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Multivariate spectral multipliers for systems of Ornstein–Uhlenbeck operators

by

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Abstract. Multivariate spectral multipliers for systems of Ornstein–Uhlenbeck operators are studied. We prove that L^p-uniform, $1 < p < \infty$, spectral multipliers extend to holomorphic functions in some subset of a polysector, depending on p. We also characterize L^1-uniform spectral multipliers and prove a Marcinkiewicz-type multiplier theorem. In the appendix we obtain analogous results for systems of Laguerre operators.

1. Introduction. In this paper we study multivariate spectral multipliers mainly associated with the system of one-dimensional Ornstein–Uhlenbeck operators

$$\mathcal{L}_n = -\frac{1}{2}\frac{\partial^2}{\partial x_n^2} + x_n\frac{\partial}{\partial x_n}, \qquad n = 1, \ldots, d.$$

More generally one could consider each of the operators $\mathcal{L}_n$, $n = 1, \ldots, d$, to be k_n-dimensional; however to avoid unnecessary notational complications, at first we restrict to the one-dimensional situation. Nevertheless, the results we prove later on actually admit arbitrary dimension of the operators $\mathcal{L}_n$ (see Corollary 3.9).

We consider two problems. The first concerns the holomorphy of joint spectral multipliers of the system $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$. We start by proving a fairly general theorem for a system $L = (L_1, \ldots, L_d)$ of non-negative operators with discrete spectra (Theorem 3.1). Namely, we show that, if, for each L_n, $n = 1, \ldots, d$, their L^p, $p > 1$, uniform spectral multipliers have the holomorphic extension property, then the same is true for the L^p, $p > 1$, uniform joint spectral multipliers of the system L. Next we prove an extension theorem for L^p-uniform, $p \in [1, \infty) \setminus \{2\}$, joint spectral multipliers of the system $\mathcal{L}$ (Theorem 3.8). Item (i) of Theorem 3.8 is a straightforward consequence of Theorem 3.1 and [2, Theorem 3.5(i)]. Item (ii) of Theorem 3.8

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is proved by enhancing the techniques from [2]. Lemmata 3.14, 3.18 used in that proof are stated in a slightly more general form. We use this later to prove an extension theorem concerning the holomorphy of multipliers of some systems built on $\mathcal{L}$, for instance, $(\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2)$ (Theorem 3.21). The material of Section 3 is to some extent a multivariate generalization of the results from [2]. It is worth mentioning that the methods presented here are, to a high degree, also applicable to the more general class of elliptic self-adjoint operators considered in [2, Section 4].

The second topic we study is a Marcinkiewicz-type multiplier theorem for the system $\mathcal{L}$ (Theorem 4.2). The results we obtain are multivariate analogues of [7] and also an application of the methods of [13] for a system of operators having 0 in their spectra.

Additionally, in the appendix we show how the techniques developed in the present paper can be utilized to obtain similar results for the system $\mathcal{L}^\alpha = (\mathcal{L}_1^{\alpha_1}, \ldots, \mathcal{L}_d^{\alpha_d})$ of Laguerre operators of orders α_n, given by

$$\mathcal{L}_n^{\alpha_n} = -x_n \frac{\partial^2}{\partial x_n^2} + (\alpha_n + 1 - x_n)\frac{\partial}{\partial x_n}, \qquad n = 1, \ldots, d.$$

Here we rely on the methods developed in [10].

2. Preliminaries. We say that self-adjoint operators $L_1, \ldots, L_d$ on some space $L^2(X,\nu)$ *commute strongly* (or that $L = (L_1, \ldots, L_d)$ is a system of strongly commuting operators on $L^2(X,\nu)$) if their spectral projections commute pairwise. This is the case if and only if their resolvents commute pairwise. If L is a system of strongly commuting operators, and m is a function defined on the joint spectrum of L, then we can consider joint spectral multipliers $m(L)$ of L, defined via spectral integration. For details the reader is referred to [6, Appendix 4].

We use the following notation. The symbol $\|\cdot\|_{L^p(X,\nu)}$ denotes either the L^p norm of a function on X with respect to a measure ν on X or the $L^p(X,\nu) \to L^p(X,\nu)$ norm of an operator. The meaning should be clear from the context. If $X = \mathbb{R}^d$, we write $L^p(\nu)$. In addition, while considering the Lebesgue measure we write L^p for short. Throughout the paper by a measure we mean a Borel measure and by a measurable function we mean a Borel measurable function.

The main results of the paper are given in the context of $L^p(\mathbb{R}^d, \gamma)$, where γ is a Gaussian measure (to be specified later on) on the fixed d-dimensional Euclidean space. The symbol p always denotes the exponent of the Lebesgue space $L^p(X,\nu)$; throughout the paper we assume $p \geq 1$.

Given a complex measure μ on $\mathbb{R}^d$ we denote by $\|\mu\|_{M(\mathbb{R}^d)}$ its total variation. The symbol $\mathbb{R}^d_+$ will denote the set $(0,\infty)^d$. We write $\mathbb{N}^d$, $\mathbb{N}^d_+$ for the sets $\mathbb{Z}^d \cap [0,\infty)^d$, $\mathbb{Z}^d \cap \mathbb{R}^d_+$, respectively.

If U is an open subset of $\mathbb{C}^d$, we write $H^\infty(U)$ for the vector space of bounded functions on U, which are holomorphic as functions of several variables. We equip this space with the supremum norm. For an angle $0 < \alpha < \pi$, we denote by $(\mathbf{S}_\alpha)^d$ the polysector

$$(\mathbf{S}_\alpha)^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \colon |\arg(z_n)| < \alpha,\ n = 1, \ldots, d\},$$

and for $\psi > 0$, we write $(\Sigma_\psi)^d$ for the polystrip

$$(\Sigma_\psi)^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \colon |\mathrm{Im}(z_n)| < \psi,\ n = 1, \ldots, d\}.$$

If $\sigma = (\sigma_1, \ldots, \sigma_d)$ and $\rho = (\rho_1, \ldots, \rho_d)$ are multi-indices we will write $\sigma < \rho$ whenever $\sigma_n < \rho_n$, $n = 1, \ldots, d$. The symbol $\mathbf{1}$ will denote the vector $(1, \ldots, 1)$. Following [2] we set $\phi_p^* = \arcsin|2/p - 1|$. We say that a tempered distribution k is an L^p *convolutor* if the convolution operator $K(f) = k * f$ extends to a bounded operator on L^p. We denote by $\|k\|_{\mathrm{conv},p}$ the L^p norm of the extension of K. If k is an L^p convolutor, we say that $\hat{k}$ is an L^p *multiplier*, and write $\|\hat{k}\|_{\mathcal{M}_p}$ for $\|K\|_{L^p}$. Throughout the paper the symbol C (possibly with a subscript) denotes a constant independent of significant quantities, whose precise value we do not need.

The operators $\mathcal{L}_n$, $n = 1, \ldots, d$, are known to be essentially self-adjoint on $L^2(\mathbb{R}, \gamma_n)$, where γ_n is the Gaussian measure $\gamma_n(x_n) = \pi^{-1/2} e^{-x_n^2}$. Moreover, their closures, which by a slight abuse of notation are denoted by the same symbols, have the spectral resolution

$$\mathcal{L}_n f(x_n) = \sum_{r\in\mathbb{N}} r\langle f, \tilde{H}_r\rangle_{L^2(\mathbb{R},\gamma_n)} \tilde{H}_r(x_n), \qquad f \in L^2(\mathbb{R}, \gamma_n),$$

defined on the usual domain

$$\mathrm{Dom}(\mathcal{L}_n) = \Big\{ f \in L^2(\mathbb{R}, \gamma_n) \colon \sum_{r\in\mathbb{N}} r^2 |\langle f, \tilde{H}_r\rangle_{L^2(\mathbb{R},\gamma_n)}|^2 < \infty \Big\}.$$

Here $\tilde{H}_r = \|H_r\|^{-1}_{L^2(\mathbb{R},\gamma_n)} H_r$, with H_r being the one-dimensional Hermite polynomial of degree r in the x_n variable (see [12]). To define the operators $\mathcal{L}_n$ on the space $L^2(\gamma)$ we use tensor products. Instead of $\mathcal{L}_n$ itself we consider the tensor product of $\mathcal{L}_n$ with $d-1$ identity operators on the spaces $L^2(\mathbb{R}, \gamma_{n'})$, $n' \neq n$, $n' = 1, \ldots, d$. To simplify notation we continue writing $\mathcal{L}_n$ for these tensor product operators. Observe that $\mathcal{L}_n$ thus defined admits the spectral resolution

$$\mathcal{L}_n f = \sum_{k\in\mathbb{N}^d} k_n \langle f, \tilde{\mathbf{H}}_k\rangle_{L^2(\gamma)} \tilde{\mathbf{H}}_k, \qquad f \in L^2(\gamma). \tag{2.1}$$

where $\tilde{\mathbf{H}}_k = \tilde{H}_{k_1} \otimes \cdots \otimes \tilde{H}_{k_d}$, while γ is the Gaussian measure on $\mathbb{R}^d$, i.e. $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_d$. The operator given by (2.1) is self-adjoint on the domain

$$\mathrm{Dom}(\mathcal{L}_n) = \Big\{ f \in L^2(\gamma) \colon \sum_{k\in\mathbb{N}^d} k_n^2 |\langle f, \tilde{\mathbf{H}}_k\rangle_{L^2(\gamma)}|^2 < \infty \Big\}.$$

It is also easy to see that $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_d)$ is a system of strongly commuting self-adjoint operators on $L^2(\gamma)$. For a bounded function $m : [0,\infty)^d \to \mathbb{C}$ define the *joint spectral multiplier operator* by

(2.2)
$$m(\mathcal{L})f = m(\mathcal{L}_1, \dots, \mathcal{L}_d)f = \sum_{k\in\mathbb{N}^d} m(k_1, \dots, k_d)\langle f, \tilde{\mathbf{H}}_k\rangle_{L^2(\gamma)}\tilde{\mathbf{H}}_k, \quad f \in L^2(\gamma).$$

In Section 3 we also need to consider joint spectral multipliers of systems in which some of the operators $\mathcal{L}_n$ are replaced by 0. These are defined similarly as in (2.2) with 0's in appropriate places as the arguments of m, for instance $m(\mathcal{L}_1, 0, \dots, 0)f = \sum_{k\in\mathbb{N}^d} m(k_1, 0, \dots, 0)\langle f, \tilde{\mathbf{H}}_k\rangle_{L^2(\gamma)}\tilde{\mathbf{H}}_k$, $f \in L^2(\gamma)$, etc.

The following is our analogue of the definition of an L^p uniform spectral multiplier from [2, p. 104].

Definition 2.3. Let $d' \in \mathbb{N}_+$ and $\phi = (\phi_1, \dots, \phi_{d'})\colon [0,\infty)^d \to [0,\infty)^{d'}$. For $t \in \mathbb{R}_+^{d'}$ set $\delta_t(x) = (t_1x_1, \dots, t_{d'}x_{d'})$, $x \in \mathbb{R}_+^{d'}$. Given $p \in [1,\infty) \setminus \{2\}$, we say that $m\colon [0,\infty)^{d'} \to \mathbb{C}$ is a *p-uniform joint spectral multiplier of the* (strongly commuting) *system* $\phi(\mathcal{L}) = (\phi_1(\mathcal{L}), \dots, \phi_{d'}(\mathcal{L}))$ if

$$A_p^{\phi(\mathcal{L})} := \sup_{t_1,\dots,t_{d'}>0} \|m \circ \delta_t \circ \phi(\mathcal{L})\|_{L^p(\gamma)} < \infty, \tag{2.4}$$

where the operator $m \circ \delta_t \circ \phi(\mathcal{L})$ is defined via joint functional calculus as

$$m \circ \delta_t \circ \phi(\mathcal{L})f = \sum_{k\in\mathbb{N}^d} m(t_1\phi_1(k), \dots, t_{d'}\phi_{d'}(k))\langle f, \tilde{\mathbf{H}}_k\rangle\tilde{\mathbf{H}}_k, \quad f \in L^2(\gamma).$$

In the case when $d' = d$ and ϕ is the identity function, condition (2.4) becomes

$$\sup_{t_1,\dots,t_d>0} \|m(t_1\mathcal{L}_1, \dots, t_d\mathcal{L}_d)\|_{L^p(\gamma)} < \infty.$$

In that case we say briefly that m is a *p-uniform joint spectral multiplier* and write A_p instead of $A_p^{\phi(\mathcal{L})}$.

3. Holomorphic extension theorems. The main goal of this paper can be stated as follows. We want to obtain some holomorphic extension properties of multiplier functions resulting in p-uniform multipliers of systems of operators built on $\mathcal{L}$. However, the first theorem we prove is more general. We give a relation between separate uniform spectral multipliers of some discrete, non-negative operators L_n, $n = 1, \dots, d$, and the uniform spectral multipliers of the system L.

Let $\{e_k\}_{k\in\mathbb{N}^d}$, $e_k = e^1_{k_1} \otimes \dots \otimes e^d_{k_d}$, be an orthonormal basis of some space $L^2(X,\nu)$, which is linearly dense in all the $L^p(X,\nu)$, $1 < p < \infty$, spaces. Here $X = X_1 \times \dots \times X_d$, $\nu = \nu_1 \otimes \dots \otimes \nu_d$, while $k = (k_1, \dots, k_d)$ is a multiindex. Assume that for each $n = 1, \dots, d$, $\{e^n_r\}_{r=0,1,\dots}$ is an eigenfunction decomposition of some self-adjoint, non-negative, non-zero, operator L_n in $L^2(X_n, \nu_n)$ with eigenvalues $0 \leq \lambda_0^n < \lambda_1^n < \cdots$, i.e.

$$L_n f = \sum_{r=0}^{\infty} \lambda_r^n \langle f, e_r^n \rangle_{L^2(X_n,\nu_n)} e_r^n$$

on the domain

$$\mathrm{Dom}(L_n) = \Big\{ f \in L^2(X_n, \nu_n) \,:\, \sum_{r=0}^{\infty} (\lambda_r^n)^2 |\langle f, e_r^n \rangle_{L^2(X_n,\nu_n)}|^2 < \infty \Big\}.$$

For a function $m_n : [0,\infty) \to \mathbb{C}$ the spectral multipliers of the operators L_n are

$$m_n(L_n) = \sum_{r \in \mathbb{N}^d} m_n(\lambda_r^n) \langle f, e_r^n \rangle_{L^2(X_n,\nu_n)} e_r^n, \qquad f \in L^2(X_n, \nu_n),\ n = 1, \dots, d.$$

For a function $m : [0,\infty)^d \to \mathbb{C}$ define the *joint spectral multipliers* of the system L by

$$m(L) = m(L_1, \dots, L_d) = \sum_{k \in \mathbb{N}^d} m(\lambda_{k_1}^1, \dots, \lambda_{k_d}^d) \langle f, e_k \rangle_{L^2(X,\nu)} e_k, \qquad f \in L^2.$$

THEOREM 3.1. *Fix $p > 1$. Assume that for each $n = 1, \dots, d$, there is an open set $E_n^p \subset \mathbb{C}$ such that every function $m_n : [0,\infty) \to \mathbb{C}$ which is bounded on $[0,\infty)$, continuous on $\mathbb{R}_+$, and satisfies $\sup_{t_n > 0} \|m_n(t_n L_n)\|_{L^p(X_n,\nu_n)} < \infty$, extends to a bounded holomorphic function in E_n^p, and*

$$\|m_n\|_{H^\infty(E_n^p)} \leq \sup_{t_n > 0} \|m_n(t_n L_n)\|_{L^p(X_n,\nu_n)}. \tag{3.2}$$

Then every function $m : [0,\infty)^d \to \mathbb{C}$ which is bounded on $[0,\infty)^d$, continuous on $\mathbb{R}_+^d$, and such that

$$A_p^L := \sup_{t_1,\dots,t_d > 0} \|m(t_1 L_1, \dots, t_d L_d)\|_{L^p(X,\nu)} < \infty, \tag{3.3}$$

extends to a bounded holomorphic function of several variables in $E^p := E_1^p \times \cdots \times E_d^p$; moreover

$$\|m\|_{H^\infty(E^p)} \leq \sup_{t_1,\dots,t_d > 0} \|m(t_1 L_1, \dots, t_d L_d)\|_{L^p(X,\nu)}. \tag{3.4}$$

Proof. Recall, that by Hartogs' theorem (see [4]) a function $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}^d$, is holomorphic as a function of several variables if and only if it is holomorphic in each variable z_n, $n = 1, \dots, d$, while the other variables are held constant. The key ingredient of the proof here is the bound (3.2). The reasoning we present has been indicated to us by Prof. Fulvio Ricci. We use induction on d. When $d = 1$, there is nothing to do. Assume that we have proved the theorem for some $d - 1$ and let $m : [0,\infty)^d \to \mathbb{C}$ be a bounded, continuous function on $[0,\infty)^d$ such that (3.3) holds.

Fix $t^{(1)} = (t_2, \dots, t_d) \in \mathbb{R}_+^{d-1}$ and

$$f, g \in \operatorname{span}\{e^2_{k_2} \otimes \cdots \otimes e^d_{k_d} \colon k_n \in \mathbb{N},\, n = 2, \dots, d\}.$$

Note that by our assumptions the latter set is dense in every $L^q(X^{(1)}, \nu^{(1)})$, $1 < q < \infty$, where $X^{(1)} = X_2 \times \cdots \times X_d$, $\nu^{(1)} = \nu_2 \otimes \cdots \otimes \nu_d$. Indeed, it can be easily verified that $\{e^2_{k_2} \otimes \cdots \otimes e^d_{k_d}\}$ satisfies the condition from [3, Corollary 2.5]. Set

$$\tilde{m}_{t^{(1)},f,g}(t_1) = \langle m(t_1, t_2L_2, \dots, t_dL_d)f, g\rangle_{L^2(X^{(1)},\nu^{(1)})}, \qquad t_1 \in [0,\infty). \tag{3.5}$$

Then it is not hard to see that for $f_1, g_1 \in L^p(\nu_1) \cap L^{p'}(\nu_1)$ we have

$$\langle \tilde{m}_{t^{(1)},f,g}(t_1L_1)f_1, g_1\rangle_{L^2(X_1,\nu_1)} = \langle m(t_1L_1, \dots, t_dL_d)(f_1 \otimes f), g_1 \otimes g\rangle_{L^2(X,\nu)}.$$

Consequently, from the assumption that m satisfies (3.3), we obtain

$$\|\tilde{m}_{t^{(1)},f,g}(t_1L_1)\|_{L^p(X_1,\nu_1)} \le A_p^L \|f\|_{L^p(X^{(1)},\nu^{(1)})} \|g\|_{L^{p'}(X^{(1)},\nu^{(1)})}.$$

Clearly, $\tilde{m}_{t^{(1)},f,g}(\cdot)$ is a bounded continuous function on $[0,\infty)$, hence from (3.2) it follows that $\tilde{m}_{t^{(1)},f,g}(t_1)$ extends to $H^\infty(E_1^p)$ and (denoting this extension by the same symbol)

$$\|\tilde{m}_{t^{(1)},f,g}(\cdot)\|_{H^\infty(E_1^p)} \le A_p^L \|f\|_{L^p(X^{(1)},\nu^{(1)}))} \|g\|_{L^{p'}(X^{(1)},\nu^{(1)})}. \tag{3.6}$$

Since in particular $\tilde{m}_{(t_2\lambda_2(\lambda_1^2)^{-1},\dots,t_d\lambda_d(\lambda_1^d)^{-1}),e_1^{(1)},e_1^{(1)}}(t_1) = m(t_1, t_2\lambda_2, \dots, t_d\lambda_d)$, where $e_1^{(1)} = e_1^2 \otimes \cdots \otimes e_1^d$, we have the bounded holomorphic extension $m(z_1, t_2\lambda_2, \dots, t_d\lambda_d)$. Moreover, for $f, g \in \operatorname{span}\{e^2_{k_2} \otimes \cdots \otimes e^d_{k_d} \colon k_n \in \mathbb{N}, n = 2, \dots, d\}$ we see that

$$z_1 \mapsto \tilde{m}_{t^{(1)},f,g}(z_1) \quad \text{and} \quad z_1 \mapsto \langle m(z_1, t_2L_2, \dots, t_dL_d)f, g\rangle_{L^2(X^{(1)},\nu^{(1)})}$$

are two holomorphic functions which agree on the positive real half-line. By uniqueness of analytic continuation,

$$\tilde{m}_{t^{(1)},f,g}(z_1) = \langle m(z_1, t_2L_2, \dots, t_dL_d)f, g\rangle_{L^2(X^{(1)},\nu^{(1)})},$$

i.e. (3.5) still holds for the extension of $\tilde{m}_{t^{(1)},f,g}$. Hence, from (3.6) we infer that

$$\sup_{t_2,\dots,t_d > 0} \|m(z_1, t_2L_2, \dots, t_dL_d)\|_{L^p(X^{(1)},\nu^{(1)})} \le A_p^L, \tag{3.7}$$

uniformly in $z_1 \in E_1^p$.

Now, from the inductive hypothesis applied separately for each $z_1 \in E_1^p$, we obtain the holomorphic function $E_2^p \times \cdots \times E_d^p \ni (z_2, \dots, z_d) \mapsto m(z_1, z_2, \dots, z_d)$, which satisfies $\|m(z_1, \cdot)\|_{H^\infty(E_2^p \times \cdots \times E_d^p)} \le A_p^L$, uniformly in $z_1 \in E_1^p$. In summary, for each $z_1 \in E_1^p$, $m(z_1, z_2, \dots, z_d)$ is a holomorphic function in $E_2^p \times \cdots \times E_d^p$, satisfying the desired bound (3.4). We also know that for each $t^{(1)} = (t_2, \dots, t_d) \in \mathbb{R}_+^{d-1}$, $m(z_1, t^{(1)})$ is holomorphic. We shall now verify that for each $z^{(1)} = (z_2, \ldots, z_d) \in E_2^p \times \cdots \times E_d^p$, $m(z_1, z^{(1)})$ is a holomorphic function in E_1^p. For an arc Γ in E_1^p define

$$h(z_2, \ldots, z_d) = \int_\Gamma m(z_1, z_2, \ldots, z_d)\, dz_1.$$

Then it is not hard to check that h is holomorphic in $E_2^p \times \cdots \times E_d^p$. Moreover, $h(t) = 0$ for $t^{(1)} \in \mathbb{R}_+^{d-1}$. Hence, by analytic continuation, $h = 0$. By Morera's theorem (see e.g. [4]), for each $z^{(1)} \in E_2^p \times \cdots \times E_d^p$, $m(\cdot, z^{(1)})$ is a holomorphic function in E_1^p. Thus, by Hartogs' theorem, we are done. ∎

Our next theorem is a holomorphic extension theorem for joint spectral multipliers of the system $\mathcal{L}$, which is a multi-dimensional generalization of [2, Theorem 3.5].

THEOREM 3.8. *Assume that* $m : [0, \infty)^d \to \mathbb{C}$ *is bounded on* $[0, \infty)^d$ *and continuous on* $\mathbb{R}_+^d$. *The following statements hold:*

(i) *if* $p \in (1, \infty) \setminus \{2\}$ *and* m *is a* p*-uniform joint spectral multiplier, then* m *extends to a bounded holomorphic function in the polysector* $(\mathbf{S}_{\phi_p^*})^d$ *and*

$$\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^d)} \le A_p;$$

(ii) m *is a* 1*-uniform joint spectral multiplier if and only if* m *extends to a bounded holomorphic function in* $(\mathbf{S}_{\pi/2})^d$ *and* $m(-2i\,\cdot\,, \ldots, -2i\,\cdot\,)$ *is the Fourier transform of a finite measure* μ *on* $\mathbb{R}^d$ *supported in* $[0, \infty)^d$; *moreover,* $\|\mu\|_{M(\mathbb{R}^d)} = A_1$.

We postpone for a moment the proof of Theorem 3.8 in order to state and prove some corollaries.

COROLLARY 3.9. *Assume* $d' \in \mathbb{N}$, $p \in [1, \infty) \setminus \{2\}$, *and* $m : [0, \infty)^{d'} \to \mathbb{C}$ *is bounded on* $[0, \infty)^{d'}$ *and continuous on* $\mathbb{R}_+^{d'}$. *The following statements hold:*

(i) *if* $1 \le n_1 < \cdots < n_{d'}$, $d = \sum_{r=1}^{d'} n_r$, *and* m *is a* p*-uniform joint spectral multiplier of the system*

$$\mathcal{L}_{n_1, \ldots, n_{d'}} = (\mathcal{L}_1 + \cdots + \mathcal{L}_{n_1}, \ldots, \mathcal{L}_{n_1 + \cdots + n_{d'-1} + 1} + \cdots + \mathcal{L}_d),$$

then m *extends to a bounded holomorphic function on* $(\mathbf{S}_{\phi_p^*})^{d'}$; *moreover,*

$$\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^{d'})} \le A_p^{\mathcal{L}_{n_1, \ldots, n_d}};$$

(ii) *if* $d' = 2$, $d = 3$, *and* m *is a* p*-uniform joint spectral multiplier of the system* $(\mathcal{L}_1 + \mathcal{L}_2, \mathcal{L}_2 + \mathcal{L}_3)$, *then* m *extends to a bounded holomorphic function in the* 2*-sector* $(\mathbf{S}_{\phi_p^*})^2$; *moreover,*

$$\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^2)} \le A_p^{(\mathcal{L}_1 + \mathcal{L}_2, \mathcal{L}_2 + \mathcal{L}_3)}.$$

Proof. First we show (ii). Let $f(x_1,x_2,x_3)=h(x_1,x_3)\tilde{H}_0(x_2)=h\otimes\tilde{H}_0$, where $\tilde{H}_0=1$ is the Hermite polynomial of degree 0 in x_2, while h is some function from a dense set in $L^p(\gamma_1\otimes\gamma_3)$, for instance, a linear combination of tensor products of Hermite polynomials in x_1,x_3. Then a short reasoning shows that

$$m(t_1(\mathcal{L}_1+\mathcal{L}_2),t_2(\mathcal{L}_2+\mathcal{L}_3))(f)(x_1,x_2,x_3)$$
$$=m(t_1\mathcal{L}_1,t_2\mathcal{L}_3)(h)(x_1,x_3)\tilde{H}_0(x_2)=[m(t_1\mathcal{L}_1,t_2\mathcal{L}_3)(h)\otimes\tilde{H}_0](x_1,x_2,x_3).$$

Hence,

$$\|m(t_1\mathcal{L}_1,t_2\mathcal{L}_3)(h)\|_{L^p(\gamma_1\otimes\gamma_3)}\|\tilde{H}_0\|_{L^p(\gamma_2)}$$
$$=\|m(t_1(\mathcal{L}_1+\mathcal{L}_2),t_2(\mathcal{L}_2+\mathcal{L}_3))(h\otimes\tilde{H}_0)\|_{L^p(\gamma)}\le A_p\|h\|_{L^p(\gamma_1\otimes\gamma_3)}\|\tilde{H}_0\|_{L^p(\gamma_2)}.$$

Consequently, m is a p-uniform joint spectral multiplier of the operators $\mathcal{L}_1,\mathcal{L}_3$, with constant A_p. The desired conclusion follows from Theorem 3.8(i).

The proof of (i) is almost the same. More precisely, we apply the preceding reasoning to functions of the form $f=h\otimes H$, where h is a function of the variables $(x_{n_1},x_{n_2},x_{n_{d'}})$, while H is a tensor product of Hermite polynomials of degree zero in the other variables (i.e. H is a tensor product of constant functions 1). Actually the proof of (ii) also shows that as far as L^p, $p\in(1,\infty)\setminus\{2\}$, multipliers are concerned, even in [2, Theorem 3.5] without loss of generality it is enough to consider a single one-dimensional Ornstein–Uhlenbeck operator. ∎

Proof of Theorem 3.8(i). This part of the theorem is an immediate consequence of [2, Theorem 3.5(i)] (in the case $d=1$) and Theorem 3.1 (with $X_n=\mathbb{R}$, $\nu_n=\gamma_n$, $L_n=\mathcal{L}_n$, $E_n^p=\mathbf{S}_{\phi_p^*}$). ∎

To prove Theorem 3.8(ii) we need several d-dimensional analogues of lemmata from [2]. Since most of their proofs are not hard to carry over to our setting, we omit them. Strictly speaking, to prove (ii) it is enough to take $p=1$ in the auxiliary lemmata below. However, since we also use these lemmata to prove Theorem 3.21, we allow general $p\in[1,\infty)\setminus\{2\}$ and (if necessary) write down some slightly more general properties. For $a=(a_1,\ldots,a_d)\in\mathbb{R}^d$ let

$$e_a(x)=e^{a_1x_1+\cdots+a_2x_2}$$

and let γ_∞ be the Borel measure given by $d\gamma_\infty=e_{-2,\ldots,-2}(x)\,dx$. Then we consider the Dirichlet forms

$$Q_n^\infty(f)=\int\limits_{\mathbb{R}^d}\left|\frac{\partial f}{\partial x_n}\right|^2d\gamma_\infty,\qquad n=1,\ldots,d.$$

Similarly to [2, p. 106], from general theory (see for instance [1, Theorem 1.2.5]) it follows that Q_n^∞ are forms of self-adjoint operators $\mathcal{L}_n^\infty$, on $L^2(\gamma_\infty)$, which on $C_c^\infty(\mathbb{R}^d)$ are given by

$$\mathcal{L}_n^\infty f = -\frac{\partial^2 f}{\partial x_n^2} + 2\frac{\partial f}{\partial x_n}, \qquad n = 1, \ldots, d.$$

The operators $\mathcal{L}_n^\infty$, $n = 1, \ldots, d$, commute on $C_c^\infty(\mathbb{R}^d)$ which is their common core. As a matter of fact, they commute strongly (see Lemma 3.11 for the proof), hence we can consider joint spectral multipliers of the system $\mathcal{L}^\infty = (\mathcal{L}_1^\infty, \ldots, \mathcal{L}_d^\infty)$. Let $\Phi(z_1, \ldots, z_d) = (z_1^2 + 1, \ldots, z_d^2 + 1)$ and for $\psi > 0$ let $(P_\psi)^d = P_\psi \times \cdots \times P_\psi$, with

$$P_\psi = \Big\{ z = x + iy \colon x > \frac{y^2}{4\psi^2} + 1 - \psi^2 \Big\} \tag{3.10}$$

(i.e., for fixed $\psi > 0$, $(P_\psi)^d$ is the image of $(\Sigma_\psi)^d$ under Φ). First we prove the following key lemma, which is an analogue of [2, Theorem 2.1].

Lemma 3.11. *Let $\mathcal{L}^\infty = (\mathcal{L}_1^\infty, \ldots, \mathcal{L}_d^\infty)$ and $m \colon \mathbb{R}_+^d \to \mathbb{C}$ be a bounded measurable function. Then, for $p \in [1, \infty) \setminus \{2\}$, the following are equivalent:*

(i) *m is an $L^p(\gamma_\infty)$ joint spectral multiplier of $\mathcal{L}^\infty$;*
(ii) *m extends to a bounded holomorphic function on the polyparabolic region $(P_{|2/p-1|})^d$ and the functions*

$$m \circ \Phi(\cdot + \varepsilon_1 |2/p - 1| i, \ldots, \cdot + \varepsilon_d |2/p - 1| i)$$

for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$ are L^p multipliers of the d-dimensional Euclidean Fourier transform on $\mathbb{R}^d$. Moreover

$$B_p := \|m(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} = \|m \circ \Phi(\cdot + i|2/p-1|, \ldots, \cdot + i|2/p-1|)\|_{\mathcal{M}_p(\mathbb{R}^d)}.$$

Proof. The reasoning is based on that of [2, Theorem 2.1], so we only point out the differences. The isometry $\mathcal{U}_p : L^p \to L^p(\gamma_\infty)$ is defined by $\mathcal{U}_p f = e_{2/p, \ldots, 2/p} f$. Easy calculations show that

$$\mathcal{U}_2^{-1} \mathcal{L}_n^\infty \mathcal{U}_2 = -\Delta_n + 1, \qquad \Delta_n = \frac{\partial^2}{\partial x_n^2}.$$

Consequently, by the spectral theorem,

$$\mathcal{U}_2^{-1} (\lambda_n - \mathcal{L}_n^\infty)^{-1} \mathcal{U}_2 = (\lambda_n - (-\Delta_n + 1))^{-1}, \qquad \lambda_n \in \mathbb{C} \setminus \mathbb{R},\ n = 1, \ldots, d.$$

Since $-\Delta_n + 1$, $n = 1, \ldots, d$, commute strongly, it follows that the same is true for $\mathcal{L}_n^\infty$, $n = 1, \ldots, d$. Hence, the expression $m(\mathcal{L}^\infty)$ makes sense and by the (multivariate) spectral theorem we have

$$\mathcal{U}_2^{-1} m(\mathcal{L}^\infty) \mathcal{U}_2 = m(-\Delta_1 + 1, \ldots, \Delta_d + 1).$$

Clearly, the operators $-\Delta_n + 1$ commute with one-dimensional translations (and among themselves), so that $m(-\Delta_1 + 1, \ldots, -\Delta_d + 1)$ commutes with d-dimensional translations. Consequently, there is a tempered distribution k

such that

$$m(-\Delta_1+1,\dots,-\Delta_d+1)f = k * f, \qquad f \in S(\mathbb{R}^d).$$

The proof that $m(\mathcal{L}^\infty)$ is $L^p(\gamma_\infty)$ bounded if and only if $e_{1-2/p,\dots,1-2/p}k$ convolves L^p into itself and that

$$\|e_{1-2/p,\dots,1-2/p}k\|_{\mathrm{conv},p} = \|m(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} \tag{3.12}$$

easily carries over from [2, p. 108] to our situation. Since $\hat{k}(\xi) = m(\xi_1^2+1,\dots,\xi_d^2+1)$, arguing as in [2, p. 108] we also show that $e_{1-2/p',\dots,1-1/p'}k$ convolves L^p into itself and

$$\|e_{1-2/p',\dots,1-2/p'}k\|_{\mathrm{conv},p} = \|e_{1-2/p,\dots,1-2/p}k\|_{\mathrm{conv},p}. \tag{3.13}$$

Now we iteratively apply Stein's complex interpolation theorem to the holomorphic family of operators

$$T_z f = e_z k * f, \qquad |\mathrm{Re}(z_n)| \le |2/p-1|, \quad n = 1,\dots,d.$$

From (3.12), (3.13) and the fact that $1-2/p' = 2/p-1$, we see that for each $t^{(1)} = (t_2,\dots,t_d) \in \mathbb{R}^{d-1}$ and $\varepsilon^{(1)} = (\varepsilon_2,\dots,\varepsilon_d) \in \{-1,1\}^{d-1}$ the holomorphic family of operators $\{R^1_{\varepsilon^{(1)},t^{(1)}}\}_{z_1}$ given by

$$R^1_{\varepsilon^{(1)},t^{(1)}}(z_1) = T_{z_1,\varepsilon_2|2/p-1|+it_2,\dots,\varepsilon_d|2/p-1|+it_d}$$

satisfies

$$\|R^1_{\varepsilon^{(1)},t^{(1)}}(\varepsilon_1|2/p-1|+it_1)\|_{L^p} \le B_p, \qquad \varepsilon_1 = \pm 1,\ t_1 \in \mathbb{R}.$$

Hence, from the first application of Stein's complex interpolation theorem, we get

$$\|R^1_{\varepsilon^{(1)},t^{(1)}}(a_1)\|_{L^p} \le B_p$$

for $a_1 \in [-|2/p-1|, |2/p-1|]$, uniformly in $\varepsilon^{(1)}$ and $t^{(1)}$. The second application, separately to each of the operators $R^2_{a_1,\varepsilon^{(2)},t^{(2)}}(z_2) = bT_{a_1,z_2,\varepsilon_3+it_3,\dots,\varepsilon_d+it_d}$,

$$a_1 \in [-|2/p-1|, |2/p-1|], \quad \varepsilon^{(2)} = (\varepsilon_3,\dots,\varepsilon_d) \in \{-1,1\}^{d-2},$$
$$t^{(2)} = (t_3,\dots,t_d) \in \mathbb{R}^{d-2},$$

gives the bound

$$\|R^2_{a_1,\varepsilon^{(2)},t^{(2)}}(a_2)\|_{L^p} \le B_p,$$

valid for $a_2 \in [-|2/p-1|, |2/p-1|]$ and uniform in a_1, $\varepsilon^{(2)}$ and $t^{(2)}$. Proceeding as indicated, in the last dth step we apply Stein's complex interpolation theorem to the holomorphic family of operators $R^d_{a_1,\dots,a_{d-1}}(z_d) = T_{a_1,\dots,a_{d-1},z_d}$. Thus, we deduce that T_a is an L^p, hence also L^2-multiplier for $a \in [-|2/p-1|, |2/p-1|]^d$. Consequently, $\hat{k} = m \circ \Phi$ extends to a holomorphic function of d variables in $(\Sigma_{|2/p-1|})^d$ satisfying the bound $|\hat{k}(z)| \le B_p$,

$z \in (\Sigma_{|2/p-1|})^d$, and it is not hard to conclude that m itself is continuous on $[1,\infty)^d$. Since $\hat{k} = m \circ \Phi$, we see that $\hat{k}(\cdot + ia_1, \ldots, \cdot + ia_d)$ is an L^p multiplier for all $a \in [-|2/p-1|, |2/p-1|]^d$, and, restating (3.12), (3.13),

$$\|m(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} = \|m \circ \Phi(\cdot + i|2/p-1|, \ldots, \cdot + i|2/p-1|)\|_{\mathcal{M}_p}.$$

It remains to verify that m itself has a holomorphic extension to $(P_{|2/p-1|})^d$. From the local invertibility of the mapping $(z_1, \ldots, z_d) \mapsto (z_1^2+1, \ldots, z_d^2+1)$ away from the (complex) coordinate axes we conclude that m has a bounded holomorphic extension to $(P_{|2/p-1|} \setminus \{1\})^d$. Now take $z^0 = (z_1^0, \ldots, z_d^0) \in (P_{|2/p-1|})^d$ such that for some $n = 1, \ldots, d$, $z_n^0 = 1$, and let γ_n be a contour in $P_{|2/p-1| \setminus \{1\}}$ enclosing the point z_n^0, $n = 1, \ldots, d$. Then, because m is bounded on $(P_{|2/p-1|} \setminus \{1\})^d$ and continuous on $[1,\infty)^d$, the function

$$(z_1, \ldots, z_d) \mapsto \frac{1}{(2\pi i)^d} \int_{\gamma_1} \cdots \int_{\gamma_d} \frac{m(w_1, \ldots, w_d)}{(z_1 - w_1) \cdots (z_d - w_d)}\, dw_1 \ldots dw_d$$

is holomorphic at z^0 and extends m. ∎

The next step is to develop an analogue of [2, Theorem 2.2]. For $m : [0,\infty)^{d'} \to \mathbb{C}$, denote

$$A_p^{\phi,\infty} = \sup_{t_1,\ldots,t_{d'}>0} \|m \circ \delta_t \circ \phi(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)}.$$

If $d' = d$ and ϕ is the identity, we write A_p^∞ for short. Using Lemma 3.11 we obtain the following.

Lemma 3.14. *Let $m : [0,\infty)^d \to \mathbb{C}$ be a bounded measurable function. The following statements hold:*

(i) *if $p \in (1,\infty) \setminus \{2\}$ and $A_p^\infty < \infty$, then m extends to a bounded holomorphic function in the polysector $(\mathbf{S}_{\phi_p^*})^d$, and $\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^d)} \le A_p^\infty$;*

(ii) *$A_1^\infty < \infty$ if and only if m extends to a bounded holomorphic function in $(\mathbf{S}_{\pi/2})^d$ and $m(-2i\cdot, \ldots, -2i\cdot)$ is the Fourier transform of a finite measure μ on $\mathbb{R}^d$ supported in $[0,\infty)^d$; furthermore, $\|\mu\|_{M(\mathbb{R}^d)} = A_1^\infty$;*

(iii) *if $p \in [1,\infty) \setminus \{2\}$, ϕ is the function from Definition 2.2, and $A_p^{\phi,\infty} < \infty$, then for each $t \in \mathbb{R}_+^{d'}$, $m \circ \delta_t \circ \phi$ extends to a bounded holomorphic function in the polyparabolic region $(P_{|2/p-1|})^d$, and*

$$\|m \circ \delta_t \circ \phi\|_{H^\infty((P_{|2/p-1|})^d)} \le A_p^{\phi,\infty}.$$

Proof (sketch). The proof is similar to the proof of [2, Theorem 2.2]. However, (iii) will be utilized in the proof of Theorem 3.21, hence we give some details. First, we establish the inequality

$$\|m(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} \ge \|m\|_{H^\infty((P_{|2/p-1|})^d)}. \tag{3.15}$$

Arguing as in [2, p. 109], we recall that an L^p multiplier of the Euclidean Fourier transform is also an L^2 multiplier and that the injection from the Banach space of L^p multipliers to the Banach space of L^2 multipliers is a non-expansive map. Therefore, from Lemma 3.11 it follows that

$$\begin{aligned}\|m(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} &= \|m\circ\Phi(\cdot+i|2/p-1|,\ldots,\cdot+i|2/p-1|)\|_{\mathcal{M}_p(\mathbb{R}^d)}\\ &\geq \|m\circ\Phi(\cdot+i|2/p-1|,\ldots,\cdot+i|2/p-1|)\|_{\mathcal{M}_2(\mathbb{R}^d)}\\ &= \|m\circ\Phi(\cdot+i|2/p-1|,\ldots,\cdot+i|2/p-1|)\|_\infty = \|m\|_{H^\infty((P_{|2/p-1|})^d)}.\end{aligned}$$

On the one hand, taking in (3.15) $m(t_1\,\cdot,\ldots,t_d\,\cdot)$ in place of m, we arrive at

$$(3.16)\quad \|m(t_1\mathcal{L}_1^\infty,\ldots,t_d\mathcal{L}_d^\infty)\|_{L^p(\gamma_\infty)} \geq \|m\|_{H^\infty((t_1^{-1}P_{|2/p-1|})\times\cdots\times(t_d^{-1}P_{|2/p-1|}))}.$$

Under the assumption $A_p^\infty<\infty$, the conclusion of (i) follows as in [2]. On the other hand, taking in (3.15) $m\circ\delta_t\circ\phi$ in place of m we also obtain

$$(3.17)\qquad \|m(t_1\phi_1(\mathcal{L}^\infty),\ldots,t_{d'}\phi_{d'}(\mathcal{L}^\infty))\|_{L^p(\gamma_\infty)} \geq \|m\circ\delta_t\circ\phi\|_{H^\infty((P_{|2/p-1|})^d)},$$

thus proving (iii).

The proof of (ii) is analogous to that of [2, Theorem 2.2(ii)]. In order to see that μ is supported in $[0,\infty)^d$, we use a multi-dimensional extension of an appropriate Paley–Wiener theorem for the distributional Fourier transform (see for instance [5, Section 3]). ∎

For $b\in\mathbb{R}^+$ define the measures

$$d\gamma_b(x)=\exp\Big(-\frac{|x|^2}{b^2}-2\sum_{n=1}^d x_n\Big)\,dx.$$

Let Q_n^b be the Dirichlet form

$$Q_n^b(f)=\int_{\mathbb{R}^d}\left|\frac{\partial f}{\partial x_n}\right|^2 d\gamma_b,\qquad n=1,\ldots,d.$$

From general theory (see [1, Theorem 1.2.5]), it follows that Q_n^b are forms of self-adjoint operators $\mathcal{L}_n^b$, on $L^2(\gamma_b)$, which on $C_c^\infty(\mathbb{R}^d)$ are given by

$$\mathcal{L}_n^b f=-\frac{\partial^2 f}{\partial x_n^2}+2\frac{\partial f}{\partial x_n}+2\frac{x_n}{b^2}\frac{\partial f}{\partial x_n},\qquad n=1,\ldots,d.$$

As in the case of $\mathcal{L}_n^\infty$, the operators $\mathcal{L}_n^b$, $n=1,\ldots,d$, commute strongly (see Lemma 3.18 for the proof), hence we can consider their joint spectral multipliers. In order to pass to systems $\mathcal{L}$ and $\phi(\mathcal{L})$ we need the following analogue of [2, Lemma 3.3].

LEMMA 3.18. *Assume $p \in [1,\infty) \setminus \{2\}$. The following statements hold:*

(i) *if $m : [0,\infty)^d \to \mathbb{C}$ is bounded and measurable, then for every $b \in \mathbb{R}_+$ and $(t_1,\dots,t_d) \in \mathbb{R}^d_+$,*

$$\|m(t_1\mathcal{L}^b_1,\dots,t_d\mathcal{L}^b_d)\|_{L^p(\gamma_b)} = \|m((t_1/2b^2)\mathcal{L}_1,\dots,(t_d/2b^2)\mathcal{L}_d)\|_{L^p(\gamma)};$$

(ii) *if $m : [0,\infty)^d \to \mathbb{C}$ is bounded on $[0,\infty)^d$ and continuous on $\mathbb{R}^d_+$, then $A^\infty_p \le A_p$;*

(iii) *if $m : [0,\infty)^{d'} \to \mathbb{C}$ is continuous on $\phi(\mathbb{R}^d_+)$, where ϕ is the function from Definition* 2.2 *and such that each ϕ_n is continuous on $\mathbb{R}^d_+$ and homogeneous of some degree l_n, $n = 1,\dots,d'$, then $A^{\phi,\infty}_p \le A^{\phi(\mathcal{L})}_p$.*

Proof (sketch). First observe that the operators $\mathcal{L}^b_n$, $n = 1,\dots,d$, commute strongly. The proof is similar to the proof of strong commutativity of $\mathcal{L}^\infty_n$. We use the multi-dimensional version of the isometry $\mathscr{V}_{b,p}$ from [2, p. 113] and the fact that the operators $\mathcal{L}_n$, $n = 1,\dots,d$, commute strongly. Then the proof of Lemma 3.18 proceeds as the proof of [2, Lemma 3.3]. We need to generalize [9, Theorems VIII.20(b), VIII.25(a)] to the d-dimensional setting. It is here that we need the continuity of m and ϕ (see [2, p. 117]). As a matter of fact these assumptions can be weakened slightly. Namely, it is enough to take ϕ continuous only on the joint spectrum of $\mathcal{L}^\infty$ (which is contained in $[1,\infty)^d$). While proving items (ii) and (iii), we arrive at (cf. [2, p. 115])

$$|\langle m(\mathcal{L}^\infty_1,\dots,\mathcal{L}^\infty_d)f,g\rangle_{L^2(\gamma_\infty)}| \le \liminf_{b\to\infty} \|m((1/2b^2)\mathcal{L}_1,\dots,(1/2b^2)\mathcal{L}_d)\|_{L^p(\gamma)} \tag{3.19}$$

for $f,g \in C^\infty_c(\mathbb{R}^d)$. Replacing in (3.19) $m(\cdot)$ by $m(t_1\cdot,\dots,t_d\cdot)$, taking the supremum over $t \in \mathbb{R}^d_+$ and $b > 0$ of the right hand side and using a density argument we obtain (ii). Moreover, replacing in (3.19) m by $m \circ \delta_t \circ \phi$ (see the proof of Lemma 3.14), we also have

$$|\langle (m\circ\delta_t\circ\phi)(\mathcal{L}^\infty)f,g\rangle_{L^2(\gamma_\infty)}| \le \liminf_{b\to\infty} \|m(t_1\phi_1(1/2b^2\mathcal{L}),\dots,t_d\phi_{d'}(1/2b^2\mathcal{L}))\|_{L^p(\gamma)}.$$

Now, if each of the functions ϕ_n, $n = 1,\dots,d'$, is homogeneous of some degree l_n, then a density argument gives

$$\|(m\circ\delta_t\circ\phi)(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} \le \liminf_{b\to\infty} \|m(t_1(2b^2)^{-l_1}\phi_1(\mathcal{L}),\dots,t_{d'}(2b^2)^{-l_{d'}}\phi_{d'}(\mathcal{L}))\|_{L^p(\gamma)} \le A^{\phi(\mathcal{L})}_p, \tag{3.20}$$

which implies (iii). ∎

Proof of Theorem 3.8(ii). We use the techniques from [2]. The proof is much more laborious than that of (i). Below we only collect the fruits of Lemmata 3.18 and 3.14.

Note that we can now also give an alternative proof of Theorem 3.8(i), following the just developed analogues of the techniques from [2]. Briefly, it is not hard to see that from Lemmata 3.18 and 3.14, for $p \in [1,\infty) \setminus \{2\}$, we have

$$\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^d)} \le A_p^\infty \le A_p,$$

yielding (i) of Theorem 3.8.

To prove Theorem 3.8(ii) assume first that $A_1 < \infty$. Then, by Lemma 3.18(ii), $A_1^\infty < \infty$. Consequently, Lemma 3.14(ii) shows that m extends to a holomorphic function in $(\mathbf{S}_{\pi/2})^d$, $m(-2i\cdot,\ldots,-2i\cdot)$ is the Fourier transform of a finite measure μ on $\mathbb{R}^d$, supported in $[0,\infty)^d$, and $\|\mu\|_{M(\mathbb{R}^d)} = A_1^\infty \le A_1$. Thus one implication of Theorem 3.8(ii) is proved.

To prove the converse, assume that m extends to a holomorphic function in $(\mathbf{S}_{\pi/2})^d$, and $m(-2i\cdot,\ldots,-2i\cdot)$ is the Fourier transform of a finite measure μ on $\mathbb{R}^d$, supported in $[0,\infty)^d$. Then by spectral theory,

$$m(t_1\mathcal{L}_1,\ldots,t_d\mathcal{L}_d) = \int_{[0,\infty]^d} e^{-(s_1t_1\mathcal{L}_1+\cdots+s_dt_d\mathcal{L}_d)}\,d\mu(s),$$

where the integral is a convergent Bochner integral in $L^1(\gamma)$ (cf. [2, p. 110]). Now, using the fact that $e^{-(t_1\mathcal{L}_1+\cdots+t_d\mathcal{L}_d)}$ is a contraction on $L^1(\gamma)$, and Minkowski's integral inequality, we finish the proof. ∎

It seems reasonable to ask about the extension properties of the multiplier function if we build the multiplier operators on some functions of the system $\mathcal{L}$, instead of building on $\mathcal{L}$ itself. The next theorem answers this question to some extent and also includes Theorem 3.8(i) as a special case.

Theorem 3.21. *Assume that $m : [0,\infty)^{d'} \to \mathbb{C}$ is bounded on $[0,\infty)^{d'}$ and continuous on $\mathbb{R}_+^{d'}$. Assume $p \in [1,\infty) \setminus \{2\}$. Let $\phi = (\phi_1,\ldots,\phi_{d'}) : [0,\infty)^d \to [0,\infty)^{d'}$ be a vector valued function such that each ϕ_n is continuous on $\mathbb{R}_+^d$ and homogeneous of some degree l_n, $n = 1,\ldots,d'$. If m is a p-uniform joint spectral multiplier of the system $\phi(\mathcal{L}) = (\phi_1(\mathcal{L}),\ldots,\phi_{d'}(\mathcal{L}))$, then each of the functions $m\circ\delta_t\circ\phi$ extends to a bounded holomorphic function in $(P_{|2/p-1|})^d$. Moreover,*

$$\|m\circ\delta_t\circ\phi\|_{H^\infty((P_{|2/p-1|})^d)} \le A_p^{\phi(\mathcal{L})}.$$

Remark 3.22. The assumption that ϕ takes values in $[0,\infty)^{d'}$ is only added for the sake of clarity of exposition. It is possible to allow more general ϕ, taking values in some other subset of $\mathbb{C}^{d'}$. Then Theorem 3.21 is still true if m is a continuous function defined on $\bigcup_{t\in\mathbb{R}_+^{d'}} \delta_t(\phi([0,\infty)^d))$, and is a p-uniform joint spectral multiplier of the system $\phi(\mathcal{L})$. A simple example with $d = d' = 2$ is to take $\phi(x_1,x_2) = (x_1 - x_2, x_2)$ (which corresponds to joint spectral multipliers of the system $(\mathcal{L}_1 - \mathcal{L}_2, \mathcal{L}_2)$). Then m needs to be defined on $\mathbb{R} \times [0, \infty)$.

REMARK 3.23. Observe that if $\delta_t \circ \phi$ can be 'holomorphically inverted' in $\mathbb{C}^d$, then Theorem 3.21 does say something about the holomorphic extension of the function m itself. That is indeed the case if $d' = d$ and ϕ is the identity, allowing us to give a different proof of Theorem 3.8(i). We postpone for a moment the proof of Theorem 3.21 to see two different examples of such a situation.

Define $U = \bigcup_{t\in\mathbb{R}^2_+} U_t$ with

$$U_t = \{(z_1, z_2) \in \mathbb{C} \colon z_1 = t_1 w_1,\ z_2 = t_2(w_1 + w_2),\ w_n \in P_{|2p-1|},\ n = 1, 2\},$$

where $P_{|2p-1|}$ is the parabolic region defined by (3.10). Observe that $U \subsetneq (\mathbf{S}_{\phi_p^*})^2$. As an almost immediate corollary of Theorem 3.21 we obtain for instance the following.

COROLLARY 3.24. *Assume $p \in [1, \infty) \setminus \{2\}$ and $m : [0, \infty)^{d'} \to \mathbb{C}$ is bounded on $[0, \infty)^{d'}$ and continuous on $\mathbb{R}^{d'}_+$. The following statements hold:*

(i) *if $d' = d = 2$, and m is a p-uniform joint spectral multiplier of the system $(\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2)$, then m extends to a bounded holomorphic function in U; moreover,*

$$\|m\|_{H^\infty(U)} \le A_p^{(\mathcal{L}_1, \mathcal{L}_1 + \mathcal{L}_2)};$$

(ii) *if $d' = 2$, $d = 1$, and m is a p-uniform joint spectral multiplier of the system $(\mathcal{L}_1, \mathcal{L}_1)$, then each of the functions $m_{t_1}(\lambda_1) = m(\lambda_1, t_1\lambda_1)$, $\lambda_1 \in [0, \infty)$, $t_1 > 0$, extends to a bounded holomorphic function on $\mathbf{S}_{\phi_p^*}$; moreover,*

$$\|m_{t_1}\|_{H^\infty(\mathbf{S}_{\phi_p^*})} \le A_p^{(\mathcal{L}_1, \mathcal{L}_1)}.$$

Proof. Both items (i) and (ii) are easily derived from Theorem 3.21. In the case of (i) we take $\phi(z_1, z_2) = (z_1, z_1 + z_2)$, and in the case of (ii), $\phi(z_1) = (z_1, z_1)$. ∎

Proof of Theorem 3.21. All the hard work has already been done in Lemmata 3.18(iii) and 3.14(iii), hence we can be very brief here. The proof shares many ingredients with the just presented proof of Theorem 3.8(ii). From Lemma 3.18(iii) we know that $\|m \circ \delta_t \circ \phi(\mathcal{L}^\infty)\|_{L^p(\gamma_\infty)} \le A_p^{\phi(\mathcal{L})}$. Hence, Lemma 3.14(iii) shows that $m \circ \delta_t \circ \phi$ extends to a bounded holomorphic function in $(P_{|2/p-1|})^d$ with the desired bound. ∎

4. A Marcinkiewicz type multiplier theorem. A question that naturally arises is: how can we find p-uniform joint spectral multipliers (in the sense of Definition 2.3) of the system $\mathcal{L}$? Fortunately, we have a

Marcinkiewicz type multiplier theorem at our disposal. We say that a function $m\colon \mathbb{R}_+^d \to \mathbb{C}$ satisfies the *d-dimensional Marcinkiewicz condition of order* $\rho = (\rho_1, \ldots, \rho_d)$ if m is bounded and for all multi-indices $\sigma = (\sigma_1, \ldots, \sigma_d) \le \rho$,

$$\sup_{R_1,\ldots,R_d>0} \int\limits_{R_1<\lambda_1<2R_1} \cdots \int\limits_{R_d<\lambda_d<2R_d} |\lambda^\sigma D^\sigma m(\lambda)|^2 \,\frac{d\lambda}{\lambda} < \infty, \tag{4.1}$$

where $\lambda^\sigma = \lambda_1^{\sigma_1} \cdots \lambda_d^{\sigma_d}$ and $\frac{d\lambda}{\lambda} = \frac{d\lambda_1}{\lambda_1} \cdots \frac{d\lambda_d}{\lambda_d}$. For a function $m\colon \overline{(\mathbf{S}_{\phi_p^*})^d} \to \mathbb{C}$ and a vector $\theta = (\theta_1, \ldots, \theta_d) \in [-\phi_p^*, \phi_p^*]^d$ set $m_\theta(\lambda) = m(e^{i\theta_1}\lambda_1, \ldots, e^{i\theta_d}\lambda_d)$, $\lambda \in \mathbb{R}_+^d$. We have the following.

Theorem 4.2. *Assume that* $m\colon \overline{(\mathbf{S}_{\phi_p^*})^d} \to \mathbb{C}$ *and* $m \in H^\infty((\mathbf{S}_{\phi_p^*})^d)$ *for some* $p \in (1,\infty) \setminus \{2\}$. *Assume also that the boundary value functions* $m_{\varepsilon_1\phi_p^*,\ldots,\varepsilon_d\phi_p^*}$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1,1\}^d$, *satisfy the d-dimensional Marcinkiewicz conditions* (4.1) *of some orders* $\rho_\varepsilon > \mathbf{1}$, *and that all the lower dimensional operators* $m({}^{\epsilon_1}\mathcal{L}_1, \ldots, {}^{\epsilon_d}\mathcal{L}_d)$ *with* ${}^0\mathcal{L}_n = 0$, ${}^1\mathcal{L}_n = \mathcal{L}_n$, $n = 1, \ldots, d$, $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d$, $\epsilon_1 + \cdots + \epsilon_d < d$, *are bounded on* $L^p(\gamma)$. *Then* $m(\mathcal{L})$ *extends to a bounded operator on* $L^p(\gamma)$.

Remark 4.3. There are several ways to ensure the L^p boundedness of the lower dimensional operators. One is to assume that m is continuous on $[0,\infty)^d$. Another is to assume that it satisfies appropriate lower dimensional Marcinkiewicz conditions of some orders greater than $\mathbf{1}$. In both cases, since the Marcinkiewicz condition is dilation invariant, the resulting multiplier operators are p-uniform.

Remark 4.4. The proof presented here also works, with the same assumptions, for multiplier operators built on sums of one-dimensional Ornstein–Uhlenbeck operators.

Proof of Theorem 4.2. For the sake of clarity of presentation we prove the theorem for $d = 2$. The proof of the general case is completely analogous. For the meaning of the symbols appearing in the present proof the reader is referred to [13]. First we apply [13, Theorem 2.2] to the operators $\mathcal{L}_{n,\varepsilon} = \mathcal{L}_n + \varepsilon I$, $n = 1, 2$, $\varepsilon > 0$. From the bound

$$\|(\mathcal{L}_{n,\varepsilon})^{iv}\|_{L^p(\gamma(x_n))} \le C_p e^{\phi_p^*|v|}, \qquad v \in \mathbb{R} \tag{4.5}$$

(see [7]) and the fact that $\arcsin x \le \pi x/2$, $0 < x < 1$, we see that the assumption (ii) of [13, Theorem 2.2] is satisfied. It remains to verify the assumption (i) of that theorem. In view of (4.5) it is enough to establish the bound

$$\sup_{T\in(0,\infty)^2} |\mathcal{M}(m_{N,T})(u)| \le C_N e^{-\phi_p^*(|u_1|+|u_2|)}(1+|u_1|)^{-\rho_1}(1+|u_2|)^{-\rho_2}. \tag{4.6}$$

We show this only for $u_1, u_2 < 0$, since the reasoning for other values of u is analogous. Changing the path of integration twice in the integral defining

$\mathcal{M}(m_{N,T})$ we see that

$$\mathcal{M}(m_{N,T})(u) = e^{(iN+u_1)\phi_p^*} e^{(iN+u_2)\phi_p^*} \times \int_0^\infty \int_0^\infty (t_1\lambda_1)^N (t_2\lambda_2)^N \exp(-e^{i\phi_p^*}(t_1\lambda_1 + t_2\lambda_2)) m_{\phi_p^*,\phi_p^*}(\lambda) \lambda_1^{iu_1} \lambda_2^{iu_2} \frac{d\lambda}{\lambda}.$$

Let ψ be a non-negative C^∞ function supported in $[1/2, 2]$ such that

$$\sum_{j=-\infty}^{\infty} \psi_j(\xi) = 1, \quad \xi > 0,$$

where $\psi_j(\xi) = \psi(2^j\xi)$. Then, obviously,

$$\sum_{j,l} \psi_{j,l}(\xi, \eta) = 1, \quad (\lambda, \eta) \in \mathbb{R}_+^2,$$

where $\psi_{j,l}(\xi, \eta) = \psi_j(\lambda)\psi_l(\eta)$. After the change of variable $t_1\lambda_1 = \eta_1$, $t_2\lambda_2 = \eta_2$ we get

$$e^{-2iN\phi_p^*} t_1^{iu_1} t_2^{iu_2} e^{-\phi_p^*(u_1+u_2)} \mathcal{M}(m_{N,T})(u) = \sum_{j,l} \int_0^\infty \int_0^\infty \eta_1^{N-iu_1} \eta_2^{N-iu_2} \times \exp(-e^{i\phi_p^*}(\eta_1 + \eta_2)) m_{\phi_p^*,\phi_p^*}(\eta_1/t_1, \eta_2/t_2) \psi_{j,l}(\eta_1, \eta_2) \frac{d\eta}{\eta}.$$

The rest of the proof of (4.6) is an easy modification of the proof of [8, Theorem 4] in the spirit of the proof of [13, Theorem 4.1]. From [13, Theorem 2.2] it now follows that $m(\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon})$ extends to a bounded operator on $L^p(\gamma)$ with the bound $\|m(\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon})\|_{L^p(\gamma)} \le C_{p,m}$, independent of $\varepsilon > 0$. Let $\mathcal{P}_0^n f(x) = \langle f, \tilde{H}_0 \rangle_{L^2(\mathbb{R},\gamma(x_n))} \tilde{H}_0(x_n)$ (that is, $\mathcal{P}_0^n$ is the projection onto the subspace spanned by $\tilde{H}_0$ applied to f as a function of x_n). Let $\mathcal{P}^n = I - \mathcal{P}_0^n$ (that is, $\mathcal{P}^n$ is the projection onto the subspace spanned by $\{\tilde{H}_{k_n}\}_{k_n>0}$). Since $\mathcal{P}_0^n$ are bounded on $L^p(\gamma)$, $\mathcal{P}^1\mathcal{P}^2$ is also bounded on $L^p(\gamma)$. Consequently,

$$\|m(\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon})\mathcal{P}^1\mathcal{P}^2\|_{L^p(\gamma)} \le C_p. \tag{4.7}$$

Observe now that

$$m(\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon})\mathcal{P}^1\mathcal{P}^2 f = \sum_{k\in\mathbb{N}_+^2} m(k_1+\varepsilon, k_2+\varepsilon) \langle f, \tilde{\mathbf{H}}_k \rangle_{L^2(\gamma)} \tilde{\mathbf{H}}_k,$$

$$m(\mathcal{L}_1, \mathcal{L}_{2,})\mathcal{P}^1\mathcal{P}^2 f = \sum_{k\in\mathbb{N}_+^2} m(k_1, k_2) \langle f, \tilde{\mathbf{H}}_k \rangle_{L^2(\gamma)} \tilde{\mathbf{H}}_k.$$

Since m is continuous and bounded on $(0,\infty)^2$, the spectral theorem implies that

$$\lim_{\varepsilon\to 0^+} m(\mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon})\mathcal{P}^1\mathcal{P}^2 f = m(\mathcal{L}_1, \mathcal{L}_2)\mathcal{P}^1\mathcal{P}^2 f, \qquad f \in L^2(\gamma).$$

From (4.7) we conclude that

$$\|m(\mathcal{L}_1, \mathcal{L}_2)\mathcal{P}^1\mathcal{P}^2 f\|_{L^p(\gamma)} \le C_p.$$

From the decomposition $I = \mathcal{P}^1\mathcal{P}^2 + \mathcal{P}_0^1 + \mathcal{P}_0^2 - \mathcal{P}_0^1\mathcal{P}_0^2$, we see that

$$m(\mathcal{L}_1, \mathcal{L}_2)f = m(\mathcal{L}_1, \mathcal{L}_2)\mathcal{P}^1\mathcal{P}^2 f + m(0, \mathcal{L}_2)\mathcal{P}_0^1 f + m(\mathcal{L}_1, 0)\mathcal{P}_0^2 f - m(0,0)\mathcal{P}_0^1\mathcal{P}_0^2.$$

It follows that $m(\mathcal{L}_1, \mathcal{L}_2)$ is indeed bounded under the assumptions of the theorem. ∎

5. Appendix: the system of Laguerre operators. Here we provide a multivariate counterpart of [10], by sketching the proofs of results analogous to those presented in our paper, this time for the system of Laguerre operators

$$\mathcal{L}_n^{\alpha_n} = -x_n \frac{\partial^2}{\partial x_n^2} + (\alpha_n + 1 - x_n)\frac{\partial}{\partial x_n}, \qquad n = 1, \dots, d.$$

These operators are essentially self-adjoint on $L^2(\mathbb{R}_+, \mu_n^{\alpha_n})$, where, for $\alpha_n > -1$,

$$d\mu_n^{\alpha_n}(x_n) = \frac{x_n^{\alpha_n} e^{-x_n}}{\Gamma(\alpha_n + 1)}\, dx_n, \qquad x_n > 0, \quad n = 1, \dots, d.$$

Moreover, their closures are given by an orthogonal expansion in terms of Laguerre polynomials of type α_n, explicitly,

$$\mathcal{L}_n^{\alpha_n} f(x_n) = \sum_{r\in\mathbb{N}} r\langle f, \tilde{L}_r\rangle_{L^2(\mathbb{R}_+, \mu_n^{\alpha_n})} \tilde{L}_r(x_n), \qquad f \in L^2(\mathbb{R}_+, \mu_n^{\alpha_n}),$$

defined on the domain

$$\mathrm{Dom}(\mathcal{L}_n^{\alpha_n}) = \Big\{ f \in L^2(\mathbb{R}_+, \mu_n^{\alpha_n}) \colon \sum_{r\in\mathbb{N}} r^2 |\langle f, \tilde{L}_r\rangle_{L^2(\mathbb{R}_+, \mu_n^{\alpha_n})}|^2 < \infty \Big\}.$$

Here $\tilde{L}_{k_n}^{\alpha_n} = \|L_{k_n}^{\alpha_n}\|^{-1}_{L^2(\mathbb{R}_+, \mu_n^{\alpha_n})} L_{k_n}^{\alpha_n}$ denotes the $L^2(\mathbb{R}_+, \mu_n^{\alpha_n})$ normalized one-dimensional Laguerre polynomial $L_{k_n}^{\alpha_n}$ of degree k_n and order α_n in the x_n variable (see [12]). A tensor product reasoning, similar to the one presented for the system of Ornstein–Uhlenbeck operators $\mathcal{L}_n$ on p. 48, suggests defining the joint spectral multipliers $m(\mathcal{L}^\alpha)$ of the system $\mathcal{L}^\alpha = (\mathcal{L}_1^{\alpha_1}, \dots, \mathcal{L}_d^{\alpha_d})$ as

(5.1)
$$m(\mathcal{L}_1^{\alpha_1}, \dots, \mathcal{L}_d^{\alpha_d})f = \sum_{k\in\mathbb{N}^d} m(k_1, \dots, k_d)\langle f, \tilde{\mathbf{L}}_k\rangle_{L^2(\mathbb{R}_+^d, \mu_\alpha)} \tilde{\mathbf{L}}_k, \quad f \in L^2(\mathbb{R}_+^d, \mu_\alpha),$$

where $m\colon \mathbb{N}^d \to \mathbb{C}$ is bounded and $\mu_\alpha = \mu_1^{\alpha_1} \otimes \cdots \otimes \mu_d^{\alpha_d}$, $\tilde{\mathbf{L}}_k = \tilde{L}_{k_1}^{\alpha_1} \otimes \cdots \otimes \tilde{L}_{k_d}^{\alpha_d}$.

Concerning the holomorphy of uniform joint spectral multipliers of the form (5.1) we have the following analogue of Theorem 3.8. Theorem 5.2 is also a multivariate extension of [10, Theorem 2].

THEOREM 5.2. *Let* $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1,\infty)^d$ *and assume that* $m : [0,\infty)^d \to \mathbb{C}$ *is bounded on* $[0,\infty)^d$ *and continuous on* $\mathbb{R}^d_+$. *The following statements hold:*

(i) *if* $p \in (1,\infty) \setminus \{2\}$ *and* $\sup_{t_1,\ldots,t_d>0} \|m(t_1\mathcal{L}_1^{\alpha_1}, \ldots, t_d\mathcal{L}_d^{\alpha_d})\|_{L^p(\mathbb{R}^d_+,\,\mu_\alpha)} < \infty$, *then* m *extends to a bounded holomorphic function in the polysector* $(\mathbf{S}_{\phi_p^*})^d$ *and*
$$\|m\|_{H^\infty((\mathbf{S}_{\phi_p^*})^d)} \le \sup_{t_1,\ldots,t_d>0} \|m(t_1\mathcal{L}_1^{\alpha_1}, \ldots, t_d\mathcal{L}_d^{\alpha_d})\|_{L^p(\mathbb{R}^d_+,\,\mu_\alpha)};$$

(ii) $\sup_{t_1,\ldots,t_d>0} \|m(t_1\mathcal{L}_1^{\alpha_1}, \ldots, t_d\mathcal{L}_d^{\alpha_d})\|_{L^1(\mathbb{R}^d_+,\,\mu_\alpha)} < \infty$ *if and only if* m *extends to a bounded holomorphic function in the polysector* $(\mathbf{S}_{\pi/2})^d$ *and* $m(-2i\,\cdot, \ldots, -2i\,\cdot\,)$ *is the Fourier transform of a finite measure* μ *on* $\mathbb{R}^d$ *supported in* $[0,\infty)^d$; *moreover,*
$$\|\mu\|_{M(\mathbb{R}^d)} = \sup_{t_1,\ldots,t_d>0} \|m(t_1\mathcal{L}_1^{\alpha_1}, \ldots, t_d\mathcal{L}_d^{\alpha_d})\|_{L^1(\mathbb{R}^d_+,\,\mu_\alpha)}.$$

Proof (outline). Item (i) is a straightforward consequence of [10, Theorem 2(i)] (in the case $d = 1$) and Theorem 3.1 (with $X_n = \mathbb{R}_+$, $\nu_n = \mu_n^{\alpha_n}$, $L_n = \mathcal{L}_n^{\alpha_n}$, $E_n^p = \mathbf{S}_{\phi_p^*}$).

To prove (ii) we proceed as in the proof of [10, Theorem 2] (with $p = 1$). First, observe that by following the arguments used to prove Theorem 3.8 with minor modifications cf. [2, Lemmata 4.1 and 4.2], we obtain a multivariate analogue of [2, Theorem 4.3], which deals with joint spectral multipliers of self-adjoint extensions of the operators

$$\mathcal{L}_{n,\varphi_n} = -\frac{\partial^2}{\partial x_n^2} + \varphi_n \frac{\partial}{\partial x_n}, \qquad f \in C_c^\infty(\mathbb{R}^d). \tag{5.3}$$

Here, φ_n is an admissible weight in the sense of [2, (4.1), p. 118], depending only on x_n. The Laguerre operators $L_n^{\alpha_n}$ are not of the form (5.3). However, by using an argument analogous to the one in [10, Section 7] we can finish the proof. The key trick is to conjugate the Laguerre operators $\mathcal{L}_n^{\alpha_n}$ with the isometry

$$L^p(\mathbb{R}^d_+, \mu_\alpha) \ni f(x) \mapsto \Psi_* f(x) = f(x_1^2, \ldots, x_d^2) \in L^p_{\mathrm{even}}((\mathbb{R}\setminus\{0\})^d, \mu^{\varphi_\alpha}),$$

where μ^{φ_α} is the push-forward measure $\Psi_*^{-1}\mu_\alpha$, while $L^p_{\mathrm{even}}((\mathbb{R}\setminus\{0\})^d, \mu^{\varphi_\alpha})$ denotes the set of $\mathbb{Z}_2^d$ symmetric functions from $L^p((\mathbb{R}\setminus\{0\})^d, \mu^{\varphi_\alpha})$. This operation allows the passage to operators of the form (5.3), with $\varphi_n = \varphi_n^{\alpha_n}$, $n = 1, \ldots, d$, being admissible weights. We also need to have in mind that every function $f\colon \mathbb{R}^d \to \mathbb{C}$ may be written as the sum of 2^d functions f_ε, $\varepsilon \in \{-1,1\}^d$, which are even in some of their variables and odd in others. We omit the details. ∎

We also have a Marcinkiewicz type multiplier theorem in this setting, which is to some extent a multivariate analogue of [10, Theorem 1].

Theorem 5.4. *Fix a parameter* $\alpha \in [0,\infty)^d$. *Assume that* $m : \overline{(\mathbf{S}_{\phi_p^*})^d} \to \mathbb{C}$, $m \in H^\infty((\mathbf{S}_{\phi_p^*})^d)$, *for some* $p \in (1,\infty) \setminus \{2\}$. *Assume also that the boundary value functions* $m_{\varepsilon_1\phi_p^*,\dots,\varepsilon_d\phi_p^*}$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_d) \in \{-1,1\}^d$, *satisfy the d-dimensional Marcinkiewicz conditions* (4.1) *of some orders* $\rho_\varepsilon > \mathbf{3}$, *and that all the lower dimensional operators* $m({}^{\epsilon_1}\mathcal{L}_1^{\alpha_1},\dots,{}^{\epsilon_d}\mathcal{L}_d^{\alpha_d})$ *with* ${}^0\mathcal{L}_n^{\alpha_n} = 0$, ${}^1\mathcal{L}_n^{\alpha_n} = \mathcal{L}_n^{\alpha_n}$, $n = 1,\dots,d$, $\epsilon = (\epsilon_1,\dots,\epsilon_d) \in \{0,1\}^d$, $\epsilon_1 + \dots + \epsilon_d < d$, *are bounded on* $L^p(\mathbb{R}_+^d, \mu_\alpha)$. *Then* $m(\mathcal{L}^\alpha)$ *extends to a bounded operator on* $L^p(\mathbb{R}_+^d, \mu_\alpha)$.

Proof. The proof is almost identical to the proof of Theorem 4.2. The only difference is that instead of (4.5) we use the bound (see [10, Proposition 2])

$$\|(\mathcal{L}_n^{\alpha_n} + \varepsilon I)^{iv}\|_{L^p(\mathbb{R}_+,\,\mu_n^{\alpha_n}(x_n))} \le C_p(1+|v|)^{5/2} e^{\phi_p^*|v|}, \qquad v \in \mathbb{R}. \ \blacksquare$$

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References

[1] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge Univ. Press, Cambridge, 1989.

[2] W. Hebisch, G. Mauceri and S. Meda, *Holomorphy of spectral multipliers of the Ornstein–Uhlenbeck operator*, J. Funct. Anal. 210 (2004), 101–124.

[3] C. Heil, *A Basis Theory Primer*, Birkhäuser/Springer, 2011.

[4] S. G. Krantz, *Function Theory of Several Complex Variables*, AMS Chelsea Publ., Providence, RI, 1992.

[5] H. A. Lauwerier, *The Hilbert problem for generalized functions*, Arch. Ration. Mech. Anal. 13 (1963), 157–166.

[6] A. Martini, *Algebras of differential operators on Lie groups and spectral multipliers*, PhD thesis, arXiv:1007.1119, 2010.

[7] G. Mauceri, S. Meda and P. Sjögren, *Sharp estimates for the Ornstein–Uhlenbeck operator*, Ann. Scuola. Norm. Super. Pisa Cl. Sci. 3 (2004), 447–480.

[8] S. Meda, *A general multiplier theorem*, Proc. Amer. Math. Soc. 110 (1990), 639–647.

[9] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I. Functional Analysis*, rev. and enlarged ed. Academic Press, New York, 1980.

[10] E. Sasso, *Functional calculus for the Laguerre operator*, Math. Z. 249 (2005), 683–711.

[11] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Ann. of Math. Stud. 63, Princeton Univ. Press, 1970.

[12] G. Szegö, *Orthogonal Polynomials*, rev. ed., Amer. Math. Soc., Providence, RI, 1959.

[13] B. Wróbel, *Multivariate spectral multipliers for tensor product orthogonal expansions*, Monatsh. Math. 168 (2012), 124–149.

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