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Łojasiewicz ideals in Denjoy–Carleman classes

by

VINCENT THILLIEZ (Lille)

Abstract. The classical notion of Łojasiewicz ideals of smooth functions is studied in the context of non-quasianalytic Denjoy–Carleman classes. In the case of principal ideals, we obtain a characterization of Łojasiewicz ideals in terms of properties of a generator. This characterization involves a certain type of estimates that differ from the usual Łojasiewicz inequality. We then show that basic properties of Łojasiewicz ideals in the $\mathcal{C}^\infty$ case have a Denjoy–Carleman counterpart.

1. Introduction. Let Ω be an open subset of $\mathbb{R}^n$, and let $\mathcal{C}^\infty(\Omega)$ be the Fréchet algebra of smooth functions in Ω. Let X be a closed subset of Ω. An element φ of $\mathcal{C}^\infty(\Omega)$ is said to satisfy the *Łojasiewicz inequality with respect to X* if, for every compact subset K of Ω, there are real constants $C > 0$ and $\nu \geq 1$ such that, for any $x \in K$, we have

$$|\varphi(x)| \geq C \operatorname{dist}(x, X)^\nu. \tag{1}$$

For example, it is well-known that any real-analytic function satisfies the Łojasiewicz inequality with respect to its zero set.

An element of $\mathcal{C}^\infty(\Omega)$ is said to be *flat on X* if it vanishes, together with all its derivatives, at each point of X. Denote by $\underline{m}_X^\infty$ the ideal of functions of $\mathcal{C}^\infty(\Omega)$ that are flat on X. The following statement appears in [15, Section V.4] and establishes a connection between the Łojasiewicz inequality and the behavior of ideals with respect to flat functions.

1.1. THEOREM. *Let $\mathcal{I}$ be a finitely generated proper ideal in $\mathcal{C}^\infty(\Omega)$, and let X be the zero set of $\mathcal{I}$. The following properties are equivalent:*

- (A) *The ideal $\mathcal{I}$ contains an element φ which satisfies the Łojasiewicz inequality with respect to X.*
- (B) $\underline{m}_X^\infty \subset \mathcal{I}$.
- (C) $\underline{m}_X^\infty = \mathcal{I}\underline{m}_X^\infty$.

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A finitely generated ideal $\mathcal{I}$ satisfying the equivalent conditions (A)–(C) is called a *Łojasiewicz ideal.* A principal ideal is Łojasiewicz if and only if condition (A) holds for a generator φ of the ideal. In the general case of a finitely generated ideal with generators $\varphi_1, \ldots, \varphi_p$, one can take $\varphi = \varphi_1^2 + \cdots + \varphi_p^2$. Łojasiewicz ideals play an important role in the study of ideals of differentiable functions; see for instance [8, 14, 15]. In particular, every closed ideal of finite type is Łojasiewicz, whereas the converse statement is false.

In the present paper, we study a possible approach to Łojasiewicz ideals in non-quasianalytic Denjoy–Carleman classes $\mathcal{C}_M(\Omega)$. While several papers have already been devoted to the study of closed ideals in $\mathcal{C}_M(\Omega)$ (see for example [10, 11, 12]), a suitable notion of Łojasiewicz ideal is still lacking, even in the case of principal ideals. This is due to the fact that if we put $\underline{m}_{X,M}^\infty = \underline{m}_X^\infty \cap \mathcal{C}_M(\Omega)$ and $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$, where φ is a given element of $\mathcal{C}_M(\Omega)$, it turns out that the usual Łojasiewicz inequality (1) is not a sufficient condition for the inclusion $\mathcal{I} \subset \underline{m}_{X,M}^\infty$, let alone for the equality $\underline{m}_{X,M}^\infty = \mathcal{I}\underline{m}_{X,M}^\infty$. Therefore, it is natural to ask for a characterization of both of these properties in terms of the generator φ, in the spirit of the characterization given by Theorem 1.1 in the $\mathcal{C}^\infty$ case.

In the case of principal ideals, a suitable characterization will be obtained in Theorem 3.4. In the statement, the Łojasiewicz inequality (1) has to be replaced by a quite different property involving successive derivatives of $1/\varphi$, which will be shown to be equivalent to the obvious Denjoy–Carleman version of property (C), that is, to the equality $\underline{m}_{X,M}^\infty = \mathcal{I}\underline{m}_{X,M}^\infty$. We are also able to get an equivalence with a corresponding version of property (B), provided we consider the inclusion $\underline{m}_{X,M}^\infty \subset \mathcal{I}$ together with a mild extra requirement on the flat points of φ.

In order to prove these results, one has to deal with the fact that the constructive techniques used by Tougeron in the classical $\mathcal{C}^\infty$ case do not seem applicable to the $\mathcal{C}_M$ setting. Thus, the main part of our proof of Theorem 3.4 is actually based on a functional-analytic argument. Once the theorem is proven, we discuss several related properties showing that basic results of the $\mathcal{C}^\infty$ case can be extended in a consistent way. For instance, we show that our $\mathcal{C}_M$ Łojasiewicz condition holds for closed principal ideals, and we also provide a non-closed example.

2. Denjoy–Carleman classes

2.1. Notation. For any multi-index $J = (j_1, \ldots, j_n)$ of $\mathbb{N}^n$, we always denote the length $j_1 + \cdots + j_n$ of J by the corresponding lower case letter j. We put $J! = j_1! \cdots j_n!$, $D^J = \partial^j/\partial x_1^{j_1} \cdots \partial x_n^{j_n}$ and $x^J = x_1^{j_1} \cdots x_n^{j_n}$. We denote by $|\cdot|$ the euclidean norm on $\mathbb{R}^n$; balls and distances in $\mathbb{R}^n$ will always be considered with respect to that norm.

If a is a point in $\mathbb{R}^n$, and if f is a smooth function in a neighborhood of a, we denote by $T_a f$ the formal Taylor series of f at a, that is, the element of $\mathbb{C}[[x_1, \ldots, x_n]]$ defined by

$$T_a f = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J f(a) x^J.$$

The function f is said to be *flat* at the point a if $T_a f = 0$.

2.2. Some properties of sequences. Let $M = (M_j)_{j \geq 0}$ be a sequence of real numbers satisfying the following assumptions:

(2) the sequence M is increasing, with $M_0 = 1$,

(3) the sequence M is *logarithmically convex*.

Property (3) amounts to saying that M_{j+1}/M_j is increasing. Together with (2), it implies

$$M_j M_k \leq M_{j+k} \quad \text{for any } (j,k) \in \mathbb{N}^2. \tag{4}$$

We say that the *moderate growth* property holds if there is a constant $A > 0$ such that, conversely,

$$M_{j+k} \leq A^{j+k} M_j M_k \quad \text{for any } (j,k) \in \mathbb{N}^2. \tag{5}$$

We say that M satisfies the *strong non-quasianalyticity* condition if there is a constant $A > 0$ such that

$$\sum_{j \geq k} \frac{M_j}{(j+1) M_{j+1}} \leq A \frac{M_k}{M_{k+1}} \quad \text{for any } k \in \mathbb{N}. \tag{6}$$

Notice that property (6) is indeed stronger than the classical Denjoy–Carleman non-quasianalyticity condition

$$\sum_{j \geq 0} \frac{M_j}{(j+1) M_{j+1}} < \infty. \tag{7}$$

The sequence M is said to be *strongly regular* if it satisfies (2), (3), (5) and (6).

2.3. Example. Let α and β be real numbers, with $\alpha > 0$. The sequence M defined by $M_j = j!^\alpha (\ln(j+e))^{\beta j}$ is strongly regular. This is the case, in particular, for the Gevrey sequences $M_j = j!^\alpha$.

With every sequence M satisfying (2) and (3) we also associate the function h_M defined by $h_M(t) = \inf_{j \geq 0} t^j M_j$ for any real $t > 0$, and $h_M(0) = 0$. From (2) and (3), it is easy to see that the function h_M is continuous, increasing, and satisfies $h_M(t) > 0$ for $t > 0$ and $h_M(t) = 1$ for $t \geq 1/M_1$. It also fully determines the sequence M, since we have $M_j = \sup_{t > 0} t^{-j} h_M(t)$.

2.4. Example. Let M be defined as in Example 2.3, and put $\eta(t) = \exp(-(t|\ln t|^{\beta})^{-1/\alpha})$ for $t > 0$. Elementary computations show that there are constants $a > 0$, $b > 0$ such that $\eta(at) \leq h_M(t) \leq \eta(bt)$ for $t \to 0$.

A technically important consequence of the moderate growth assumption (5) is the existence of a constant $\rho \geq 1$, depending only on M, such that

$$h_M(t) \leq (h_M(\rho t))^2 \quad \text{for any } t \geq 0. \tag{8}$$

We refer to [3] for a proof that (2), (3) and (5) imply (8).

2.5. Denjoy–Carleman classes. Let Ω be an open subset of $\mathbb{R}^n$, and let M be a sequence of real numbers satisfying (2) and (3). We define $\mathcal{C}_M(\Omega)$ as the space of functions f belonging to $\mathcal{C}^\infty(\Omega)$ and satisfying the following condition: for any compact subset K of Ω, one can find a real $\sigma > 0$ and a constant $C > 0$ such that

$$|D^J f(x)| \leq C\sigma^j j! M_j \quad \text{for any } J \in \mathbb{N}^n \text{ and } x \in K. \tag{9}$$

Given a function f in $\mathcal{C}^\infty(\Omega)$, a compact subset K of Ω and a real number $\sigma > 0$, put

$$\|f\|_{K,\sigma} = \sup_{x \in K,\, J \in \mathbb{N}^n} \frac{|D^J f(x)|}{\sigma^j j! M_j}.$$

We see that f belongs to $\mathcal{C}_M(\Omega)$ if and only if, for any compact subset K of Ω, one can find a real $\sigma > 0$ such that $\|f\|_{K,\sigma}$ is finite ($\|f\|_{K,\sigma}$ then coincides with the smallest constant C for which (9) holds). The function space $\mathcal{C}_M(\Omega)$ is called the *Denjoy–Carleman class of functions of class $\mathcal{C}_M$ in the sense of Roumieu* (which corresponds to $\mathcal{E}_{\{j!M_j\}}(\Omega)$ in the notation of [5]).

From now on, we will assume that the sequence M is strongly regular. In particular, it satisfies (7), which implies that $\mathcal{C}_M(\Omega)$ contains compactly supported functions. We denote by $\mathcal{D}_M(\Omega)$ the space of elements of $\mathcal{C}_M(\Omega)$ with compact support in Ω.

For the reader's convenience, we now recall some basic topological facts about $\mathcal{C}_M(\Omega)$ and $\mathcal{D}_M(\Omega)$, without proof (we refer to [5] for the details). With each Whitney 1-regular compact subset K of Ω, and each integer $\nu \geq 1$, we associate the vector space $\mathcal{C}_{M,K,\nu}$ of all functions f which are $\mathcal{C}^\infty$-smooth on K in the sense of Whitney, and such that $\|f\|_{K,\nu} < \infty$. Then $\mathcal{C}_{M,K,\nu}$ is a Banach space for the norm $\|\cdot\|_{K,\nu}$ and it can be shown that for $\nu < \nu'$, the inclusion $\mathcal{C}_{M,K',\nu} \hookrightarrow \mathcal{C}_{M,K',\nu'}$ is compact. We define the Denjoy–Carleman class $\mathcal{C}_M(K)$ as the reunion of all spaces $\mathcal{C}_{M,K,\nu}$ with $\nu \geq 1$. Endowed with the inductive topology, $\mathcal{C}_M(K)$ is a (DFS)-space (or *Silva space*). Given an exhaustion $(K_j)_{j\geq 1}$ of Ω by Whitney 1-regular compact subsets, the Denjoy–Carleman class $\mathcal{C}_M(\Omega)$ can be identified with the projective limit of all (DFS)-spaces $\mathcal{C}_M(K_j)$.

Similarly, denote by $\mathcal{D}_{M,K,\nu}$ the space of all functions $f \in \mathcal{C}^\infty(\Omega)$ such that $\operatorname{supp} f \subset K$ and $\|f\|_{K,\nu} < \infty$. Then $\mathcal{D}_{M,K,\nu}$ is a Banach space and we have the following properties: for $K \subset K'$, the space $\mathcal{D}_{M,K,\nu}$ is a closed subspace of $\mathcal{D}_{M,K',\nu}$, and for $\nu < \nu'$, the inclusion $\mathcal{D}_{M,K',\nu} \hookrightarrow \mathcal{D}_{M,K',\nu'}$ is compact. For any integer $\nu \geq 1$, put $\mathcal{D}_\nu = \mathcal{D}_{M,K_\nu,\nu}$, $\|\cdot\|_\nu = \|\cdot\|_{K_\nu,\nu}$, and notice that $\mathcal{D}_M(\Omega) = \bigcup_{\nu\geq 1} \mathcal{D}_\nu$ as a set. By the preceding remarks, we have a compact injection $\mathcal{D}_\nu \hookrightarrow \mathcal{D}_{\nu+1}$. Thus, the space $\mathcal{D}_M(\Omega)$ is another (DFS)-space for the corresponding inductive limit topology.

2.6. Some basic properties of $\mathcal{C}_M(\Omega)$. Properties (2) and (3) of the sequence M ensure that $\mathcal{C}_M(\Omega)$ is an algebra containing the algebra of real-analytic functions, and that $\mathcal{C}_M$ regularity is stable under composition [9]. This implies, in particular, the following invertibility property.

2.7. LEMMA ([9]). *If the function f belongs to $\mathcal{C}_M(\Omega)$ and has no zero in Ω, then the function $1/f$ belongs to $\mathcal{C}_M(\Omega)$.*

It is also known that the implicit function theorem holds within the framework of $\mathcal{C}_M$ regularity [6]. Thus, $\mathcal{C}_M$ manifolds and submanifolds can be defined in the usual way.

The strong regularity assumption on M ensures that suitable versions of Whitney's extension theorem and Whitney's spectral theorem hold in $\mathcal{C}_M(\Omega)$; see [1, 2, 3, 4]. The extension result relies on a crucial construction of cutoff functions whose successive derivatives satisfy a certain type of optimal estimates. This construction is due to Bruna [2]; see also [3, Proposition 4]. Up to a rescaling in the statement of [3], the result can be written as follows.

2.8. LEMMA ([2, 3]). *There is a constant $c > 0$ such that, for any real numbers $r > 0$ and $\sigma > 0$, one can find a function $\chi_{r,\sigma}$ belonging to $\mathcal{C}_M(\mathbb{R}^n)$, compactly supported in the ball $B = B(0, r)$, and such that $0 \leq \chi_{r,\sigma} \leq 1$, $\chi_{r,\sigma}(t) = 1$ for $|t| \leq r/2$ and $\|\chi_{r,\sigma}\|_{\overline{B},c\sigma} \leq (h_M(\sigma r))^{-1}$.*

We shall also need a basic result on flat functions. Given a closed subset Z of Ω, recall that $\underline{m}^\infty_{Z,M}$ denotes the ideal of functions of $\mathcal{C}_M(\Omega)$ which are flat at each point of Z.

2.9. LEMMA. *Let f be an element of $\underline{m}^\infty_{Z,M}$. For any compact subset K of Ω, there are positive constants c_1 and c_2 such that, for any multi-index I in $\mathbb{N}^n$ and any x in K, we have*

$$|D^I f(x)| \leq c_1 c_2^i i! M_i h_M(c_2 \operatorname{dist}(x, Z)). \tag{10}$$

Proof. For any real $r > 0$, put $K_r = \{y \in \Omega : \operatorname{dist}(y, K) \leq r\}$. If r is chosen small enough, K_r is a compact subset of Ω. Thus, there is a constant $\sigma > 0$ such that, for any $y \in K_r$, $I \in \mathbb{N}^n$ and $J \in \mathbb{N}^n$, we have $|D^{I+J} f(y)| \leq \|f\|_{K_r,\sigma} \sigma^{i+j} (i + j)! M_{i+j}$. Using (5) and the elementary estimate $(i + j)! \leq$

$2^{i+j}i!j!$, we get

$$|D^{I+J}f(y)| \le c_1 c_2^i i! M_i c_2^j j! M_j \tag{11}$$

with $c_1 = \|f\|_{K_r,\sigma}$ and $c_2 = 2A\sigma$. Now let x be a point in K, and let z be a point in Z such that

$$|x - z| = \operatorname{dist}(x, Z). \tag{12}$$

If $\operatorname{dist}(x, Z) \le r$, then the segment $[x, z]$ is contained in K_r. Let j be an integer. Since $D^I f$ is flat at z, the Taylor formula easily yields $|D^I f(x)| \le n^j \sup_{|J|=j,\, y\in K_r} |D^{I+J}(y)|\, |x - z|^j/j!$. Using (11) and (12), and taking the infimum with respect to j, we obtain (10) up to the replacement of c_2 by nc_2. If $\operatorname{dist}(x, Z) > r$, the estimate is a simple consequence of the definition of $\mathcal{C}_M(\Omega)$, up to another modification of c_1 and c_2. ∎

3. Łojasiewicz ideals. The following notion will serve as a replacement for the standard Łojasiewicz inequality.

3.1. Definition. Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ and let X be the zero set of φ. We say that φ *satisfies the $\mathcal{C}_M$ Łojasiewicz condition* if, for any compact subset K of Ω and any real $\lambda > 0$, one can find positive constants C and σ (depending on K and λ) such that, for any multi-index $J \in \mathbb{N}^n$ and any $x \in K \setminus X$, we have

$$\left|D^J(1/\varphi)(x)\right| \le \frac{C\sigma^j j! M_j}{h_M(\lambda \operatorname{dist}(x, X))}. \tag{13}$$

3.2. Remark. From the basic properties of h_M in Section 2.2, we see that, on a given open subset $\{x \in \Omega : \operatorname{dist}(x, X) > \delta\}$ with $\delta > 0$, the $\mathcal{C}_M$ Łojasiewicz condition amounts to nothing more than the conclusion of Lemma 2.7. It is relevant only as a bound on the explosion of $1/\varphi$ and its derivatives in a neighborhood of the zeros of φ.

In Section 4, we will provide examples of functions for which the $\mathcal{C}_M$ Łojasiewicz condition holds. Lemma 3.3 below shows that such functions cannot have "too many flat points" on the boundary of their zero set.

3.3. Lemma. *Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ and let X be its zero set. Assume that φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition, and let $X_\infty = \{a \in X : T_a\varphi = 0\}$ be the set of points of flatness of φ. Then $X \setminus X_\infty$ is dense in the boundary ∂X of X.*

Proof. Notice that φ is necessarily flat at each interior point of X, hence the inclusion $X \setminus X_\infty \subset \partial X$. We prove the density property by contradiction. If the property is not true, there are a point a in ∂X and an open neighborhood ω of a in Ω such that φ is flat on $\omega \cap \partial X$. Put $K = \overline{B(a, r)}$

with $r = \frac{1}{2}\operatorname{dist}(a, X \setminus \omega)$. Then K is a compact subset of ω and we have

$$\operatorname{dist}(x, \omega \cap \partial X) = \operatorname{dist}(x, \partial X) = \operatorname{dist}(x, X) \quad \text{for any } x \in K. \tag{14}$$

Using Lemma 2.9 on the open set ω, with $f = \varphi_{|\omega}$, $Z = \omega \cap \partial X$ and $I = 0$, we see that there are constants c_1 and c_2 such that we have $|\varphi(x)| \le c_1 h_M(c_2 \operatorname{dist}(x, \omega \cap \partial X))$ for any $x \in K$. Taking property (8) into account, we obtain, for any $x \in K$,

$$|\varphi(x)| \le c_1 h_M(c_3 \operatorname{dist}(x, \omega \cap \partial X))^2 \tag{15}$$

with $c_3 = \rho c_2$. On the other hand, using the $\mathcal{C}_M$ Łojasiewicz condition with $\lambda = c_3$ and $J = 0$, we obtain a constant $c_4 > 0$ such that, for any $x \in K \setminus X$,

$$|\varphi(x)| \ge c_4 h_M(c_3 \operatorname{dist}(x, X)). \tag{16}$$

Gathering (14), (15) and (16), we obtain $h_M(c_3 d(x, X)) \ge c_4/c_1$ for any $x \in K \setminus X$, which is impossible since $K \setminus X$ has an accumulation point on X, namely the point a. ■

We are now able to state the main result.

3.4. Theorem. *Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$, let X be its zero set, and let X_∞ be its set of points of flatness. Put $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$. The following properties are equivalent:*

(A′) *The function φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition.*
(B′) *$\underline{m}^\infty_{X,M} \subset \mathcal{I}$ and $X \setminus X_\infty$ is dense in ∂X.*
(C′) *$\underline{m}^\infty_{X,M} = \mathcal{I}\underline{m}^\infty_{X,M}$.*

Proof. We prove the implication (C′)⇒(A′) first. We use the (DFS)-space $\mathcal{D}_M(\Omega) = \varinjlim \mathcal{D}_\nu$ defined in Section 2.5. The intersection $\mathcal{D}_M(\Omega) \cap \underline{m}^\infty_{X,M}$ is obviously closed in $\mathcal{D}_M(\Omega)$, hence it is also a (DFS)-space with step spaces $\mathcal{E}_\nu = \mathcal{D}_\nu \cap \underline{m}^\infty_{X,M}$.

It is easy to see that the map $\Lambda : \mathcal{D}_M(\Omega) \cap \underline{m}^\infty_{X,M} \to \mathcal{D}_M(\Omega) \cap \underline{m}^\infty_{X,M}$ defined by $\Lambda(f) = \varphi f$ is continuous. Furthermore, given an element g of $\mathcal{D}_M(\Omega) \cap \underline{m}^\infty_{X,M}$, the assumption implies that it can be written φh for some $h \in \underline{m}^\infty_{X,M}$. If χ is an element of $\mathcal{D}_M(\Omega)$ such that $\chi = 1$ on $\operatorname{supp} g$, then we have $g = \chi g = \varphi f$ with $f = \chi h \in \mathcal{D}_M(\Omega) \cap \underline{m}^\infty_{X,M}$. Thus, Λ is also surjective.

We can therefore apply the De Wilde open mapping theorem ([7, Chapter 24]), which yields the following property: for any $\nu \ge 1$, there exist an integer $\mu_\nu \ge 1$ and a real constant $C_\nu > 0$ such that, for any $g \in \mathcal{E}_\nu$, one can find an element f of $\mathcal{E}_{\mu_\nu}$ such that

$$\varphi f = g \quad \text{and} \quad \|f\|_{\mu_\nu} \le C_\nu \|g\|_\nu. \tag{17}$$

Now, let x be a point in $K \setminus X$, let d_K be a real number such that $0 < d_K < \operatorname{dist}(K, \mathbb{R}^n \setminus \Omega)$, and put $r_x = \min(\operatorname{dist}(x, X), d_K)$. Given $\lambda > 0$, we apply Lemma 2.8 with $r = 2r_x/3$ and $\sigma = 3\lambda/2$. We set $g_x(y) = \chi_{r,\sigma}(y - x)$. Then g_x belongs to $\mathcal{C}_M(\Omega)$ and is compactly supported in the ball $B_x = B(x, 2r_x/3)$.

Obviously B_x is contained in $K' = \{y \in \Omega : \operatorname{dist}(y, K) \le 2d_K/3\}$, which is a compact subset of Ω. For a sufficiently large integer ν, depending only on K and λ, we have $\nu \ge c\sigma$ and $K' \subset K_\nu$, so that g_x belongs to $\mathcal{E}_\nu$ and

$$\|g_x\|_\nu = \|g_x\|_{\overline{B_x},\nu} \le \|g_x\|_{\overline{B_x},c\sigma} \le (h_M(\lambda r_x))^{-1}. \tag{18}$$

Since $h_M(\lambda r_x)$ equals either $h_M(\lambda \operatorname{dist}(x, X))$ or $h_M(\lambda d_K)$, and since we have $h_M(t) \le 1$ for every $t > 0$, we see that

$$h_M(\lambda r_x) \ge h_M(\lambda d_K) h_M(\lambda \operatorname{dist}(x, X)). \tag{19}$$

Now, if f_x denotes the element of $\mathcal{E}_{\mu_\nu}$ associated with g_x by property (17), we therefore have $\varphi f_x = g_x$ and, thanks to (18) and (19),

$$\|f_x\|_{\mu_\nu} \le C'_\nu \big(h_M(\lambda \operatorname{dist}(x, X))\big)^{-1} \tag{20}$$

with $C'_\nu = C_\nu / h_M(\lambda d_K)$. For any y in $B'_x = B(x, r_x/3)$, we have $g_x(y) = 1$, hence

$$f_x(y) = 1/\varphi(y). \tag{21}$$

In particular, $f_x(y) \ne 0$. Thus, we derive $B'_x \subset \operatorname{supp} f_x \subset K_{\mu_\nu}$, which implies, for any $y \in B'_x$ and any multi-index J,

$$|D^J f_x(y)| \le \|f_x\|_{\mu_\nu} (\mu_\nu)^j j! M_j. \tag{22}$$

Combining (20), (21) and (22), we get the desired estimate (13) with suitable constants $A = C'_\nu$ and $B = \mu_\nu$ depending only on ν, hence only on K and λ.

We now prove the implication (A′)⇒(B′). By Lemma 3.3, the assumption implies that $X \setminus X_\infty$ is dense in ∂X. The proof of the inclusion $\underline{m}^\infty_{X,M} \subset \mathcal{I}$ is a variant of the proof of [10, Theorem 2.3]; we give some details for the reader's convenience. Let f be an element of $\underline{m}^\infty_{X,M}$. For any $x \in \Omega \setminus X$ and any multi-index $P \in \mathbb{N}^n$, the Leibniz formula yields

$$D^P(f/\varphi)(x) = \sum_{I+J=P} \frac{P!}{I!J!} D^I f(x) D^J(1/\varphi)(x). \tag{23}$$

Let K be a compact subset of Ω. For $x \in K \setminus X$, we combine the $\mathcal{C}_M$ Łojasiewicz condition with Lemma 2.9 in order to obtain an estimate for all the terms $D^I f(x) D^J(1/\varphi)(x)$ that appear in (23). Lemma 2.9, together with (8), yields $|D^I f(x)| \le c_1 c_2^i i! M_i \big(h_M(c_3 \operatorname{dist}(x, X))\big)^2$ with $c_3 = \rho c_2$. Applying the $\mathcal{C}_M$ Łojasiewicz condition with $\lambda = c_3$, we therefore get $|D^I f(x) D^J(1/\varphi)(x)| \le c_2 C c_2^i \sigma^j i! j! M_i M_j h_M(c_3 \operatorname{dist}(x, X))$. Since $i + j = p$, we have $i!j! \le p!$, as well as $M_i M_j \le M_p$ by (4). Inserting these estimates in (23), we obtain, for every multi-index P and every $x \in K \setminus X$,

$$|D^P(f/\varphi)(x)| \le c_5 c_6^p p! M_p h_M(c_3 \operatorname{dist}(x, X)) \tag{24}$$

with $c_5 = c_2 C$ and $c_6 = c_2 + \sigma$. Using (24) and the Hestenes lemma, we see that the function g defined by $g(x) = f(x)/\varphi(x)$ for $x \in \Omega \setminus X$ and $g(x) = 0$ for $x \in X$ belongs to $\mathcal{C}_M(\Omega)$. Obviously, $f = \varphi g$, hence $f \in \mathcal{I}$.

Finally, we prove the implication (B′)⇒(C′). Let $f \in \underline{m}^\infty_{X,M}$. By assumption, there is $g \in \mathcal{C}_M(\Omega)$ such that $f = \varphi g$. Let a be a point of $X \setminus X_\infty$. In the ring of formal power series, we have $0 = T_a f = (T_a\varphi)(T_a g)$ with $T_a\varphi \neq 0$, which implies $T_a g = 0$. Thus, g is flat on $X \setminus X_\infty$, hence on ∂X since it is assumed that $X \setminus X_\infty$ is dense in ∂X. Put $\tilde{g}(x) = g(x)$ for $x \in \Omega \setminus X$ and $\tilde{g}(x) = 0$ for $x \in X$. By the Hestenes lemma, it is then readily seen that $\tilde{g} \in \underline{m}^\infty_{X,M}$. Moreover, we have $f = \varphi\tilde{g}$, hence $f \in \mathcal{I}\underline{m}^\infty_{X,M}$, and the proof is complete. ∎

3.5. Remark. We do not know whether the implication (B′)⇒(C′) still holds without the additional assumption on $X \setminus X_\infty$ in (B′). This is true when X is a real-analytic submanifold of Ω: indeed, according to [13, Theorem 4.2.4] ([1]), we then have $\underline{m}^\infty_{X,M} = \underline{m}^\infty_{X,M}\underline{m}^\infty_{X,M}$. Thus, in this case, the inclusion $\underline{m}^\infty_{X,M} \subset \mathcal{I}$ easily implies (C′).

3.6. Remark. Using the equivalence (A′)⇔(C′), we see that if φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition and if h is an invertible element of the algebra $\mathcal{C}_M(\Omega)$, so that φ and $h\varphi$ generate the same ideal $\mathcal{I}$, then $h\varphi$ also satisfies the $\mathcal{C}_M$ Łojasiewicz condition. This can also be checked by a direct computation with the Leibniz formula.

4. Additional properties and examples

4.1. On the zero set. We have a Denjoy–Carleman counterpart of [15, Proposition V.4.6].

4.2. Proposition. *Let φ be an element of $\mathcal{C}_M(\Omega)$ that satisfies the $\mathcal{C}_M$ Łojasiewicz condition, and let X be its zero set. Then there is a $\mathcal{C}_M$-smooth submanifold Y of Ω such that $X = \overline{Y}$.*

Proof. We notice first that the conclusion of Lemma 3.3 only requires a weaker property than the $\mathcal{C}_M$ Łojasiewicz condition: more precisely, the proof remains valid as soon as, for any compact subset K of Ω and any real $\lambda > 0$, one can find a constant $C > 0$ such that the inequality $|\varphi(x)| \geq Ch_M(\lambda \operatorname{dist}(x, X))$ holds for any $x \in K$. It is then fairly easy to check that the proof by induction given in [15] for the usual Łojasiewicz inequality on $\mathcal{C}^\infty$ functions remains valid in the $\mathcal{C}_M$ case, up to minor modifications. ∎

4.3. Connection with closedness. In this section, we show that the $\mathcal{C}_M$ Łojasiewicz condition behaves as expected with respect to closedness properties of ideals.

([1]) The result in [13] is actually a local version of the statement we give, but it can be globalized, using partitions of unity.

4.4. Proposition. *Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ that generates a closed ideal in $\mathcal{C}_M(\Omega)$. Then φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition. Moreover, both properties are equivalent when the zeros of φ are isolated.*

Proof. We use the same notation as in the proof of the implication (C′)⇒(A′) of Theorem 3.4. Put $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$ and assume that $\mathcal{I}$ is closed in $\mathcal{C}_M(\Omega)$. Since the inclusion $\mathcal{D}_M(\Omega) \hookrightarrow \mathcal{C}_M(\Omega)$ is continuous, $\mathcal{I} \cap \mathcal{D}_M(\Omega)$ is closed in $\mathcal{D}_M(\Omega)$. Using cutoff functions, it is also easy to see that $\mathcal{I}\cap\mathcal{D}_M(\Omega) = \varphi\mathcal{D}_M(\Omega)$. It is then possible to duplicate the proof of the implication (C′)⇒(A′), the only difference being that the map $f \mapsto \varphi f$ is now considered as a map from the (DFS)-space $\mathcal{D}_M(\Omega)$ onto its closed subspace $\varphi\mathcal{D}_M(\Omega)$.

The converse in the case of isolated zeros is based on a variant of the argument leading to [10, Proposition 4.1] (which deals with a singleton). Assume that φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition and that its zero set X consists of isolated points, so that X is a countable subset $\{a_j : j \geq 1\}$ of Ω. Put $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$ and let f be an element of the closure $\overline{\mathcal{I}}$. By the $\mathcal{C}_M$ version of Whitney's spectral theorem [4], for every $j \geq 1$ there is a function g_j of $\mathcal{C}_M(\Omega)$ such that $f - \varphi g_j$ is flat at a_j. Let $(\chi_j)_{j\geq1}$ be a sequence of compactly supported elements of $\mathcal{C}_M(\Omega)$ such that $\chi_j = 1$ in a neigborhood of a_j and $\operatorname{supp}\chi_j \cap \operatorname{supp}\chi_k = \emptyset$ for $k \neq j$. Then the (locally finite) series $g = \sum_{j\geq1} \chi_j g_j$ defines an element of $\mathcal{C}_M(\Omega)$ and we have $f - \varphi g \in \underline{m}^\infty_{X,M}$. Since (B′) holds, this yields $f \in \mathcal{I}$, hence the result. ■

4.5. Example. According to Proposition 4.4 and the results in [10, 12], examples of functions φ which satisfy the $\mathcal{C}_M$ Łojasiewicz condition will include any homogeneous polynomial with an isolated real critical point at 0, as well as real-analytic functions whose germs of complex zeros intersect $\mathbb{R}^n$ at isolated points with Łojasiewicz exponent 1 for the regular separation property. On the other hand, some analytic functions do not satisfy the $\mathcal{C}_M$ Łojasiewicz condition: for instance, given an integer $k \geq 2$, the polynomial $\psi(x) = x_1^2 + x_2^{2k}$ does not satisfy the $\mathcal{C}_M$ Łojasiewicz condition in $\mathbb{R}^2$, as can be seen from the results in [10] (property (B′) fails).

We now give an example showing that the converse to Proposition 4.4 is false without the assumption of isolated zeros. In particular, the $\mathcal{C}_M$ Łojasiewicz condition does not imply closedness in general.

4.6. Example. We put $n = 2$, $\Omega = \mathbb{R}^2$, and $\varphi(x) = x_1\psi(x)$ where ψ is the polynomial mentioned in Example 4.5. We then write $X = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and observe that $\operatorname{dist}(x, X) = |x_1|$ for all x. Let $x \in \mathbb{R}^2 \setminus X$. For any $v = (v_1, v_2) \in \mathbb{C}^2$, we have

$$|\psi(x+v) - \psi(x)| \leq 2|x_1|\,|v_1| + |v_1|^2 + \sum_{p=1}^{2k} \binom{2k}{p} |x_2|^{2k-p}|v_2|^p.$$

We also have the obvious inequalities $|x_1| \le (\psi(x))^{1/2}$ and $|x_2| \le (\psi(x))^{1/2k}$. Thus, if we assume $|v_1| \le \delta(\psi(x))^{1/2}$ and $|v_2| \le \delta(\psi(x))^{1/2k}$ for some real number δ with $0 < \delta < 1$, we get

$$|\psi(x+v) - \psi(x)| \le \left(2\delta + \delta^2 + \sum_{p=1}^{2k} \binom{2k}{p} \delta^p\right)\psi(x) \le (2^{2k}+2)\delta\psi(x).$$

Setting $\delta = (2^{2k+1}+4)^{-1}$, we obtain $|\psi(\zeta)| \ge \frac{1}{2}\psi(x)$ for every point ζ in the bidisc $\{\zeta \in \mathbb{C}^2 : |\zeta_1 - x_1| \le \delta(\psi(x))^{1/2},\ |\zeta_2 - x_2| \le \delta(\psi(x))^{1/2k}\}$. The Cauchy formula then yields, for every $(i,j) \in \mathbb{N}^2$,

$$\left|\frac{\partial^{i+j}}{\partial x_1^i x_2^j}\left(\frac{1}{\psi(x)}\right)\right| \le 2\delta^{-(i+j)} i!j!(\psi(x))^{-(\frac{i}{2}+\frac{j}{2k}+1)},$$

which easily implies

$$\left|\frac{\partial^{i+j}}{\partial x_1^i x_2^j}\left(\frac{1}{\psi(x)}\right)\right| \le 2\delta^{-(i+j)} i!j!|x_1|^{-(i+j+2)} \tag{25}$$

provided we assume $|x_1| < 1$. Using (25), the definition of φ, and the Leibniz formula, we then get

$$\left|\frac{\partial^{i+j}}{\partial x_1^i x_2^j}\left(\frac{1}{\varphi(x)}\right)\right| \le B^{i+j+1} i!j!|x_1|^{-(i+j+2)}$$

for some suitable constant $B > 0$. Let λ be a given positive real number. We write

$$|x_1|^{-(i+j+2)} = \frac{\lambda^{i+j+2} M_{i+j+2}}{(\lambda|x_1|)^{i+j+2} M_{i+j+2}}.$$

The definition of h_M implies

$$(\lambda|x_1|)^{i+j+2} M_{i+j+2} \ge h_M(\lambda|x_1|) = h_M(\lambda \operatorname{dist}(x,X)),$$

whereas (5) yields $M_{i+j+2} \le A^{i+j+2} M_2 M_{i+j}$. Gathering these inequalities, we eventually obtain $|x_1|^{-(i+j+2)} \le (A\lambda)^{i+j+2}\big(h_M(\lambda \operatorname{dist}(x,X))\big)^{-1}$ and

$$\left|\frac{\partial^{i+j}}{\partial x_1^i x_2^j}\left(\frac{1}{\varphi(x)}\right)\right| \le \frac{C\sigma^{i+j}(i+j)!M_{i+j}}{h_M(\lambda \operatorname{dist}(x,X))}$$

with $C = A^2 B\lambda^2$ and $\sigma = AB\lambda$. Thus, we have established the desired estimate for $|x_1| = \operatorname{dist}(x,X) < 1$, which suffices to conclude that φ satisfies the $\mathcal{C}_M$ Łojasiewicz condition (see Remark 3.2). However, the ideal $\mathcal{I} = \varphi\mathcal{C}_M(\mathbb{R}^2)$ is not closed for $k \ge 2$. Indeed, in this case, it has been shown in [10] that the ideal $\mathcal{J} = \psi\mathcal{C}_M(\mathbb{R}^2)$ is not closed. Since $\mathcal{J}$ is the preimage of $\mathcal{I}$ under the continuous mapping $\Pi : \mathcal{C}_M(\mathbb{R}^2) \to \mathcal{C}_M(\mathbb{R}^2)$ defined by $\Pi(f)(x) = x_1 f(x)$, we see that $\mathcal{I}$ is not closed either.

We conclude with a natural question.

Problem. Is it possible to extend the above results to the general case of finitely generated ideals? A first idea is to mimic the definition of Łojasiewicz ideals in the $\mathcal{C}^\infty$ case, and say that a finitely generated ideal of $\mathcal{C}_M(\Omega)$ is Łojasiewicz if it contains an element φ which satisfies the $\mathcal{C}_M$ Łojasiewicz condition. However, this definition does not seem to allow an immediate extension of the crucial implication (C′)⇒(A′), whose proof is quite different from the $\mathcal{C}^\infty$ case and does not seem easily adaptable to the case of several generators.

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Vincent Thilliez
Laboratoire Paul Painlevé
Mathématiques – Bâtiment M2
Université Lille 1
F-59655 Villeneuve d'Ascq Cedex, France
E-mail: thilliez@math.univ-lille1.fr

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