K}(Q)(y)\,d\mu(y)\right)^{p'-1} \sigma(Q)K(Q)\chi_Q(x).$$

For recent developments on this problem we refer to [9, 5, 15, 13]. In [14], the author and Terasawa develop a theory of weights for positive operators and generalized Doob maximal operators in a filtered measure space.

In the general situation, where we do not assume that the pair (K,σ) satisfies the DLBO condition, some sufficient condition for the trace inequality (1.3) to hold was obtained by Cascante, Ortega and Verbitsky:

Proposition 1.2 ([1, Theorem 2.7]). *Let $K:\mathcal{D}\to\mathbb{R}^+$ and $1<q<p<\infty$. Let μ and σ be nonnegative Borel measures on $\mathbb{R}^n$. Then the condition*

$$\overline{\mathcal{W}}^{\mathcal{D}}_{K,\sigma}[\mu]^{1/p'}\in L^r(d\mu),\qquad 1/q=1/r+1/p,$$

where

$$\overline{\mathcal{W}}^{\mathcal{D}}_{K,\sigma}[\mu](x)=\sum_{Q\in\mathcal{D}}\Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'-1}\sigma(Q)\overline{K}(Q)(x),$$

is sufficient for the trace inequality (1.3) *to hold.*

In this paper we shall establish two-weight extensions of Proposition 1.2.

Let $K:\mathcal{D}\to\mathbb{R}^+$. We define the integral operator $T_K[f\,d\sigma]$ by

$$T_K[f\,d\sigma](x)=\sum_{Q\in\mathcal{D}}K(Q)\int_{3Q}f(y)\,d\sigma(y)\,\chi_Q(x).$$

For $0<\alpha<n$ we notice that, by Fubini's theorem,

$$I_\alpha[|f|](x)\approx\sum_{Q\in\mathcal{D}}\frac{|Q|^{\alpha/n}}{|Q|}\int_{3Q}|f(y)|\,dy\,\chi_Q(x)\quad\text{a.e. } x\in\mathbb{R}^n \tag{1.4}$$

and

$$\frac{|Q|^{\alpha/n}}{|Q|}\chi_Q(x)\approx\frac{1}{|Q|}\sum_{Q'\subset Q}|Q'|^{\alpha/n}\chi_{Q'}(x)\quad\text{a.e. } x\in Q\in\mathcal{D}. \tag{1.5}$$

In this paper we shall prove the following theorem:

Theorem 1.3. *Let $K:\mathcal{D}\to\mathbb{R}^+$, $a>1$, $0<q<p<\infty$ and $1<p<\infty$. Let μ be a nonnegative Borel measure, σ be a positive "doubling" Borel measure and w be a weight on $\mathbb{R}^n$. Then:*

(a) *There exists $C>0$ such that whenever f is a nonnegative Borel measurable function on $\mathbb{R}^n$, the two-weight norm inequality*

$$\|T_K[f\,d\sigma]\|_{L^q(d\mu)}\le C\|\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w]^{1/p'}\|_{L^r(d\mu)}\|fw\|_{L^p(d\sigma)},\qquad 1/q=1/r+1/p,$$

holds, where

$$\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](x)=\sum_{Q\in\mathcal{D}}\left(\frac{\int_{3Q}w(y)^{-ap'}\,d\sigma(y)}{\sigma(Q)}\right)^{1/a}\times\left(\int_Q\overline{K}(Q)(y)\,d\mu(y)\right)^{p'-1}\sigma(Q)\overline{K}(Q)(x).$$

(b) *There exists $C>0$ such that whenever f is a nonnegative Borel measurable function on $\mathbb{R}^n$, the two-weight norm inequality*

$$\|T_K[f\,d\sigma]\|_{L^q(d\mu)}\le C\|\overline{\mathcal{W}}^{\mathcal{D};p}_{K,\sigma}[\mu,w]^{1/p'}\|_{L^r(d\mu)}\|(M_\sigma f)w\|_{L^p(d\sigma)},\quad 1/q=1/r+1/p,$$

holds, where

$$\overline{\mathcal{W}}^{\mathcal{D};p}_{K,\sigma}[\mu,w](x)=\sum_{Q\in\mathcal{D}}\int_Q w(y)^{-p'}\,d\sigma(y)\times\left(\int_Q\overline{K}(Q)(y)\,d\mu(y)\right)^{p'-1}\overline{K}(Q)(x).$$

Here, M_σ stands for the Hardy–Littlewood maximal operator given by

$$M_\sigma f(x)=\sup_{x\in Q\in\mathcal{Q}}\frac{1}{\sigma(Q)}\int_Q|f(y)|\,d\sigma(y).$$

In the last section, we will discuss the necessity of our sufficient conditions.

In view of (1.4) and (1.5), we have the following:

Corollary 1.4. *Let $0<\alpha<n$, $a>1$, $0<q<p<\infty$ and $1<p<\infty$. Let μ be a nonnegative Borel measure and w be a weight on $\mathbb{R}^n$. Then:*

(a) *There exists $C>0$ such that whenever f is a nonnegative measurable function on $\mathbb{R}^n$, the two-weight norm inequality*

$$\|I_\alpha f\|_{L^q(d\mu)}\le C\|\mathcal{W}^{\mathcal{D};p,a}_\alpha[\mu,w]^{1/p'}\|_{L^r(d\mu)}\|fw\|_{L^p(dx)},\quad 1/q=1/r+1/p,$$

holds, where

$$\mathcal{W}^{\mathcal{D};p,a}_\alpha[\mu,w](x)=\sum_{Q\in\mathcal{D}}\left(\frac{\int_{3Q}w(y)^{-ap'}\,dy}{|Q|}\right)^{1/a}\left(\frac{|Q|^{\alpha p/n}}{|Q|}\mu(Q)\right)^{p'-1}\chi_Q(x).$$

(b) *There exists $C>0$ such that, if $w^p\in A_p$, whenever f is a nonnegative measurable function on $\mathbb{R}^n$, the two-weight norm inequality*

$$\|I_\alpha f\|_{L^q(d\mu)}\le C\|\mathcal{W}^{\mathcal{D};p}_\alpha[\mu,w]^{1/p'}\|_{L^r(d\mu)}\|fw\|_{L^p(dx)},\quad 1/q=1/r+1/p,$$

holds, where

$$\mathcal{W}_{\alpha}^{\mathcal{D};p}[\mu,w](x)=\sum_{Q\in\mathcal{D}}\int_{Q} w(y)^{-p'}\,dy\biggl\{\biggl(\frac{|Q|^{\alpha/n}}{|Q|}\biggr)^{p}\mu(Q)\biggr\}^{p'-1}\chi_Q(x).$$

Remark 1.5. For nonnegative and locally σ-integrable functions Φ and f on $\mathbb{R}^n$, we define

$$T_{\Phi}f(x)=\int_{\mathbb{R}^n}\Phi(x-y)f(y)\,d\sigma(y).$$

We observe that

$$\begin{aligned}T_{\Phi}f(x)&=\sum_{k\in\mathbb{Z}}\int_{2^{k-1}<|x-y|\le 2^k}\Phi(x-y)f(y)\,d\sigma(y)\\&\le\sum_{k\in\mathbb{Z}}\Bigl(\sup_{2^{k-1}<|y|\le 2^k}\Phi(y)\Bigr)\int_{|x-y|\le 2^k}f(y)\,d\sigma(y)\\&\le\sum_{Q\in\mathcal{D}}\Bigl(\sup_{\ell(Q)/2<|y|\le\ell(Q)}\Phi(y)\Bigr)\int_{3Q}f(y)\,d\sigma(y)\chi_Q(x),\end{aligned}$$

where we have used the geometric fact that if $x\in Q\in\mathcal{D}$ and $\ell(Q)=2^k$, then $3Q\supset\{y\in\mathbb{R}^n:|x-y|\le 2^k\}$.

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A\approx B$ we mean that $c^{-1}B\le A\le cB$ with some positive constant c independent of appropriate quantities.

2. Proof of Theorem 1.3. The theorem follows easily from the following lemma.

Lemma 2.1. *Let $K:\mathcal{D}\to\mathbb{R}^+$, $a>1$ and $1<p<\infty$. Let μ be a nonnegative Borel measure, σ be a positive doubling Borel measure and w be a weight on $\mathbb{R}^n$. Define*

$$\mu_1(x)=\frac{d\mu(x)}{\overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w](x)^{p-1}}\quad\textit{and}\quad\mu_2(x)=\frac{d\mu(x)}{\overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p}[\mu,w](x)^{p-1}}.$$

Then, if f is nonnegative, bounded and has compact support, we have

$$\|T_K[f\,d\sigma]\|_{L^p(d\mu_1)}\le C\|fw\|_{L^p(d\sigma)},\tag{2.1}$$

$$\|T_K[f\,d\sigma]\|_{L^p(d\mu_2)}\le C\|(M_{\sigma}f)w\|_{L^p(d\sigma)}.\tag{2.2}$$

Proof. For simplicity, we will use the notation

$$m_Q(f)=\frac{1}{\sigma(Q)}\int_{3Q}f(x)\,d\sigma(x),\qquad Q\in\mathcal{D}.$$

We set

$$C_0 = \sup_{Q \in \mathcal{Q}} \frac{\sigma(3Q)}{\sigma(Q)}.$$

Since σ is a positive doubling Borel measure, i.e.,

$$\sup_{Q \in \mathcal{Q}} \frac{\sigma(2Q)}{\sigma(Q)} < \infty,$$

C_0 is positive and finite.

We remember that f is nonnegative, bounded and has compact support. Letting $A = 2C_0^3$, we define

$$D_k = \bigcup \{Q \in \mathcal{D} : m_Q(f) > A^k\} \quad \text{for } k \in \mathbb{Z}.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_j^k,$$

where the cubes $\{Q_j^k\}$ are nonoverlapping. By the maximality of Q_j^k,

$$A^k < m_{Q_k^j}(f) \le C_0 A^k. \tag{2.3}$$

Let $E_j^k = Q_j^k \setminus D_{k+1}$. We need the following properties:

(2.4) $\{E_j^k\}$ is a disjoint cover of the support of $T_K[f\,d\sigma]$,

$$\sigma(Q_j^k) \le 2\sigma(E_j^k). \tag{2.5}$$

The claim (2.4) is clear. The inequality (2.5) can be verified as follows:

By (2.3) we see that, if $Q_i^{k+1} \subset Q_j^k$, then for any $x \in Q_i^{k+1}$,

$$M_\sigma[\chi_{3Q_j^k} f](x) \ge \frac{1}{\sigma(3Q_i^{k+1})} \int_{3Q_i^{k+1}} f(y)\,d\sigma(y) \ge \frac{m_{Q_i^{k+1}}(f)}{C_0} \ge \frac{A^{k+1}}{C_0},$$

where we have used $\sigma(3Q_i^{k+1}) \le C_0 \sigma(Q_i^{k+1})$. This gives

$$Q_j^k \cap D_{k+1} \subset \{x \in Q_j^k : M_\sigma[\chi_{3Q_j^k} f](x) > A^{k+1}/C_0\}.$$

Using the weak-$(1,1)$ boundedness of M_σ, which is justified since σ is a doubling measure, we have

$$\sigma(Q_j^k \cap D_{k+1}) \le \frac{C_0^2}{A^{k+1}} \int_{3Q_j^k} f(y)\,d\sigma(y) \le \frac{C_0^3}{A} \sigma(Q_j^k) = \frac{1}{2}\sigma(Q_j^k),$$

where we have used (2.3) again. This clearly implies (2.5).

We set

$$\mathcal{D}_j^k = \{Q \in \mathcal{D} : Q \subset Q_j^k,\ A^k < m_Q(f) \le A^{k+1}\}.$$

Then we obtain

$$\{Q \in \mathcal{D} : m_Q(f) > 0\} = \bigcup_{k,j} \mathcal{D}_j^k. \tag{2.6}$$

We shall prove the lemma by a duality argument. We first verify (2.1). To this end, we take $g \geq 0$ satisfying

$$\|g\|_{L^{p'}(d\mu_1)} \leq 1 \tag{2.7}$$

and evaluate

$$\mathrm{I} = \sum_{Q \in \mathcal{D}} m_Q(f)\sigma(Q)K(Q) \int_Q g(x)\, d\mu_1(x).$$

Noticing (2.6), we shall estimate

$$\mathrm{II} = \sum_{Q \in \mathcal{D}_j^k} m_Q(f)\sigma(Q)K(Q) \int_Q g(x)\, d\mu_1(x).$$

From the definition of $\mathcal{D}_j^k$,

$$\begin{aligned}
\frac{\mathrm{II}}{A^{k+1}} &\leq \sum_{Q \in \mathcal{D}_j^k} \sigma(Q)K(Q) \int_Q g(x)\, d\mu_1(x) \leq \sum_{Q \subset Q_j^k} \sigma(Q)K(Q) \int_Q g(x)\, d\mu_1(x) \\
&= \int_{Q_j^k} \Big(\sum_{Q \subset Q_j^k} \sigma(Q)K(Q)\chi_Q(x) \Big) g(x)\, d\mu_1(x) \\
&= \int_{Q_j^k} \overline{K}(Q_j^k)(x) g(x)\, d\mu_1(x)\, \sigma(Q_j^k),
\end{aligned}$$

where we have used the definition of $\overline{K}(Q_j^k)(x)$. From (2.3) and (2.5),

$$\begin{aligned}
\mathrm{II} &\leq C m_{Q_j^k}(f) \int_{Q_j^k} \overline{K}(Q_j^k)(x) g(x)\, d\mu_1(x)\, \sigma(Q_j^k) \\
&\leq C m_{Q_j^k}(f) \int_{Q_j^k} \overline{K}(Q_j^k)(x) g(x)\, d\mu_1(x)\, \sigma(E_j^k).
\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}
m_{Q_j^k}(f) &\leq \left(\frac{1}{\sigma(Q_j^k)} \int_{3Q_j^k} w(y)^{-ap'}\, d\sigma(y) \right)^{1/ap'} \\
&\quad \times \left(\frac{1}{\sigma(Q_j^k)} \int_{3Q_j^k} (f(y)w(y))^{bp}\, d\sigma(y) \right)^{1/bp},
\end{aligned}$$

where $b<1$ satisfies $(bp)'=ap'$. These imply, as $Q_j^k \supset E_j^k$,

$$\mathrm{II} \le C \int_{E_j^k} M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x) M_\sigma[(fw)^{bp}](x)^{1/bp}\, d\sigma(x), \tag{2.8}$$

where

$$M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x) = \sup_{x\in Q\in\mathcal{D}} \left(\frac{1}{\sigma(Q)}\int_{3Q} w(y)^{-ap'}\, d\sigma(y)\right)^{1/ap'} \int_Q \overline{K}(Q)(y)g(y)\, d\mu_1(y).$$

Such maximal operators are essentially used in [6] and [4]. From (2.4), (2.6) and (2.8), we obtain

$$\mathrm{I} \le C \int_{\mathbb{R}^n} M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x) M_\sigma[(fw)^{bp}](x)^{1/bp}\, d\sigma(x).$$

By Hölder's inequality,

$$\mathrm{I} \le C\Big\{ \int_{\mathbb{R}^n} M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x)^{p'}\, d\sigma(x)\Big\}^{1/p'} \times \Big\{ \int_{\mathbb{R}^n} M_\sigma[(fw)^{bp}](x)^{1/b}\, d\sigma(x)\Big\}^{1/p}. \tag{2.9}$$

Since the $L^{1/b}(d\sigma)$-boundedness of M_σ yields

$$\Big\{ \int_{\mathbb{R}^n} M_\sigma[(fw)^{bp}](x)^{1/b}\, d\sigma(x)\Big\}^{1/p} \le C\|fw\|_{L^p(d\sigma)}, \tag{2.10}$$

we need only evaluate

$$(\mathrm{III})^{p'} = \int_{\mathbb{R}^n} M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x)^{p'}\, d\sigma(x).$$

It is clear that for any $x\in\mathbb{R}^n$,

$$M_{K,\sigma}^{\mathcal{D};p,a}[g d\mu_1, w](x)^{p'} \le \sum_{Q\in\mathcal{D}} \left(\frac{1}{\sigma(Q)}\int_{3Q} w(y)^{-ap'}\, d\sigma(y)\right)^{1/a} \Big(\int_Q \overline{K}(Q)(y)g(y)\, d\mu_1(y)\Big)^{p'} \chi_Q(x).$$

This simple inequality shows that $(\mathrm{III})^{p'}$ is bounded by

$$\sum_{Q\in\mathcal{D}} \left(\frac{1}{\sigma(Q)}\int_{3Q} w(y)^{-ap'}\, d\sigma(y)\right)^{1/a} \sigma(Q)\Big(\int_Q \overline{K}(Q)(y)g(y)\, d\mu_1(y)\Big)^{p'}.$$

It follows from the definition of μ_1 and Hölder's inequality that

$$\begin{aligned}\Big(\int_Q \overline{K}(Q)(y)g(y)\,d\mu_1(y)\Big)^{p'} &= \Big(\int_Q \overline{K}(Q)(y)^{1/p}\cdot g(y)\frac{\overline{K}(Q)(y)^{1/p'}}{\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](y)^{p-1}}\,d\mu(y)\Big)^{p'} \\ &\le \Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'-1}\int_Q \frac{\overline{K}(Q)(y)g(y)^{p'}}{\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](y)^{p}}\,d\mu(y) \\ &= \Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'-1}\int_Q \frac{\overline{K}(Q)(y)g(y)^{p'}}{\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](y)}\,d\mu_1(y).\end{aligned}$$

This yields, by the definition of $\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w]$,

$$(\mathrm{III})^{p'} \le \int_{\mathbb{R}^n} \frac{\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](y)}{\overline{\mathcal{W}}^{\mathcal{D};p,a}_{K,\sigma}[\mu,w](y)} g(y)^{p'}\,d\mu_1(y).$$

Hence, by (2.7) we have

$$\mathrm{III} \le C. \tag{2.11}$$

The inequalities (2.9)–(2.11) yield (2.1).

We next verify (2.2). We use the above estimates with μ_2 in place of μ_1. It follows by noticing $Q_j^k \supset E_j^k$ that

$$\mathrm{II} \le C\int_{E_j^k} M^{\mathcal{D}}_{K,\sigma}[g d\mu_2](x) M_\sigma f(x)\,d\sigma(x), \tag{2.12}$$

where

$$M^{\mathcal{D}}_{K,\sigma}[g d\mu_2](x) = \sup_{x\in Q\in\mathcal{D}} \int_Q \overline{K}(Q)(y)g(y)\,d\mu_2(y).$$

From (2.4), (2.6) and (2.12), we have

$$\mathrm{I} \le C\int_{\mathbb{R}^n} M^{\mathcal{D}}_{K,\sigma}[g d\mu_2](x) M_\sigma f(x)\,d\sigma(x).$$

By Hölder's inequality,

$$\begin{aligned}\mathrm{I} \le C\Big\{\int_{\mathbb{R}^n} M^{\mathcal{D}}_{K,\sigma}[g d\mu_2](x)^{p'} w(x)^{-p'}\,d\sigma(x)\Big\}^{1/p'} \\ \times\Big\{\int_{\mathbb{R}^n} (M_\sigma f(x) w(x))^p\,d\sigma(x)\Big\}^{1/p}.\end{aligned} \tag{2.13}$$

Now, we need only evaluate

$$(\mathrm{IV})^{p'} = \int_{\mathbb{R}^n} M_{K,\sigma}^{\mathcal{D}}[g d\mu_2](x)^{p'} w(x)^{-p'}\, d\sigma(x).$$

It is clear that for any $x \in \mathbb{R}^n$,

$$M_{K,\sigma}^{\mathcal{D}}[g d\mu_2](x)^{p'} \le \sum_{Q\in\mathcal{D}} \Big(\int_Q \overline{K}(Q)(y) g(y)\, d\mu_2(y)\Big)^{p'} \chi_Q(x).$$

This inequality shows that $(\mathrm{IV})^{p'}$ is bounded by

$$\begin{aligned}(2.14)\qquad &\sum_{Q\in\mathcal{D}} \int_Q w(y)^{-p'}\, d\sigma(y) \Big(\int_Q \overline{K}(Q)(y) g(y)\, d\mu_2(y)\Big)^{p'} \\ &= \sum_{Q\in\mathcal{D}} \Big(\frac{1}{\sigma(Q)} \int_Q w(y)^{-p'}\, d\sigma(y)\Big) \sigma(Q) \Big(\int_Q \overline{K}(Q)(y) g(y)\, d\mu_2(y)\Big)^{p'}.\end{aligned}$$

Once we have verified (2.14), in the same manner as above, we obtain

$$\mathrm{IV} \le C. \tag{2.15}$$

The inequalities (2.13)–(2.15) complete the proof of (2.2). ■

Proof of Theorem 1.3. First, we recall that $r = pq/(p-q)$. It follows from Lemma 2.1 and Hölder's inequality with exponent $(p-q)/p + q/p = 1$ that

$$\begin{aligned}&\|T_K[f d\sigma]\|_{L^q(d\mu)} \\ &\quad= \Big\{ \int_{\mathbb{R}^n} \overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w](x)^{q/p'} \overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w](x)^{-q/p'} T_K[f d\sigma]^q\, d\mu(x)\Big\}^{1/q} \\ &\quad\le \|\overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w]^{1/p'}\|_{L^r(d\mu)} \|T_K[f d\sigma]\|_{L^p(d\mu_1)} \\ &\quad\le C\|\overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w]^{1/p'}\|_{L^r(d\mu)} \|f w\|_{L^p(d\sigma)}.\end{aligned}$$

This proves Theorem 1.3(a). Theorem 1.3(b) can be verified similarly. ■

REMARK 2.2. Following [11], one can replace the norm

$$\Big(\frac{1}{\sigma(Q)} \int_{3Q} w(y)^{-ap'}\, d\sigma(y)\Big)^{1/ap'}$$

in $\overline{\mathcal{W}}_{K,\sigma}^{\mathcal{D};p,a}[\mu,w]$ by weaker norms which are defined in terms of certain mapping properties of appropriate maximal operators associated to each norm.

3. Appendix. In this appendix, we show that our sufficient conditions are necessary in some special cases.

Proposition 3.1. *Let $K : \mathcal{D} \to \mathbb{R}^+$ and $1 < q < p < \infty$. Let μ be a nonnegative Borel measure on $\mathbb{R}^n$ and $w^p \in A_p$. Suppose that for any $Q \in \mathcal{D}$,*

$$\sup_{x\in Q} \overline{K}(Q)(x) \le CK(Q). \tag{3.1}$$

Then the two-weight norm inequality

$$\|T_K[f\,dx]\|_{L^q(d\mu)} \le C\|fw\|_{L^p(dx)}$$

holds if and only if

$$\overline{\mathcal{W}}_K^{\mathcal{D};p}[\mu,w]^{1/p'} \in L^r(d\mu), \qquad 1/q = 1/r + 1/p,$$

where

$$\overline{\mathcal{W}}_K^{\mathcal{D};p}[\mu,w](x) = \sum_{Q\in\mathcal{D}} \int_Q w(y)^{-p'}\,dy \Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'-1} \overline{K}(Q)(x).$$

To prove the proposition we need the following lemma.

Lemma 3.2. *Let $K : \mathcal{D} \to \mathbb{R}^+$ and $1 < s < \infty$. Let ν be a nonnegative Borel measure on $\mathbb{R}^n$ and $w^s \in A_s$. Suppose that* (3.1) *holds for any $Q \in \mathcal{D}$. Then*

$$\|(T_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)} \approx \|(\overline{M}_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)} \approx \|(\overline{\mathcal{W}}_K^{\mathcal{D};s}[\nu])w^s\|_{L^1(dx)}^{1/s},$$

where

$$T_K^{\mathcal{D}}[\nu](x) = \sum_{Q\in\mathcal{D}} K(Q)\nu(Q)\chi_Q(x),$$

$$\overline{M}_K^{\mathcal{D}}[\nu](x) = \sup_{x\in Q\in\mathcal{D}} \int_Q \overline{K}(Q)(y)\,d\nu(y),$$

$$\overline{\mathcal{W}}_K^{\mathcal{D};s}[\nu](x) = \sum_{Q\in\mathcal{D}} \Big(\int_Q \overline{K}(Q)(y)\,d\nu(y)\Big)^s \chi_Q(x).$$

Proof. Given an $f \in L^1_{\mathrm{loc}}(dx)$ and a $Q \in \mathcal{D}$, let f_Q denote the average of f over Q:

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx.$$

Define the *dyadic sharp maximal function* of f by

$$M^{\sharp,\mathcal{D}}f(x) = \sup_{x\in Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy.$$

Since $w^s \in A_s$, one knows that

$$\|fw\|_{L^s(dx)} \approx \|(M^{\sharp,\mathcal{D}}f)w\|_{L^s(dx)}. \tag{3.2}$$

It follows that, for any $Q \in \mathcal{D}$,

$$\frac{1}{|Q|}\int_Q |T_K^{\mathcal{D}}[\nu](x) - (T_K^{\mathcal{D}}[\nu])_Q|\,dx \le 2\int_Q \overline{K}(Q)(x)\,d\nu(x).$$

By (3.2) this yields

$$\|(T_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)} \approx \|(\overline{M}_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)}. \tag{3.3}$$

We see clearly that

$$\|(\overline{M}_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)} \le \|(\overline{\mathcal{W}}_K^{\mathcal{D};s}[\nu])w^s\|_{L^1(dx)}^{1/s}.$$

Now, we shall verify the converse. We have

$$\begin{aligned}
&\|(\overline{\mathcal{W}}_K^{\mathcal{D};s}[\nu])w^s\|_{L^1(dx)} \\
&\quad= \int_{\mathbb{R}^n}\Big\{\sum_{Q\in\mathcal{D}}\Big(\int_Q \overline{K}(Q)(y)\,d\nu(y)\Big)^s\chi_Q(x)\Big\}w(x)^s\,dx \\
&\quad\le \int_{\mathbb{R}^n}\overline{M}_K^{\mathcal{D}}[\nu](x)^{s-1}\Big\{\sum_{Q\in\mathcal{D}}\int_Q \overline{K}(Q)(y)\,d\nu(y)\,\chi_Q(x)\Big\}w(x)^s\,dx.
\end{aligned}$$

From (3.1) and Hölder's inequality, the above is

$$\begin{aligned}
&\le C\int_{\mathbb{R}^n}\overline{M}_K^{\mathcal{D}}[\nu](x)^{s-1}\Big\{\sum_{Q\in\mathcal{D}}K(Q)\mu(Q)\chi_Q(x)\Big\}w(x)^s\,dx \\
&\le C\|(\overline{M}_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)}^{s-1}\|(T_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)}.
\end{aligned}$$

Thus, by (3.3) we obtain

$$\|(\overline{\mathcal{W}}_K^{\mathcal{D};s}[\nu])w^s\|_{L^1(dx)}^{1/s} \le C\|(\overline{M}_K^{\mathcal{D}}[\nu])w\|_{L^s(dx)}.$$

This completes the proof. ■

Proof of Proposition 3.1. We need only verify the "only if" part. We now assume that

$$\|T_K[f\,dx]\|_{L^q(d\mu)} \le C\|fw\|_{L^p(dx)}.$$

This implies

$$\|T_K^{\mathcal{D}}[f\,dx]\|_{L^q(d\mu)} \le C\|fw\|_{L^p(dx)}$$

and, by duality,

$$\|T_K^{\mathcal{D}}[g\,d\mu]w^{-1}\|_{L^{p'}(dx)} \le C\|g\|_{L^{q'}(d\mu)}.$$

Noticing $w^{-p'} \in A_{p'}$ and applying Lemma 3.2, we have

$$\|\overline{\mathcal{W}}_K^{\mathcal{D};p'}[g\,d\mu]w^{-p'}\|_{L^1(dx)} \le C_1^{p'}\|g\|_{L^{q'}(d\mu)}^{p'}. \tag{3.4}$$

Let

$$c_Q = \int_Q w(y)^{-p'}\,dy\Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'}, \qquad Q \in \mathcal{D}.$$

For any nonnegative $\psi \in L^{q'/p'}(d\mu)$, by use of (3.4), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\sum_{Q\in\mathcal{D}}\frac{c_Q}{\mu(Q)}\chi_Q(x)\Big)\psi(x)\,d\mu(x) &= \sum_{Q\in\mathcal{D}}\frac{c_Q}{\mu(Q)}\int_Q \psi(x)\,d\mu(x)\\ &\le \sum_{Q\in\mathcal{D}}\Big(\int_Q \overline{K}(Q)(x)M_\mu^{\mathcal{D}}\psi(x)^{1/p'}\,d\mu(x)\Big)^{p'}\int_Q w(x)^{-p'}dx\\ &= \|\overline{\mathcal{W}}_K^{\mathcal{D};p'}[(M_\mu^{\mathcal{D}}\psi)^{1/p'}d\mu]w^{-p'}\|_{L^1(dx)} \le C_1^{p'}\|(M_\mu^{\mathcal{D}}\psi)^{1/p'}\|_{L^{q'}(d\mu)}^{p'},\end{aligned}$$

where

$$M_\mu^{\mathcal{D}}\psi(x) = \sup_{x\in Q\in\mathcal{D}}\frac{1}{\mu(Q)}\int_Q \psi(y)\,d\mu(y), \qquad x\in\mathbb{R}^n.$$

The $L^{q'/p'}(d\mu)$-boundedness of $M_\mu^{\mathcal{D}}$ yields

$$\int_{\mathbb{R}^n}\Big(\sum_{Q\in\mathcal{D}}\frac{c_Q}{\mu(Q)}\chi_Q(x)\Big)\psi(x)\,d\mu(x) \le C\|\psi\|_{L^{q'/p'}(d\mu)}.$$

Thus, by duality again,

$$\Big\|\sum_{Q\in\mathcal{D}}\frac{c_Q}{\mu(Q)}\chi_Q\Big\|_{L^{q(p-1)/(p-q)}(d\mu)} \le CC_1^{p'}.$$

It follows from (3.1) that

$$\frac{c_Q}{\mu(Q)} \approx \int_Q w(y)^{-p'}\,dy\Big(\int_Q \overline{K}(Q)(y)\,d\mu(y)\Big)^{p'-1}\overline{K}(Q)(x), \qquad x\in Q\in\mathcal{D}.$$

Thus, we obtain

$$\|\overline{\mathcal{W}}_K^{\mathcal{D};p}[\mu,w]^{1/p'}\|_{L^r(d\mu)} \le CC_1,$$

where we have used

$$r = \frac{pq}{p-q} = \frac{p}{p-1}\cdot\frac{q(p-1)}{p-q}.$$

This proves the proposition.

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