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A numerical radius inequality involving the generalized Aluthge transform

by

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Dedicated to Rajendra Bhatia on the occasion of his sixtieth birthday

Abstract. A spectral radius inequality is given. An application of this inequality to prove a numerical radius inequality that involves the generalized Aluthge transform is also provided. Our results improve earlier results by Kittaneh and Yamazaki.

1. Introduction. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. For $A \in \mathcal{B}(\mathcal{H})$, let $r(A)$, $w(A)$, and $\|A\|$ denote the spectral radius, the numerical radius, and the operator norm of A, respectively. Recall that $w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. It is well-known that $w(A)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $A \in \mathcal{B}(\mathcal{H})$,

$$\tfrac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. For proofs and more facts about the numerical radius, we refer the reader to [GR] and [H].

Kittaneh has shown in [K1] that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w(A) \le \tfrac{1}{2}(\|A\| + \|A^2\|^{1/2}). \tag{1.2}$$

Obviously, the inequality (1.2) is sharper than the second inequality in (1.1).

For $A \in \mathcal{B}(\mathcal{H})$, let $A = U|A|$ be the polar decomposition of A, where U is a partial isometry such that $\ker U = \ker A$ and $|A| = (A^*A)^{1/2}$. The *Aluthge transform* of A, denoted by $\tilde{A}$, was first defined by Aluthge [A] as

$$\tilde{A} = |A|^{1/2}U|A|^{1/2}.$$

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The Aluthge transform has received much attention in recent years, and various connections between operators and their Aluthge transforms, including relations between various spectra and numerical ranges, have been established. The following are among the well-known relations: $\sigma(\tilde{A}) = \sigma(A)$ and $\overline{W(\tilde{A})} \subseteq \overline{W(A)}$. From these it follows that $r(\tilde{A}) = r(A)$ and $w(\tilde{A}) \leq w(A)$. Also, it follows from the definition of $\tilde{A}$ that $\|\tilde{A}\| \leq \|A^2\|^{1/2}$. For more material about the Aluthge transform, see, e.g., [JKP], [W], and the references therein.

Yamazaki [Y] has used the Aluthge transform to improve the inequality (1.2) so that

$$w(A) \leq \tfrac{1}{2}(\|A\| + w(\tilde{A})). \tag{1.3}$$

It is well-known that if $A, B \in \mathcal{B}(\mathcal{H})$ are such that $AB = BA$, then

$$r(A+B) \leq r(A) + r(B) \quad \text{and} \quad r(AB) \leq r(A)r(B).$$

However, for noncommuting operators, the two-dimensional example $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ shows that the spectral radius is neither subadditive nor submultiplicative.

Kittaneh [K2] has established a general spectral radius inequality which yields spectral radius inequalities for sums, products, and commutators of operators. In fact, Kittaneh has shown that if $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned} r(A_1B_1 + A_2B_2) \leq {} & \tfrac{1}{2}(\|B_1A_1\| + \|B_2A_2\|) \\ & + \tfrac{1}{2}\sqrt{(\|B_1A_1\| - \|B_2A_2\|)^2 + 4\|B_1A_2\|\,\|B_2A_1\|}. \end{aligned} \tag{1.4}$$

In Section 2, we prove a spectral radius inequality that refines the inequality (1.4), and similarly yields spectral radius inequalities for sums, commutators, and products of operators. In Section 3, we use the main result of Section 2 and the generalized Aluthge transform (defined below) to prove a numerical radius inequality, which generalizes and improves Yamazaki's inequality (1.3). Our new inequalities in this paper are sharp. Before we move to Section 2, we need the following basic facts about the spectral radius of an operator. For a comprehensive account, see [B] and [H].

It is well-known that for every $A \in \mathcal{B}(\mathcal{H})$,

$$r(A) \leq w(A), \tag{1.5}$$

with equality if A is normal. In addition to (1.5), an important property of the spectral radius is a commutativity property, which asserts that

$$r(AB) = r(BA) \quad \text{for every } A, B \in \mathcal{B}(\mathcal{H}). \tag{1.6}$$

Also, it is well-known (see [B, p. 10]) that if $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$, then

$$r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{r(A), r(B)\}. \tag{1.7}$$

2. A general spectral radius inequality. In order to establish our new spectral radius inequality, we need the following lemma, which contains a special case of a more general inequality given in [AK], and improves a related inequality given in [HD]. This special case has also been observed in [PB].

LEMMA 2.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be an operator matrix with $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in \mathcal{B}(\mathcal{H}_2)$. Then*

$$\begin{aligned} w(T) &\le w\left(\begin{bmatrix} w(A) & \|B\| \\ \|C\| & w(D) \end{bmatrix}\right) \\ &= \tfrac{1}{2}(w(A) + w(D)) + \tfrac{1}{2}\sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2}. \end{aligned}$$

Here $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ is the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$.

Now, we are in a position to present our desired spectral radius inequality.

THEOREM 2.2. *Let $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} r(A_1B_1 + A_2B_2) &\le \tfrac{1}{2}(w(B_1A_1) + w(B_2A_2)) \\ &\quad + \tfrac{1}{2}\sqrt{(w(B_1A_1) - w(B_2A_2))^2 + 4\|B_1A_2\|\,\|B_2A_1\|}. \end{aligned} \tag{2.1}$$

Proof. By using the basic properties (1.5)–(1.7), we have

$$\begin{aligned} r(A_1B_1 + A_2B_2) &= r\left(\begin{bmatrix} A_1B_1 + A_2B_2 & 0 \\ 0 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}\right) \\ &\le w\left(\begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}\right), \end{aligned}$$

where the operator matrices $\left[\begin{smallmatrix} A_1B_1+A_2B_2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} A_1 & A_2 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} B_1 & 0 \\ B_2 & 0 \end{smallmatrix}\right]$ act on $\mathcal{H} \oplus \mathcal{H}$. Hence, by Lemma 2.1, we have

$$\begin{aligned} r(A_1B_1 + A_2B_2) &\le \tfrac{1}{2}(w(B_1A_1) + w(B_2A_2)) \\ &\quad + \tfrac{1}{2}\sqrt{(w(B_1A_1) - w(B_2A_2))^2 + (\|B_1A_2\| + \|B_2A_1\|)^2}. \end{aligned}$$

The desired inequality now follows by replacing A_1 and B_1 by tA_1 and $\frac{1}{t}B_1$, respectively, and then taking the infimum over $t > 0$. ∎

Remark 2.3. To see that the inequality (2.1) refines the inequality (1.4), let

$$b=\tfrac{1}{2}(w(B_1A_1)+w(B_2A_2))+\tfrac{1}{2}\sqrt{(w(B_1A_1)-w(B_2A_2))^2+4\|B_1A_2\|\,\|B_2A_1\|},$$
$$b_K=\tfrac{1}{2}(\|B_1A_1\|+\|B_2A_2\|)+\tfrac{1}{2}\sqrt{(\|B_1A_1\|-\|B_2A_2\|)^2+4\|B_1A_2\|\,\|B_2A_1\|}.$$

Then it can be easily seen that

$$b=\left\|\begin{bmatrix} w(B_1A_1) & \sqrt{\|B_1A_2\|\,\|B_2A_1\|} \\ \sqrt{\|B_1A_2\|\,\|B_2A_1\|} & w(B_2A_2)\end{bmatrix}\right\|,$$
$$b_K=\left\|\begin{bmatrix} \|B_1A_1\| & \sqrt{\|B_1A_2\|\,\|B_2A_1\|} \\ \sqrt{\|B_1A_2\|\,\|B_2A_1\|} & \|B_2A_2\|\end{bmatrix}\right\|,$$

and so by the norm monotonicity of matrices with nonnegative entries,

$$b\le b_K.$$

Corollary 2.4. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$r(A+B)\le\tfrac{1}{2}(w(A)+w(B))+\tfrac{1}{2}\sqrt{(w(A)-w(B))^2+4\min\{\|AB\|,\|BA\|\}}.$$

Proof. Letting $A_1=A$, $A_2=B_1=I$, and $B_2=B$ in Theorem 2.2, we have

$$r(A+B)\le\tfrac{1}{2}(w(A)+w(B))+\tfrac{1}{2}\sqrt{(w(A)-w(B))^2+4\|BA\|}. \tag{2.2}$$

By symmetry, it follows from (2.2) that

$$r(A+B)\le\tfrac{1}{2}(w(A)+w(B))+\tfrac{1}{2}\sqrt{(w(A)-w(B))^2+4\|AB\|}. \tag{2.3}$$

The desired inequality now follows from (2.2) and (2.3). ∎

Corollary 2.5. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$r(AB\pm BA)\le\tfrac{1}{2}(w(AB)+w(BA))+\tfrac{1}{2}\sqrt{(w(AB)-w(BA))^2+4\|A^2\|\|B^2\|}.$$

Proof. The desired inequality follows from Theorem 2.2 by letting $A_1=B_2=A$, $B_1=B$, and $A_2=\pm B$. ∎

Corollary 2.6. *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$r(AB\pm BA)\le w(AB)+\sqrt{\min\{\|A\|\,\|AB^2\|,\|B\|\,\|A^2B\|\}}, \tag{2.4}$$
$$r(AB\pm BA)\le w(BA)+\sqrt{\min\{\|A\|\,\|B^2A\|,\|B\|\,\|BA^2\|\}}. \tag{2.5}$$

Proof. Letting $A_1=I$, $A_2=B$, $B_1=AB$, and $B_2=\pm A$ in Theorem 2.2, we have

$$r(AB\pm BA)\le w(AB)+\sqrt{\|A\|\,\|AB^2\|}. \tag{2.6}$$

Similarly, letting $A_1 = AB$, $A_2 = B$, $B_1 = I$, and $B_2 = \pm A$ in Theorem 2.2, we have

$$r(AB \pm BA) \le w(AB) + \sqrt{\|B\|\,\|A^2B\|}. \tag{2.7}$$

The inequality (2.4) now follows from (2.6) and (2.7). The inequality (2.5) follows from (2.4) by symmetry. ■

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$r(AB) \le \tfrac{1}{4}(w(AB) + w(BA)) + \tfrac{1}{4}\sqrt{(w(AB) - w(BA))^2 + 4\min\{\|A\|\,\|BAB\|, \|B\|\,\|ABA\|\}}.$$

Proof. Letting $A_1 = \frac{1}{2}A$, $A_2 = \frac{1}{2}AB$, $B_1 = B$, and $B_2 = I$ in Theorem 2.2, we have

$$r(AB) \le \tfrac{1}{4}(w(AB) + w(BA)) + \tfrac{1}{4}\sqrt{(w(AB) - w(BA))^2 + 4\|A\|\,\|BAB\|}. \tag{2.8}$$

By symmetry, it follows from (2.8) that

$$r(AB) \le \tfrac{1}{4}(w(AB) + w(BA)) + \tfrac{1}{4}\sqrt{(w(AB) - w(BA))^2 + 4\|B\|\,\|ABA\|}. \tag{2.9}$$

The desired inequality now follows from (2.8) and (2.9). ■

3. The generalized Aluthge transform and a generalized numerical radius inequality. Let $A \in \mathcal{B}(\mathcal{H})$, and let $A = U|A|$ be the polar decomposition of A. The *generalized Aluthge transform*, denoted by $\tilde{A}_t$, is defined as

$$\tilde{A}_t = |A|^t U |A|^{1-t} \quad \text{for } t \in [0,1].$$

In particular, $\tilde{A}_0 = U^*U^2|A|$, $\tilde{A}_1 = |A|UU^*U = |A|U$, and $\tilde{A}_{1/2} = |A|^{1/2}U|A|^{1/2} = \tilde{A}$ (the Aluthge transform of A). Here $|A|^0$ is defined as U^*U (see, e.g., [CT]).

The first lemma in this section is well-known (see, e.g., [Y]). It gives a useful characterization of the numerical radius.

Lemma 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w(A) = \sup_{\theta\in\mathbb{R}} \|\operatorname{Re}(e^{i\theta}A)\|. \tag{3.1}$$

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$, and let $A = U|A|$ be the polar decomposition of A. Then*

$$w(A) \le \tfrac{1}{2}(\|A\| + w(\tilde{A}_t)) \tag{3.2}$$

for all $t \in [0,1]$. In particular,

$$w(A) \le \tfrac{1}{2}(\|A\| + w(\tilde{A})).$$

Proof. Let $A = U|A|$ be the polar decomposition of A, and let $t \in [0,1]$. Then for every $\theta \in \mathbb{R}$, we have

$$\begin{aligned}\|\mathrm{Re}(e^{i\theta}A)\| &= r(\mathrm{Re}(e^{i\theta}A)) \\ &= \tfrac{1}{2} r(e^{i\theta}A + e^{-i\theta}A^*) = \tfrac{1}{2} r(e^{i\theta}U|A| + e^{-i\theta}|A|U^*) \\ &= \tfrac{1}{2} r(e^{i\theta}U|A|^{1-t}|A|^t + e^{-i\theta}|A|^t|A|^{1-t}U^*).\end{aligned}$$

By letting $A_1 = e^{i\theta}U|A|^{1-t}$, $A_2 = e^{-i\theta}|A|^t$, $B_1 = |A|^t$, and $B_2 = |A|^{1-t}U^*$ in Theorem 2.2, we have

$$\begin{aligned}\|\mathrm{Re}(e^{i\theta}A)\| &\le \tfrac{1}{2} w(\tilde{A}_t) + \tfrac{1}{2}\sqrt{\||A|^{2t}\|\||A|^{1-t}U^*U|A|^{1-t}\|} \\ &= \tfrac{1}{2} w(\tilde{A}_t) + \tfrac{1}{2}\sqrt{\|A\|^{2t}\|A\|^{2-2t}} = \tfrac{1}{2} w(\tilde{A}_t) + \tfrac{1}{2}\|A\|.\end{aligned}$$

This, together with Lemma 3.1, implies that

$$w(A) = \sup_{\theta \in \mathbb{R}} \|\mathrm{Re}(e^{i\theta}A)\| \le \tfrac{1}{2}(\|A\| + w(\tilde{A}_t)),$$

as required. ∎

It follows from Theorem 3.2 that

$$w(A) \le \tfrac{1}{2}\Big(\|A\| + \min_{0 \le t \le 1} w(\tilde{A}_t)\Big).$$

In order to appreciate our inequality (3.2), we give the following example, which is due to T. Yamazaki. It shows that $w(\tilde{A}) \ne \min_{0 \le t \le 1} w(\tilde{A}_t)$ and that the inequality (3.2) is a nontrivial improvement of (1.3).

Example 3.3. Let

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$|A| = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(where U is the partial isometry part in the polar decomposition of A), and so

$$\tilde{A}_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2^t \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad w(\tilde{A}_t) = 2^{t-1}.$$

Thus,

$$\min_{0 \le t \le 1} w(\tilde{A}_t) = w(\tilde{A}_0) = \tfrac{1}{2} < \tfrac{1}{\sqrt{2}} = w(\tilde{A}).$$

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