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New spectral multiplicities for ergodic actions

by

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Abstract. Let G be a locally compact second countable Abelian group. Given a measure preserving action T of G on a standard probability space (X,μ), let $\mathcal{M}(T)$ denote the set of essential values of the spectral multiplicity function of the Koopman representation U_T of G defined in $L^2(X,\mu)\ominus\mathbb{C}$ by $U_T(g)f := f\circ T_{-g}$. If G is either a discrete countable Abelian group or $\mathbb{R}^n$, $n\geq 1$, it is shown that the sets of the form $\{p,q,pq\}$, $\{p,q,r,pq,pr,qr,pqr\}$ etc. or any multiplicative (and additive) subsemigroup of $\mathbb{N}$ are realizable as $\mathcal{M}(T)$ for a weakly mixing G-action T.

0. Introduction. Let G be a locally compact second countable Abelian group and let $T=(T_g)_{g\in G}$ be a measure preserving action of G on a standard probability space $(X,\mathfrak{B},\mu)$. Denote by U_T the induced Koopman unitary representation of G in $L^2_0(X,\mu) := L^2(X,\mu)\ominus\mathbb{C}$ given by

$$U_T(g)f := f\circ T_{-g}.$$

By the spectral theorem, there is a probability measure σ on the dual group $\widehat{G}$ called a *measure of maximal spectral type* of U_T and a measurable field of Hilbert spaces $\widehat{G}\ni\omega\mapsto\mathcal{H}_\omega$ such that

$$L^2_0(X,\mu)=\int_{\widehat{G}}^{\oplus}\mathcal{H}_\omega\,d\sigma(\omega)\quad\text{and}\quad U_T(g)=\int_{\widehat{G}}^{\oplus}\omega(g)I_\omega\,d\sigma(\omega),\qquad g\in G,$$

where I_ω is the identity operator on $\mathcal{H}_\omega$ [Nai]. The map $m_T\colon\widehat{G}\ni\omega\mapsto\dim\mathcal{H}_\omega\in\mathbb{N}\cup\{\infty\}$ is called the *spectral multiplicity function* of U_T. Let $\mathcal{M}(T)$ stand for the set of essential values of m_T. We are interested in the following *spectral multiplicity problem*:

(**Pr**) Which subsets $E\subset\mathbb{N}$ are realizable as $\mathcal{M}(T)$ for an ergodic (or weakly mixing) G-action T?

This problem was studied by a number of authors (see the recent survey [Da1] and the references therein) mainly in the case $G=\mathbb{Z}$. It is known, in

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particular, that a subset $E \subset \mathbb{N}$ is realizable in each of the following cases:

- $1 \in E$ ([KwL] for $G = \mathbb{Z}$, [DL] for $G = \mathbb{R}$),
- $2 \in E$ ([KaL] for $G = \mathbb{Z}$, [DL] for $G = \mathbb{R}$),
- $E = \{p\}$ for arbitrary $p \in \mathbb{N}$ ([Ag], [Ry], [Da2] for $G = \mathbb{Z}$, [DS] for $\mathbb{R}^n$ and an arbitrary discrete countable Abelian group),
- $E = n \cdot F$ for arbitrary $F \ni 1$ and $n > 1$ ([Da2] for $G = \mathbb{Z}$).

Our aim is to obtain some new spectral multiplicities that first appeared in [Ry] for $G = \mathbb{Z}$. Given $E, F \subset \mathbb{N}$, let $E \diamond F := E \cup F \cup EF$ ([1]). In this notation, $\{p\} \diamond \{q\} = \{p, q, pq\}$, $\{p\} \diamond \{q\} \diamond \{r\} = \{p, q, r, pq, pr, qr, pqr\}$ etc.

Theorem 0.1. *Let G be either a discrete countable Abelian group or $\mathbb{R}^m$ with $m \geq 1$. Given a (finite or infinite) sequence of positive integers $p_1, p_2, \ldots$, there exists a weakly mixing probability measure preserving G-action T such that $\mathcal{M}(T) = \{p_1\} \diamond \{p_2\} \diamond \cdots$.*

Since any multiplicative subsemigroup of $\mathbb{N}$ can be represented in the form $\{p_1\} \diamond \{p_2\} \diamond \cdots$, we obtain the following

Corollary 0.2. *Any multiplicative (and hence any additive) subsemigroup E of $\mathbb{N}$ is realizable as $\mathcal{M}(T)$ for a weakly mixing G-action T.*

To prove Theorem 0.1 we adapt the idea from [Ry]. The required action is the product $T_1 \times T_2 \times \cdots$, where T_i is a weakly mixing G-action with homogeneous spectrum of multiplicity p_i. The existence of such actions was proved in [DS] via a 'generic' argument originating in [Ag]. To 'control' the spectral multiplicities of Cartesian products of such actions we furnish T_i with certain asymptotical operator properties using both 'generic' arguments and the (C, F)-technique.

In Section 1 we list some basic definitions and facts that will be used to prove the main theorem. Subsection 1.1 contains the detailed proofs of some results on spectral multiplicities for unitary representations. In Subsection 1.3 we briefly outline the (C, F)-construction of measure preserving actions which is an algebraic counterpart of the classical geometric 'cutting-and-stacking' technique, and in 1.4 we recall the definition and some basic properties of the Poisson suspension, which allows us to obtain finite measure preserving actions from infinite measure preserving ones. Both techniques are used to explicitly construct rigid actions in Lemmata 2.3 and 3.2. In Section 2 we prove Theorem 0.1 in the case of $G = \mathbb{R}^m$. In general, the proof goes along the lines developed in [Ry]. To prove Theorem 0.1 for an arbitrary discrete countable Abelian group we need some modification of this scheme. This is done in Section 3. Though both proofs can be given in the spirit of

([1]) Given $E, F \subset \mathbb{N}$, we denote by EF their algebraic product $\{ef \mid e \in E,\ f \in F\}$.

Section 3, the constraints in Section 3 seem to be artificial and this is the main reason why we consider separately the two cases for G.

1. Preliminaries

1.1. Unitary representations. Denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on a separable Hilbert space $\mathcal{H}$. We endow $\mathcal{U}(\mathcal{H})$ with the (Polish) strong operator topology (which on $\mathcal{U}(\mathcal{H})$ is also the weak operator topology). Given a locally compact second countable group Γ, we furnish the product space $\mathcal{U}(\mathcal{H})^{\Gamma}$ with the (Polish) topology of uniform convergence on compact subsets in Γ. Denote by $\mathcal{U}_{\Gamma}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})^{\Gamma}$ the subset of all unitary representations of Γ in $\mathcal{H}$. Obviously, $\mathcal{U}_{\Gamma}(\mathcal{H})$ is closed in $\mathcal{U}(\mathcal{H})^{\Gamma}$ and hence Polish in the induced topology. Let $\mathcal{B}(\mathcal{H})$ stand for the set of all bounded linear operators on $\mathcal{H}$ endowed with the weak operator topology. By a *unitary polynomial on* Γ we mean a mapping $P\colon \mathcal{U}_{\Gamma}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of the form

$$P(U) = \alpha_1 U(g_1) + \cdots + \alpha_n U(g_n), \qquad \alpha_i \in \mathbb{C},\ g_i \in \Gamma,\ U \in \mathcal{U}_{\Gamma}(\mathcal{H}).$$

We now list some lemmata that will be needed when proving the main theorem.

LEMMA 1.1. *Given a unitary polynomial $P\colon \mathcal{U}_{\Gamma}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and a sequence $(g_n)_{n=1}^{\infty}$ in Γ, the set*

$$\mathcal{P} := \{U \in U_{\Gamma}(\mathcal{H}) \mid P(U) \text{ is a limit point of } \{U(g_n)\}_{n\in\mathbb{N}}\}$$

is a G_δ subset in $U_{\Gamma}(\mathcal{H})$.

Proof. Let d stand for a metric compatible with the weak topology on $\mathcal{B}(\mathcal{H})$. Then

$$\mathcal{P} = \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{U \in U_{\Gamma}(\mathcal{H}) \mid d(P(U), U(g_n)) < 1/m\}.$$

Obviously, the sets $\{U \in U_{\Gamma}(\mathcal{H}) \mid d(P(U), U(g_n)) < 1/m\}$ are open in $U_{\Gamma}(\mathcal{H})$. ∎

Recall that two unitary representations $U, V \in \mathcal{U}_G(\mathcal{H})$ of an Abelian group G are called *spectrally disjoint* if their measures of maximal spectral type, σ_U and σ_V, are mutually singular: $\sigma_U \perp \sigma_V$. We denote by $\mathcal{M}(U)$ the essential image of the spectral multiplicity function of U. It is clear that if U and V are spectrally disjoint then $\mathcal{M}(U \oplus V) = \mathcal{M}(U) \cup \mathcal{M}(V)$. Lemma 1.2 gives us a useful sufficient condition of spectral disjointness.

LEMMA 1.2. *Let G be a locally compact second countable Abelian group. Let $U, V \in \mathcal{U}_G(\mathcal{H})$. If there is a sequence $(g_n)_{n=1}^{\infty} \subset G$ such that*

$$U(g_n) \to I \text{ and } V(g_n) \to 0 \qquad \text{as } n \to \infty,$$

then U and V are spectrally disjoint.

Proof. Let σ_U and σ_V be the measures of maximal spectral type of U and V respectively. By the spectral theorem,

$$\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\omega^{(1)}\, d\sigma_U(\omega), \quad U(g) = \int_{\widehat{G}}^{\oplus} \omega(g) I_\omega\, d\sigma_U(\omega),$$
$$\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\omega^{(2)}\, d\sigma_V(\omega), \quad V(g) = \int_{\widehat{G}}^{\oplus} \omega(g) I_\omega\, d\sigma_V(\omega).$$

Suppose σ_U is equivalent to σ_V on some subset $A \subset \widehat{G}$ with $\sigma_U(A) > 0$. Take any $0 \neq f \in \mathcal{H}$ with $\operatorname{supp} f \subset A$. Then on the one hand,

$$\int_{\widehat{G}} \langle \omega(g_n) f(\omega), f(\omega)\rangle\, d\sigma_U(\omega) = \langle U(g_n) f, f\rangle \to \|f\|^2 \neq 0.$$

On the other hand,

$$\begin{aligned}&\int_{\widehat{G}} \langle \omega(g_n) f(\omega), f(\omega)\rangle\, d\sigma_U(\omega) \\ &\quad = \int_{\widehat{G}} \langle \omega(g_n) f(\omega), f(\omega)\rangle \frac{d\sigma_U}{d\sigma_V}(\omega)\, d\sigma_V(\omega) = \left\langle V(g_n) f, \frac{d\sigma_U}{d\sigma_V} f\right\rangle \to 0.\end{aligned}$$

This contradiction proves that $\sigma_U \perp \sigma_V$. ∎

Given $U, V \in \mathcal{U}_G(\mathcal{H})$, by their *tensor product* we mean the unitary representation $U \otimes V$ of G in $\mathcal{H} \otimes \mathcal{H}$ defined by $(U \otimes V)(g) := U(g) \otimes V(g)$. If σ_U and σ_V are measures of maximal spectral type of U and V, then the convolution $\sigma_U * \sigma_V$ is a measure of maximal spectral type of $U \otimes V$. Let

$$\sigma_U \times \sigma_V = \int_{\widehat{G}} \sigma_\omega\, d(\sigma_U * \sigma_V)(\omega)$$

stand for the disintegration of $\sigma_U \times \sigma_V$ with respect to the projection map $\widehat{G} \times \widehat{G} \ni (\omega_1, \omega_2) \mapsto \omega_1\omega_2 \in \widehat{G}$. Then the map $\widehat{G} \ni \omega \mapsto \dim(L^2(\widehat{G} \times \widehat{G}, \sigma_\omega))$ is the multiplicity function of $U \otimes V$.

The following lemma, which is an obvious generalization of [Ry, Lemma 3.1], allows us to 'control' the spectral multiplicities of tensor products. Recall that a unitary representation $U \in \mathcal{U}_G(\mathcal{H})$ has *simple spectrum* (that is, $\mathcal{M}(U) = \{1\}$) if and only if there is $\varphi \in \mathcal{H}$ (called a *cyclic vector* for U) such that the smallest closed subspace $\mathcal{H}_\varphi$ of $\mathcal{H}$ containing all the vectors $U(g)\varphi$, $g \in G$, is the entire $\mathcal{H}$. Then $\mathcal{H}_\varphi$ is called the *cyclic subspace* of φ.

LEMMA 1.3. *Let G be a locally compact second countable Abelian group and let $U, V \in \mathcal{U}_G(\mathcal{H})$. Suppose there exists a sequence $(g_n)_{n=1}^\infty \subset G$ and subsequences $(g_{n_k(i)})_{k=1}^\infty$, $i \in J$, such that*

(a) $U(g_n) \to I$ *as* $n \to \infty$,
(b) $V(g_{n_k(i)}) \to V(d_i)$ *as* $k \to \infty$ *for each* $i \in J$,

where $\{d_i\}_{i\in J} \subset G$ *is an at most countable subset such that* $\langle d_i\rangle_{i\in J}$ (2) *is dense in* G. *Then*

(1) *if* U *and* V *have simple spectrum, so does* $U \otimes V$,
(2) $\mathcal{M}(U \otimes V) = \mathcal{M}(U)\mathcal{M}(V)$.

Proof. (1) Let φ and ψ be cyclic vectors for U and V respectively. We claim that $\varphi \otimes \psi$ is a cyclic vector for $U \otimes V$. Indeed, the cyclic subspace $\mathcal{H}_{\varphi\otimes\psi}$ of $\varphi \otimes \psi$ is weakly closed (3), invariant under $U(g) \otimes V(g)$ for each $g \in G$ and contains all the vectors $U(g)\varphi \otimes V(g)\psi$, $g \in G$. Hence by (a) and (b) it contains all the weak limits

$$\begin{aligned}\varphi \otimes V(d_i)\psi &= \lim_{k\to\infty} U(g_k(i))\varphi \otimes V(g_k(i))\psi,\\ \varphi \otimes V(d_i + d_j)\psi &= \lim_{k\to\infty} U(g_k(j))\varphi \otimes V(g_k(j))V(d_i)\psi, \quad \text{etc.}\end{aligned}$$

The space $\mathcal{H}_{\varphi\otimes\psi}$ contains therefore all the vectors $\varphi \otimes V(d)\psi$, $d \in \langle d_i\rangle_{i\in J}$. Since $\mathcal{H}_{\varphi\otimes\psi}$ is invariant under $U(g) \otimes V(g)$ for each $g \in G$ it contains all the vectors $U(g)\varphi \otimes V(d + g)\psi$, $g \in G$, $d \in \langle d_i\rangle_{i\in J}$, which form a total system in $\mathcal{H} \otimes \mathcal{H}$. Hence $U \otimes V$ has simple spectrum.

(2) Let

$$U = \bigoplus_{p\in\mathcal{M}(U)} pU^{(p)} \quad \text{and} \quad V = \bigoplus_{q\in\mathcal{M}(V)} qV^{(q)},$$

where $U^{(p)}$, $p \in \mathcal{M}(U)$, and $V^{(q)}$, $q \in \mathcal{M}(V)$, are spectrally disjoint and have simple spectrum. In other words, $\bigoplus_p U^{(p)}$ and $\bigoplus_q V^{(q)}$ have simple spectrum. Then for $U \otimes V$ we have the following decomposition:

$$U \otimes V = \bigoplus_{\substack{p\in\mathcal{M}(U)\\ q\in\mathcal{M}(V)}} pq(U^{(p)} \otimes V^{(q)}).$$

As already shown in (1), $\bigoplus_{p,q} U^{(p)} \otimes V^{(q)} = \bigoplus_p U^{(p)} \otimes \bigoplus_q V^{(q)}$ has simple spectrum. This means that $U^{(p)}\otimes V^{(q)}$, $(p, q) \in \mathcal{M}(U)\times\mathcal{M}(V)$, are spectrally disjoint and have simple spectrum. Hence $\mathcal{M}(U \otimes V) = \mathcal{M}(U)\mathcal{M}(V)$. ∎

Following [Ry], we will say that U and V are *strongly disjoint* if the map $(\widehat{G} \times \widehat{G}, \sigma_U \times \sigma_V) \ni (\omega_1, \omega_2) \mapsto \omega_1\omega_2 \in (\widehat{G}, \sigma_U * \sigma_V)$ is one-to-one mod 0. If U and V have simple spectrum then they are strongly disjoint if and only if $U \otimes V$ has simple spectrum, and hance for any two strongly disjoint unitary representations U and V we have $\mathcal{M}(U \otimes V) = \mathcal{M}(U)\mathcal{M}(V)$. In fact, Lemma 1.3 gives a sufficient condition of strong disjointness for unitary representations.

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(2) Given a subset $A \subset G$, we denote by $\langle A\rangle$ the smallest subgroup of G containing A.
(3) Since any (strongly) closed convex set is weakly closed.

1.2. Group actions. Let Γ be a locally compact second countable group. Given a standard non-atomic probability space $(X, \mathfrak{B}, \mu)$, denote by $\mathrm{Aut}(X, \mu)$ the group of invertible μ-preserving transformations of X. By an *action* T of Γ we mean a continuous group homomorphism $T\colon \Gamma \ni g \mapsto T_g \in \mathrm{Aut}(X, \mu)$. Denote by $\mathcal{A}_\Gamma \subset \mathrm{Aut}(X, \mu)^\Gamma$ the subset of all measure preserving actions of Γ on $(X, \mathfrak{B}, \mu)$. Recall that U_T denotes the Koopman representation of Γ associated with $T \in \mathcal{A}_\Gamma$. We endow $\mathcal{A}_\Gamma$ with the weakest topology which makes the mapping

$$\mathcal{A}_\Gamma \ni T \mapsto U_T \in \mathcal{U}_\Gamma(L^2_0(X, \mu)).$$

continuous. Then $\mathcal{A}_\Gamma$ is Polish. It is easy to verify that a sequence $T^{(n)}$ of Γ-actions converges to T if and only if $\sup_{g\in K} \mu(T^{(n)}_g A \triangle T_g A) \to 0$ as $n \to \infty$ for each compact $K \subset \Gamma$ and $A \in \mathfrak{B}$. There is a natural action of $\mathrm{Aut}(X, \mu)$ on $\mathcal{A}_\Gamma$ by conjugation:

$$(R \cdot T)_g = RT_gR^{-1} \quad \text{for } R \in \mathrm{Aut}(X, \mu),\ T \in \mathcal{A}_\Gamma,\ g \in \Gamma,$$

and this action is obviously continuous.

If $\mu(X) = \infty$ we define the Polish space $\mathcal{A}_\Gamma(X, \mu)$ of all infinite measure preserving Γ-actions in a similar way. Notice that for μ infinite the Koopman representation associated with $T \in \mathcal{A}_\Gamma(X, \mu)$ is considered in the entire space $L^2(X, \mu)$.

1.3. (C, F)-construction. We now briefly outline the (C, F)-construction of measure preserving actions for locally compact groups. For details see [Da3] and the references therein.

Let Γ be a unimodular locally compact second countable amenable group. Fix a (σ-finite) left Haar measure λ on it. Given two subsets $E, F \subset \Gamma$, we denote by EF their algebraic product, i.e. $EF = \{ef \mid e \in E,\ f \in F\}$. The set $\{e^{-1} \mid e \in E\}$ is denoted by E^{-1}. If E is a singleton, say $E = \{e\}$, then we will write eF for EF.

To define a (C, F)-action of Γ we need two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ of subsets in Γ such that the following conditions are satisfied:

(1.1) $\quad (F_n)_{n=0}^\infty$ is a Følner sequence in Γ,

(1.2) $\quad C_n$ is finite and $\#C_n > 1$,

(1.3) $\quad F_nC_{n+1} \subset F_{n+1}$,

(1.4) $\quad F_nc \cap F_nc' = \emptyset$ for all $c \neq c' \in C_{n+1}$.

We equip F_n with the measure $(\#C_1 \cdots \#C_n)^{-1}\lambda{\restriction}F_n$ and endow C_n with the equidistributed probability measure. Let $X_n := F_n \times \prod_{k>n} C_k$ stand for the product of measure spaces. Define an embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_nc_{n+1}, c_{n+2}, \ldots).$$

It is easy to see that it is measure preserving. Then $X_1 \subset X_2 \subset \cdots$. Let $X := \bigcup_{n=0}^{\infty} X_n$ denote the inductive limit of this sequence of measure spaces and let $\mathfrak{B}$ and μ denote the corresponding Borel σ-algebra and measure on X. Then X is a standard Borel space and μ is a σ-finite measure on it. It is finite if

$$\prod_{n=1}^{\infty} \frac{\lambda(F_{n+1})}{\lambda(F_n)\#C_{n+1}} < \infty, \tag{1.5}$$

and infinite if

$$\prod_{n=1}^{\infty} \frac{\lambda(F_{n+1})}{\lambda(F_n)\#C_{n+1}} = \infty. \tag{1.6}$$

If (1.5) is satisfied then we choose (i.e., normalize) λ in such a way that $\mu(X) = 1$. Given a Borel subset $A \subset F_n$, we put

$$[A]_n := \{x \in X \mid x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n \text{ and } f_n \in A\}$$

and call this set an *n-cylinder*. It is clear that the σ-algebra $\mathfrak{B}$ is generated by the family of all cylinders.

To construct a μ-preserving action of Γ on $(X, \mathfrak{B}, \mu)$, fix a filtration $K_1 \subset K_2 \subset \cdots$ of Γ by compact subsets. Thus $\bigcup_{m=1}^{\infty} K_m = \Gamma$. Given $n, m \in \mathbb{N}$, we set

$$L_m^{(n)} := \Big(\bigcap_{k \in K_m} (k^{-1}F_n) \cap F_n\Big) \times \prod_{k>n} C_k \subset X_n,$$
$$R_m^{(n)} := \Big(\bigcap_{k \in K_m} (kF_n) \cap F_n\Big) \times \prod_{k>n} C_k \subset X_n.$$

It is easy to verify that $L_{m+1}^{(n)} \subset L_m^{(n)} \subset L_m^{(n+1)}$ and $R_{m+1}^{(n)} \subset R_m^{(n)} \subset R_m^{(n+1)}$. We define a Borel mapping $K_m \times L_m^{(n)} \ni (g, x) \mapsto T_{m,g}^{(n)} x \in R_m^{(n)}$ by setting for $x = (f_n, c_{n+1}, c_{n+2}, \ldots)$,

$$T_{m,g}^{(n)}(f_n, c_{n+1}, c_{n+2}, \ldots) := (gf_n, c_{n+1}, c_{n+2}, \ldots).$$

Now let $L_m := \bigcup_{n=1}^{\infty} L_m^{(n)}$ and $R_m := \bigcup_{n=1}^{\infty} R_m^{(n)}$. Then a Borel one-to-one mapping $T_{m,g} \colon K_m \times L_m \ni (g, x) \mapsto T_{m,g} x \in R_m$ is well defined by $T_{m,g} \restriction L_m^{(n)} = T_{m,g}^{(n)}$ for $g \in K_m$ and $n \ge 1$. It is easy to see that $L_m \supset L_{m+1}$, $R_m \supset R_{m+1}$ and $T_{m,g} \restriction L_{m+1} = T_{m+1,g}$ for all m. It follows from (1.1) that $\mu(L_m) = \mu(R_m) = 1$ for all $m \in \mathbb{N}$. Finally we set $\widehat{X} := \bigcap_{m=1}^{\infty} L_m \cap \bigcap_{m=1}^{\infty} R_m$ and define a Borel mapping $T \colon \Gamma \times \widehat{X} \ni (g, x) \mapsto T_g x \in \widehat{X}$ by setting $T_g x := T_{m,g} x$ for some (and hence any) m such that $g \in K_m$. It is clear that $\mu(\widehat{X}) = 1$. Thus $T = (T_g)_{g \in \Gamma}$ is a free Borel measure preserving action of Γ on a conull subset of the standard Borel space $(X, \mathfrak{B}, \mu)$. It is easy to verify that T does not depend on the choice of the filtration $(K_m)_{m=1}^{\infty}$. The action T is called the *(C,F)-action of Γ associated with* $(C_{n+1},F_n)_{n\geq 0}$.

We now recall some basic properties of $(X,\mathfrak{B},\mu,T)$. Given Borel subsets $A,B\subset F_n$, we have

$$\begin{aligned}
&[A\cap B]_n=[A]_n\cap[B]_n, \qquad [A\cup B]_n=[A]_n\cup[B]_n,\\
&[A]_n=[AC_{n+1}]_{n+1}=\bigsqcup_{c\in C_{n+1}}[Ac]_{n+1},\\
&T_g[A]_n=[gA]_n \quad \text{ if } gA\subset F_n.
\end{aligned}$$

Each (C,F)-action is of rank one. Note also that the (C,F)-construction respects Cartesian products. Namely, the product of (C,F)-actions $(T_g^{(i)})_{g\in G_i}$ associated with $(C_n^{(i)},F_n^{(i)})_n$, $i=1,2$, is the (C,F)-action of $G_1\times G_2$ associated with $(C_n^{(1)}\times C_n^{(2)},F_n^{(1)}\times F_n^{(2)})_n$.

1.4. Poisson suspension. Let $(X,\mathcal{B})$ be a standard Borel space and let μ be an infinite σ-finite non-atomic measure on X. Fix an increasing sequence of Borel subsets $X_1\subset X_2\subset\cdots$ with $\bigcup_{i=1}^{\infty}X_i=X$ and $\mu(X_i)<\infty$ for each i. A Borel subset is called *bounded* if it is contained in some X_i. Let $\widetilde{X}_i$ denote the space of finite measures on X_i. For each bounded subset $A\subset X_i$, let N_A stand for the map

$$\widetilde{X}_i\ni\omega\mapsto\omega(A)\in\mathbb{R}.$$

Denote by $\widetilde{\mathcal{B}}_i$ the smallest σ-algebra on $\widetilde{X}_i$ in which all the maps N_A, $A\in\mathcal{B}\cap X_i$, are measurable. It is well known that $(\widetilde{X}_i,\widetilde{\mathcal{B}}_i)$ is a standard Borel space. Denote by $(\widetilde{X},\widetilde{\mathcal{B}})$ the projective limit of the sequence

$$(\widetilde{X}_1,\widetilde{\mathcal{B}}_1)\leftarrow(\widetilde{X}_2,\widetilde{\mathcal{B}}_2)\leftarrow\cdots,$$

where the arrows denote the (Borel) natural restriction maps. Then $(\widetilde{X},\widetilde{\mathcal{B}})$ is a standard Borel space. To put it in another way, $\widetilde{X}$ is the space of measures on X which are σ-finite along $(X_i)_{i>0}$. Then there is a unique probability measure $\widetilde{\mu}$ on $(\widetilde{X},\widetilde{\mathcal{B}})$ such that

(1) N_A maps $\widetilde{\mu}$ to the Poisson distribution with parameter $\mu(A)$, i.e.
$$\widetilde{\mu}(\{\omega\mid N_A(\omega)=j\})=\frac{\mu(A)^j\exp(-\mu(A))}{j!}$$
for all bounded $A\subset X$ and integer $j\geq 0$ and

(2) if A and B are disjoint bounded subsets of X then the random variables N_A and N_B on $(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu})$ are independent.

Let G be a locally compact second countable group and let T be a μ-preserving action of G on X such that T_g preserves the subclass of bounded subsets for each $g\in G$. Then T induces a $\widetilde{\mu}$-preserving action $\widetilde{T}$ of G on $\widetilde{X}$ by the formula $\widetilde{T}_g\omega := \omega\circ T_{-g}$. We recall that the dynamical system $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is called the *Poisson suspension* of $(X, \mathcal{B}, \mu, T)$ (see [CFS], [Roy] for the case $G = \mathbb{Z}$).

The well known Fock representation of $L^2(\widetilde{X}, \widetilde{\mu})$ gives an isomorphism

$$L^2(\widetilde{X}, \widetilde{\mu}) \simeq \bigoplus_{n=0}^{\infty} L^2(X, \mu)^{\odot n},$$

where $L^2(X, \mu)^{\odot n}$ is the nth symmetric tensor power of $L^2(X, \mu)$, with $L^2(X, \mu)^{\odot 0} = \mathbb{C}$. The Koopman representation $U_{\widetilde{T}} \oplus P_0$ (considered on $L^2(\widetilde{X}, \widetilde{\mu})$) is unitarily equivalent to the exponential of U_T:

$$U_{\widetilde{T}} \oplus P_0 \simeq \exp U_T = \bigoplus_{n=0}^{\infty} U_T^{\odot n},$$

where P_0 is the orthogonal projection on $\mathbb{C} \subset L^2(\widetilde{X}, \widetilde{\mu})$ and $U_T^{\odot n}$ is the nth symmetric tensor power of U_T [Ne]. Recall that since μ is infinite, we consider U_T in the entire space $L^2(X, \mu)$. It follows, in particular, that the mapping $\mathcal{A}_\Gamma(X, \mu) \ni T \mapsto \widetilde{T} \in \mathcal{A}_\Gamma(\widetilde{X}, \widetilde{\mu})$ is continuous. $\widetilde{T}$ is rigid (for the sequence g_n) if and only if T is rigid (for g_n). If T has no invariant subsets of finite positive measure then $\widetilde{T}$ is weakly mixing [Roy].

2. $\mathbb{R}^m$-actions. In this section we prove Theorem 0.1 in the case when $G = \mathbb{R}^m$.

For given $p > 1$, let $A\colon \mathbb{Z}^p \to \mathbb{Z}^p$ denote the 'cyclic' group automorphism

$$A(x_1, x_2, \dots, x_p) = (x_p, x_1, \dots, x_{p-1}).$$

Following [DS], denote by Γ the semidirect product (4)

$$\Gamma := G \times \mathbb{Z}^p \rtimes_A \mathbb{Z}(p)$$

with multiplication law

$$(g, x, n)(h, y, k) := (g + h, x + A^n y, n + k)$$

for $g, h \in G$, $x, y \in \mathbb{Z}^p$, $n, k \in \mathbb{Z}(p)$. We will identify G with the subgroup $\{(g, 0, 0) \mid g \in G\} \subset \Gamma$. Let $\mathcal{E}_\Gamma \subset \mathcal{A}_\Gamma$ stand for the subset of all free ergodic Γ-actions. Then $\mathcal{E}_\Gamma$ is a G_δ subset in $\mathcal{A}_\Gamma$ and hence it is Polish space with the induced topology [DS]. To prove Theorem 0.1 we will use a 'generic' argument and the following facts will be needed.

Lemma 2.1 ([DS, Theorem 2.8]). *For a generic action $T \in \mathcal{E}_\Gamma$ the action $T{\restriction}G$ is weakly mixing and $\mathcal{M}(T{\restriction}G) = \{p\}$.*

Lemma 2.2 ([DS, Lemma 2.4]). *The* Aut(X, μ)*-orbit of any action $T \in \mathcal{E}_\Gamma$ is dense in $\mathcal{E}_\Gamma$.*

(4) By $\mathbb{Z}(p)$ we denote a cyclic group of order p, i.e. $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$.

We will apply Lemma 2.2 to show that the set of Γ-actions with certain properties is dense in $\mathcal{E}_\Gamma$. However to apply this lemma we will need at least one action in this set. This single action is constructed explicitly in Lemma 2.3.

LEMMA 2.3. *For any sequence $(g_k)_{k=1}^\infty \subset G$ with $g_k \to \infty$, there exists a (C,F)-action $T \in \mathcal{E}_\Gamma$ such that $U_T(g_{k_n}) \to I$ for some subsequence $(g_{k_n})_{n=1}^\infty$.*

Proof. To construct the (C,F)-action we shall determine a sequence $(C_{n+1}, F_n)_{n=0}^\infty$. This will be done inductively. Let $F_n = F_n' \times F_n''$ and $C_n = C_n' \times C_n''$, where $F_n', C_n' \subset G$, $F_n'', C_n'' \subset \mathbb{Z}^p \rtimes \mathbb{Z}(p)$.

First, we claim that the sets $F_n', C_n' \subset G = \mathbb{R}^m$ and a subsequence $(g_{k_n})_{n=1}^\infty$ can be chosen in such a way that

$$\lim_{n\to\infty} \frac{\#(C_n' \cap (C_n' - g_{k_n}))}{\#C_n'} = 1.$$

To show this select a subsequence $(g_{k_n})_{n=1}^\infty$ such that $g_{k_n}/|g_{k_n}|$ converges (to some point of the unit sphere) as $n \to \infty$. From now on we will write g_n instead of g_{k_n} for short. Let $g_n = (g_n^{(1)}, \dots, g_n^{(m)}) \in \mathbb{R}^m$. Without loss of generality we may assume that $g_n^{(i)} > 0$, $i = 1, \dots, m$, and $g_n^{(1)} \to \infty$. In the other cases the proof is similar. Fix a sequence of positive numbers α_n with $\sum_{n=1}^\infty \alpha_n < \infty$. By replacing $(g_n)_{n=1}^\infty$ with a subsequence if necessary, we may assume that

$$\frac{g_{n+1}^{(1)}}{g_n^{(1)}} > \frac{1}{\alpha_n} + 1.$$

We will construct C_n' and F_n' inductively. Choose C_1' and F_0' arbitrarily. Now suppose that we already have C_{n-1}' and $F_{n-1}' = (-a_{n-1}^{(1)}, a_{n-1}^{(1)}) \times \dots \times (-a_{n-1}^{(m)}, a_{n-1}^{(m)})$, where $a_{n-1}^{(i)} > 0$ and $a_{n-1}^{(1)} = g_n^{(1)}/2$. Our purpose is to define C_n' and F_n'. Set

$$h_n := \left\lfloor \frac{g_{n+1}^{(1)} - g_n^{(1)}}{2g_n^{(1)}} \right\rfloor > \frac{1}{\alpha_n} - \frac{1}{2}.$$

In particular, $(2h_n+1)g_n^{(1)} < g_{n+1}^{(1)} < (2h_n+1)g_n^{(1)} + 2g_n^{(1)}$ and $2h_n+1 > 2/\alpha_n$. Select integers $w_n^{(2)}, \dots, w_n^{(m)} > 0$ in such a way that

$$\frac{h_n g_n^{(i)}}{a_{n-1}^{(1)}(2w_n^{(i)} + 1)} < \alpha_n.$$

We set

$$A_n := \{(0, 2l^{(2)}a_{n-1}^{(2)}, \dots, 2l^{(m)}a_{n-1}^{(m)}) \mid l^{(i)} \in \mathbb{Z},\ -w_n^{(i)} \le l^{(i)} \le w_n^{(i)}\} \subset \mathbb{R}^m,$$

$$C_n' := \bigsqcup_{k=-h_n}^{h_n} (A_n + kg_n).$$

Let also $a_n^{(1)} := g_{n+1}^{(1)}/2$ and $a_n^{(i)} := (2w_n^{(i)}+1)a_{n-1}^{(i)} + h_n g_n^{(i)}$, $i = 2, \dots, m$. Set $F_n' := (-a_n^{(1)}, a_n^{(1)}) \times \dots \times (-a_n^{(m)}, a_n^{(m)})$. Then, by construction,

$$\frac{\lambda(F_n')}{\lambda(F_{n-1}')\#C_n'} = \frac{g_{n+1}^{(1)}}{g_n^{(1)}(2h_n+1)} \prod_{i=2}^{m} \frac{2a_n^{(i)}}{2a_{n-1}^{(i)}(2w_n^{(i)}+1)}$$
$$< \left(1+\frac{2}{2h_n+1}\right)\prod_{i=2}^{m}\left(1+\frac{h_n g_n^{(i)}}{2w_n^{(i)}+1}\right) < (1+\alpha_n)^m.$$

Thus the conditions (1.1)–(1.5) hold for $(F_n', C_n')_n$. It also follows from the definition of C_n' that

$$\frac{\#(C_n' \cap (C_n' - g_n))}{\#C_n'} = \frac{2h_n}{2h_n+1} \to 1.$$

Secondly, let C_n'' and F_n'' be any subsets of $\mathbb{Z}^p \rtimes \mathbb{Z}(p)$ satisfying (1.1)–(1.5). For instance, set

$$F_n'' := \{-(3^n-1)/2, \dots, (3^n-1)/2\}^p \times \mathbb{Z}(p) \subset \mathbb{Z}^p \rtimes \mathbb{Z}(p),$$
$$C_n'' := \{-3^{n-1}, 0, 3^{n-1}\}^p \times \{0\} \subset \mathbb{Z}^p \rtimes \mathbb{Z}(p).$$

Let T be the (C,F)-action associated with $(C_n, F_n)_n = (C_n' \times C_n'', F_n' \times F_n'')_n$. As was mentioned in Section 1.3, T is then the product of the two (C,F)-actions $T^{(1)} = (T_g^{(1)})_{g\in G}$ and $T^{(2)} = (T_z^{(2)})_{z\in\mathbb{Z}^p\rtimes\mathbb{Z}(p)}$ associated with $(C_n', F_n')_n$ and $(C_n'', F_n'')_n$ respectively. Since $g_n \in G$, we have

$$\lim_{n\to\infty} \frac{\#(C_n \cap g_n^{-1}C_n)}{\#C_n} = 1.$$

We claim that $\lim_{n\to\infty} \mu(T_{g_n}A \triangle A) = 0$ for any $A \in \mathfrak{B}$. It suffices to consider the cylinders $[A]_n$, $A \subset F_n$. Fix $\varepsilon > 0$ and select n such that

$$\frac{\#(C_n \setminus g_n^{-1}C_n)}{\#C_n} < \varepsilon. \tag{2.1}$$

Let $A \subset F_{n-1}$. Notice that g_n commutes with all the elements of Γ. Thus

$$[A]_{n-1} = \bigsqcup_{c\in C_n} [Ac]_n = A_1 \sqcup \bigsqcup_{c\in C_n\cap g_nC_n} [Ac]_n = A_2 \sqcup \bigsqcup_{c\in C_n\cap g_n^{-1}C_n} [Ac]_n,$$

where $A_1 := \bigsqcup_{c\in C_n\setminus g_nC_n}[Ac]_n$, $A_2 := \bigsqcup_{c\in C_n\setminus g_n^{-1}C_n}[Ac]_n$ and $\mu(A_i) < \varepsilon$ by (2.1). On the other hand,

$$T_{g_n}[A]_{n-1} = T_{g_n}A_2 \sqcup \bigsqcup_{c\in C_n\cap g_n^{-1}C_n} T_{g_n}[Ac]_n = T_{g_n}A_2 \sqcup \bigsqcup_{c\in C_n\cap g_n^{-1}C_n} [g_nAc]_n$$
$$= T_{g_n}A_2 \sqcup \bigsqcup_{c\in C_n\cap g_nC_n} [Ac]_n.$$

Hence $T_{g_n}[A]_{n-1} \triangle [A]_{n-1} \subset A_1 \cup T_{g_n} A_2$ and $\mu(T_{g_n}[A]_{n-1} \triangle [A]_{n-1}) < 2\varepsilon$. The claim is proven. It follows that $U_T(g_n) \to I$ as $n \to \infty$.

Since any (C, F)-action is free and ergodic, $T \in \mathcal{E}_\Gamma$. ■

As was mentioned above, to prove the main result we will apply the Baire category theorem, so the following lemma will be useful.

LEMMA 2.4. *Given a sequence $g_n \to \infty$ in G, the following subsets are residual in $\mathcal{E}_\Gamma$:*

$$\mathcal{I} := \{T \in \mathcal{E}_\Gamma \mid I \textit{ is a limit point of } \{U_T(g_n)\}_{n=1}^\infty\},$$
$$\mathcal{O} := \{T \in \mathcal{E}_\Gamma \mid 0 \textit{ is a limit point of } \{U_T(g_n)\}_{n=1}^\infty\}.$$

Proof. It follows from Lemma 1.1 that $\mathcal{I}$ and $\mathcal{O}$ are both G_δ in $\mathcal{E}_\Gamma$. Notice also that they are both $\mathrm{Aut}(X, \mu)$-invariant. Therefore in view of Lemma 2.2 it remains to show that they each contain at least one free ergodic action. The set $\mathcal{I}$ is non-empty by Lemma 2.3. Consider the action of Γ on itself by translations. This action preserves the (σ-finite, infinite) Haar measure. The corresponding Poisson suspension of this action (see Section 1.4) is a probability preserving free Γ-action and it belongs to $\mathcal{O}$ (see [OW]). ■

Lemma 2.5 will be the main ingredient in the proof of Theorem 0.1. In general, its proof goes along the lines developed in [Ry] for $\mathbb{Z}$-actions.

LEMMA 2.5. *Given a rigid weakly mixing $S \in \mathcal{A}_G$ and $p > 0$, there exists a weakly mixing $T \in \mathcal{A}_G$ such that $S \times T$ is rigid, weakly mixing and satisfies $\mathcal{M}(S \times T) = \mathcal{M}(S) \diamond \{p\}$.*

Moreover, if $(r_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ are sequences in G such that $U_S(r_n) \to I$ and $U_S(g_n) \to 0$, then $U_{S\times T}(r'_n) \to I$ and $U_{S\times T}(g'_n) \to 0$ as $n \to \infty$ for some subsequences $(r'_n)_{k=1}^\infty$ and $(g'_n)_{k=1}^\infty$.

Proof. Fix sequences $(r_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ in G such that

$$U_S(r_n) \to I \quad \text{and} \quad U_S(g_n) \to 0. \tag{2.2}$$

Let also $\langle d_1, \ldots, d_{2m} \rangle$ be dense in G. Let $\Gamma := G \times \mathbb{Z}^p \rtimes_A \mathbb{Z}(p)$ stand for the auxiliary non-Abelian group defined above. We claim that for a generic $\widetilde{T} \in \mathcal{E}_\Gamma$, the G-action $T := \widetilde{T} \restriction G$ has the following properties:

(1) T is weakly mixing,
(2) $\mathcal{M}(T) = \{p\}$,
(3) $0, I$ and $U_T(d_1), \ldots, U_T(d_{2m})$ are limit points of the set $\{U_T(r_n)\}_{n\in\mathbb{N}}$,
(4) 0 and I are limit points of $\{U_T(g_n)\}_{n\in\mathbb{N}}$.

The properties (1)–(2) are generic by Lemma 2.1. Since $U_T(d)$ is a limit point of $\{U_T(r_n)\}_{n=1}^\infty$ if and only if I is a limit point of $\{U_T(r_n - d)\}_{n=1}^\infty$, Lemma 2.4 implies (3)–(4) for a generic $\widetilde{T} \in \mathcal{E}_\Gamma$. Hence there is an action satisfying all of these conditions.

Now let us show that T is the required action. Lemma 1.3, in view of (2.2) and (3), implies that $\mathcal{M}(U_S \otimes U_T) = p\mathcal{M}(U_S)$. Since the Koopman representation is considered on the space $L^2(X,\mu) \ominus \mathbb{C}$, we have the decomposition

$$U_{S\times T} = (1 \otimes U_T) \oplus (U_S \otimes U_T) \oplus (U_S \otimes 1), \tag{2.3}$$

where 1 denotes the identity operator on $\mathbb{C}$. If $1 \otimes U_T$, $U_S \otimes U_T$, $U_S \otimes 1$ are pairwise spectrally disjoint then

$$\mathcal{M}(S \times T) = \{p\} \cup p\mathcal{M}(S) \cup \mathcal{M}(S) = \mathcal{M}(S) \diamond \{p\}.$$

Apply (3) and (4) and fix a subsequence $(r_n'')_{n=1}^\infty$ of $(r_n)_{n=1}^\infty$ and a subsequence $(g_n'')_{n=1}^\infty$ of $(g_n)_{n=1}^\infty$ such that $U_T(r_n'') \to 0$ and $U_T(g_n'') \to I$ as $n \to \infty$. The spectral disjointness for each pair of terms in (2.3) follows from Lemma 1.2, since

$$\begin{aligned}
(U_S \otimes 1)(r_n'') &\to I, & (U_S \otimes U_T)(r_n'') &\to 0,\\
(1 \otimes U_T)(g_n'') &\to I, & (U_S \otimes U_T)(g_n'') &\to 0,\\
(1 \otimes U_T)(g_n'') &\to I, & (U_S \otimes 1)(g_n'') &\to 0.
\end{aligned}$$

It is clear that $S \times T$ is weakly mixing. By (3) and (4) there are subsequences $(r_n')_{n=1}^\infty$ and $(g_n')_{n=1}^\infty$ of $(r_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ such that $U_T(r_n') \to I$ and $U_T(g_n') \to 0$. Hence $U_{S\times T}(r_n') \to I$ and $U_{S\times T}(g_n') \to 0$. ∎

Proof of Theorem 0.1 for $G = \mathbb{R}^m$. Consider the auxiliary group $\varGamma_1 := G\times\mathbb{Z}^{p_1} \rtimes \mathbb{Z}(p_1)$ defined above. Let $\widetilde{T}_1 \in \mathcal{E}_{\varGamma_1}$ be such that $T_1 := \widetilde{T}_1 \restriction G$ is weakly mixing, $\mathcal{M}(T_1) = \{p_1\}$ and $U_{T_1}(r_{n,1}) \to I$, $U_{T_1}(g_{n,1}) \to 0$ as $n \to \infty$, where $(r_{n,1})_{n=1}^\infty$, $(g_{n,1})_{n=1}^\infty$ are some sequences in G. Since all these properties are generic for the actions from $\mathcal{E}_{\varGamma_1}$ by Lemmata 2.1 and 2.4, there is an action $\widetilde{T}_1$ possessing all of them.

Now we apply Lemma 2.5 and choose a weakly mixing $T_2 \in \mathcal{A}_G$ such that $\mathcal{M}(T_1 \times T_2) = \{p_1\} \diamond \{p_2\}$ and $U_{T_1\times T_2}(r_{n,2}) \to I$, $U_{T_1\times T_2}(g_{n,2}) \to 0$ as $n \to \infty$, where $(r_{n,2})_{n=1}^\infty$ and $(g_{n,2})_{n=1}^\infty$ are subsequences of $(r_{n,1})_{n=1}^\infty$ and $(g_{n,1})_{n=1}^\infty$ respectively.

By induction, given a weakly mixing G-action $T_1 \times \cdots \times T_{k-1}$ with

$$\mathcal{M}(T_1 \times \cdots \times T_{k-1}) = \{p_1\} \diamond \cdots \diamond \{p_{k-1}\},$$
$$U_{T_1\times\cdots\times T_{k-1}}(r_{n,k-1}) \to I, \quad U_{T_1\times\cdots\times T_{k-1}}(g_{n,k-1}) \to 0,$$

by Lemma 2.5 there exists a weakly mixing $T_k \in \mathcal{A}_G$ such that

$$\mathcal{M}(T_1 \times \cdots \times T_k) = \{p_1\} \diamond \cdots \diamond \{p_k\}, \tag{2.4}$$

$$U_{T_1\times\cdots\times T_k}(r_{n,k}) \to I, \quad U_{T_1\times\cdots\times T_k}(g_{n,k}) \to 0, \tag{2.5}$$

where $(r_{n,k})_{n=1}^\infty$ and $(g_{n,k})_{n=1}^\infty$ are suitable subsequences of $(r_{n,k-1})_{n=1}^\infty$ and $(g_{n,k-1})_{n=1}^\infty$ respectively. This proves the theorem if the sequence $p_1, p_2, \ldots$ is finite. Otherwise we obtain an infinite sequence of weakly mixing G-actions

T_k satisfying (2.4)–(2.5). It is clear that the product $T := T_1 \times T_2 \times \cdots$ is weakly mixing and $\mathcal{M}(T) = \{p_1\} \diamond \{p_2\} \diamond \cdots$. ∎

The following simple lemma (stated in [DL] without proof) shows how to extend the result of Theorem 0.1 from $\mathbb{R}$ to any torsion free discrete countable Abelian group (Corollary 2.7).

LEMMA 2.6. *Let G and H be locally compact second countable Abelian groups and let $\varphi\colon G \to H$ be a continuous one-to-one homomorphism with $\overline{\varphi(G)} = H$. Given an H-action $T = (T_h)_{h\in H}$, the composition $T \circ \varphi = (T_{\varphi(g)})_{g\in G}$ is a G-action with $\mathcal{M}(T \circ \varphi) = \mathcal{M}(T)$.*

Proof. Let σ be a measure of maximal spectral type and $m\colon \widehat{H} \to \mathbb{N}\cup\{\infty\}$ be the spectral multiplicity function of U_T:

$$(2.6)\quad L_0^2(X,\mu) = \int_{\widehat{H}}^{\oplus} \mathcal{H}_\chi \, d\sigma(\chi) \quad \text{and} \quad U_T(h)f(\chi) = \chi(h)f(\chi), \qquad h \in H,$$

for each $f\colon \widehat{H} \ni \chi \mapsto f(\chi) \in \mathcal{H}_\chi$ with $\int_{\widehat{H}} \|f(\chi)\|^2 \, d\sigma(\chi) < \infty$, $\dim \mathcal{H}_\chi = m(\chi)$. Let $\widehat{\varphi}\colon \widehat{H} \to \widehat{G}$ stand for the homomorphism dual to φ and $\widehat{\sigma} := \sigma \circ \widehat{\varphi}^{-1}$ be the image of σ under $\widehat{\varphi}$. Clearly, $\widehat{\sigma}(\widehat{\varphi}(\widehat{H})) = 1$. Let $\sigma = \int_{\widehat{G}} \sigma_\omega \, d\widehat{\sigma}(\omega)$ denote the disintegration of σ relative to $\widehat{\varphi}$. Then we derive from (2.6) that

$$L_0^2(X,\mu) = \int_{\widehat{G}}^{\oplus} \mathcal{H}'_\omega \, d\widehat{\sigma}(\omega) = \int_{\widehat{\varphi}(\widehat{H})}^{\oplus} \mathcal{H}'_\omega \, d\widehat{\sigma}(\omega),$$

where $\mathcal{H}'_\omega := \int_{\widehat{H}}^{\oplus} \mathcal{H}_\chi \, d\sigma_\omega(\chi)$. Let $l(\omega) := \dim \mathcal{H}'_\omega$, $\omega \in \widehat{G}$. Then

$$l(\omega) = \begin{cases} \sum_{\sigma_\omega(\chi)>0} m(\chi) & \text{if } \sigma_\omega \text{ is purely atomic,} \\ \infty & \text{otherwise.} \end{cases}$$

Since $\overline{\varphi(G)} = H$, $\widehat{\varphi}$ is one-to-one and hence $\mathcal{H}'_{\widehat{\varphi}(\chi)} = \mathcal{H}_\chi$ for any $\chi \in \widehat{H}$. In particular, $l(\widehat{\varphi}(\chi)) = m(\chi)$ for $\chi \in \widehat{H}$. It follows from (2.6) that for any $\omega = \widehat{\varphi}(\chi) \in \widehat{G}$,

$$\begin{aligned} U_{T\circ\varphi}(g)f(\omega) &= U_T(\varphi(g))f(\widehat{\varphi}(\chi)) = U_T(\varphi(g))(f \circ \widehat{\varphi})(\chi) \\ &= \chi(\varphi(g))(f \circ \widehat{\varphi})(\chi) = (\widehat{\varphi}(\chi))(g)f(\widehat{\varphi}(\chi)) = \omega(g)f(\omega). \end{aligned}$$

This means that $\widehat{\sigma}$ is a measure of maximal spectral type and l is the spectral multiplicity function of $U_{T\circ\varphi}$. Hence $\mathcal{M}(T \circ \varphi) = \mathcal{M}(T)$. ∎

COROLLARY 2.7. *Let G be a torsion free discrete countable Abelian group. Given a sequence of positive integers $p_1, p_2, \ldots$, there exists a weakly mixing probability preserving G-action S such that $\mathcal{M}(S) = \{p_1\} \diamond \{p_2\} \diamond \cdots$.*

Proof. For $G = \mathbb{Z}$ see [Ry] or Section 3. Suppose $G \neq \mathbb{Z}$. In view of Lemma 2.6 it suffices to show that there is an embedding $\varphi\colon G \to \mathbb{R}$ such that $\overline{\varphi(G)} = \mathbb{R}$. Indeed, G can be embedded into $\mathbb{Q}^{\mathbb{N}}$ (see [HR]). In turn, the last group obviously embeds into $\mathbb{R}$. It remains to note that if an infinite subgroup of $\mathbb{R}$ is not isomorphic to $\mathbb{Z}$ then it is dense in $\mathbb{R}$.

By Theorem 0.1 for $G = \mathbb{R}$, there is a weakly mixing $\mathbb{R}$-action T such that $\mathcal{M}(T) = \{p_1\} \diamond \{p_2\} \diamond \cdots$. Then by Lemma 2.6 the composition $T \circ \varphi = (T_{\varphi(g)})_{g \in G}$ is a weakly mixing G-action with $\mathcal{M}(T \circ \varphi) = \mathcal{M}(T) = \{p_1\} \diamond \{p_2\} \diamond \cdots$. ∎

3. Discrete countable Abelian group actions. In this section we prove Theorem 0.1 in the case when G is an infinite discrete countable Abelian group.

As in the previous section, given a countable discrete Abelian group J and $p > 1$, we denote by Γ the semidirect product $G \times J^p \rtimes_A \mathbb{Z}(p)$, where $A\colon J^p \to J^p$ is the same (as in Section2) 'cyclic' group automorphism. From now on we will identify G with the corresponding subgroup in Γ.

LEMMA 3.1 ([DS, Theorem 1.7]). *Given G and $p > 1$, there is J such that for a generic action T from $\mathcal{A}_\Gamma$ the action $T \restriction G$ is weakly mixing and $\mathcal{M}(T \restriction G) = \{p\}$.*

Notice that we can choose J to be either $\mathbb{Z}$ or $\mathbb{Z}(q)^{\oplus \mathbb{N}}$, $q > 1$ [DS, Section 1].

Let $(g_n)_{n=1}^\infty$ be a sequence in G. We will say that $(g_n)_{n=1}^\infty$ is *good* if $g_n \to \infty$ and one of the following is satisfied:

(1) there is $g_0 \in G$ such that $g_n \in \langle g_0 \rangle$ for each n (it follows that g_0 has infinite order),
(2) each g_n is of finite order and the orders of g_n are unbounded,
(3) the orders of g_n are bounded from above and g_n are independent ([5]).

It is clear that G always contains a good sequence. Notice also that any subsequence of a good sequence is good. We need this notion to be able to apply the (C, F)-construction in the proof of Lemma 3.2, which is the analog of Lemma 2.3.

LEMMA 3.2. *Let $(g_k)_{k=1}^\infty$ be a good sequence in G. For any $d \in G$ there exists a free action $S \in \mathcal{A}_\Gamma$ such that $U_S(g_{k_n}) \to U_S(d)$ for some subsequence $(g_{k_n})_{n=1}^\infty$ of $(g_k)_{k=1}^\infty$.*

Proof. Fix $d \in G$. First, we claim that there is an *infinite* measure preserving action T of Γ and subsequence $(g_{k_n})_{n=1}^\infty$ such that $U_T(g_{k_n}) \to U_S(d)$. Recall that for μ infinite, we consider U_T in the entire space $L^2(X, \mu)$. We will construct T as $T = T^{(1)} \times T^{(2)}$, where $T^{(1)}$ and $T^{(2)}$ are (C, F)-actions of G and $J^p \rtimes \mathbb{Z}(p)$ respectively.

([5]) That is, the subgroups $\langle g_n \rangle$ are independent.

To construct $T^{(1)}$ we will select subsets $C_n, F_n \subset G$ and a subsequence $(g_{k_n})_{n=1}^{\infty}$ of $(g_k)_{k=1}^{\infty}$ in such a way that

$$\lim_{n\to\infty} \frac{\#(C_n \cap (C_n - (g_{k_n} - d)))}{\#C_n} = 1. \tag{3.1}$$

Then, arguing as in the proof of Lemma 2.3, the reader can easily deduce that

$$\lim_{n\to\infty} \mu(T_{g_{k_n}-d} A \bigtriangleup A) = 0$$

for any $A \in \mathfrak{B}$, and hence $U_T(g_{k_n} - d) \to I$ as $n \to \infty$.

Thus our aim is to select C_n, F_n and k_n satisfying (1.1)–(1.4), (1.6) and (3.1). This will be done inductively. Fix an increasing sequence of positive integers h_n. Suppose that we already have F_{n-1} and k_{n-1}. To satisfy (3.1) we want C_n to be an arithmetic progression with common difference $g_{k_n} - d$ long enough. We also need C_n to be independent of F_{n-1}. Consider separately three possible cases for $(g_k)_{k=1}^{\infty}$.

(i) *There is $g_0 \in G$ such that $g_k = m_k g_0$, $m_k \in \mathbb{Z}$, $k \in \mathbb{N}$.* Without loss of generality we may assume that $m_k > 0$ and $m_{k+1} > m_k$, $k \in \mathbb{N}$. Then let $k_n := \max\{k \mid g_k \in F_{n-1} - F_{n-1}\} + 1$. Clearly, $l g_{k_n} \notin F_{n-1} - F_{n-1}$ for any $l > 0$ and hence $l(g_{k_n} - d) + F_{n-1} \cap l'(g_{k_n} - d) + F_{n-1} = \emptyset$ for $l \neq l'$.

(ii) *Each g_k is of finite order and the orders are unbounded.* Without loss of generality we may assume that $\#\{k \mid \operatorname{ord} g_k < N\} < \infty$ for each $N > 0$. Given $0 \neq f \in F_{n-1} - F_{n-1}$ and $0 < l \leq h_n$, let $D_{n,l}^f := \{k > k_{n-1} \mid l(g_k - d) = f\}$. We claim that each $D_{n,l}^f$ is finite. Indeed, if $l(g_k - d) = f$ for some k then for any k' with $\operatorname{ord} g_{k'} > l \operatorname{ord} g_k$ we have $\operatorname{ord}(g_k - g_{k'}) > l$ and hence $l(g_{k'} - d) \neq l(g_k - d) = f$. Since there are only finitely many k' with $\operatorname{ord} g_{k'} \leq l \operatorname{ord} g_k$, the set $D_{n,l}^f$ is finite and we can choose $k_n > k_{n-1}$ such that $k_n \notin D_{n,l}^f$ for $0 \neq f \in F_{n-1} - F_{n-1}$, $0 < l \leq h_n$. Then $l g_{k_n} \notin F_{n-1} - F_{n-1}$, $0 < l \leq h_n$. In particular, $l(g_{k_n} - d) + F_{n-1} \cap l'(g_{k_n} - d) + F_{n-1} = \emptyset$ for $0 \leq l < l' \leq h_n$.

(iii) *The orders of g_k are bounded from above and g_k are independent.* In this case for any $0 \neq f \in F_{n-1} - F_{n-1}$ and $l > 0$ there is at most one k with $l g_k = f$. Hence we can select $k_n > k_{n-1}$ in such a way that $l g_{k_n} \notin F_{n-1} - F_{n-1}$ whenever $l g_{k_n} \neq 0$.

In each of these three cases we set

$$C_n := \begin{cases} \{0, (g_{k_n} - d), 2(g_{k_n} - d), \ldots, h_n(g_{k_n} - d)\} & \text{if } \operatorname{ord}(g_{k_n} - d) > h_n, \\ \langle g_{k_n} - d \rangle & \text{otherwise.} \end{cases}$$

It follows that C_n and F_{n-1} are independent. Since

$$\frac{\#(C_n \cap (C_n - (g_{k_n} - d)))}{\#C_n} \leq \frac{h_n}{h_n + 1},$$

the C_n satisfy (3.1). Let $F_n \subset G$ be any subset satisfying (1.1), (1.4) and (1.6). Let $T^{(1)}$ be the (C,F)-action associated with $(C_n, F_n)_n$.

$T^{(2)}$ may be any (C,F)-action of $J^p \rtimes \mathbb{Z}(p)$. In view of the structure of J which is either $\mathbb{Z}$ or $\mathbb{Z}(q)^{\oplus\mathbb{N}}$, $q > 1$, such an action can be easily constructed. For instance, set

$$F'_n := \{-(3^n-1)/2, \ldots, (3^n-1)/2\}^p \times \mathbb{Z}(p) \subset J^p \rtimes \mathbb{Z}(p),$$
$$C'_n := \{-3^{n-1}, 0, 3^{n-1}\}^p \times \{0\} \subset J^p \rtimes \mathbb{Z}(p),$$

if $J = \mathbb{Z}$, and

$$F'_n := (\underbrace{\mathbb{Z}(q) \oplus \cdots \oplus \mathbb{Z}(q)}_{n} \oplus \{0\} \oplus \cdots)^p \times \mathbb{Z}(p) \subset J^p \rtimes \mathbb{Z}(p),$$
$$C'_n := (\underbrace{\{0\} \oplus \cdots \oplus \{0\}}_{n-1} \oplus \mathbb{Z}(p) \oplus \{0\} \oplus \cdots)^p \times \{0\} \subset J^p \rtimes \mathbb{Z}(p)$$

if $J = \bigoplus_{n=1}^{\infty} \mathbb{Z}(q)$, $q > 1$. Clearly, $(C'_n, F'_n)_n$ satisfy (1.1)–(1.5). Let $T^{(2)}$ be the (C,F)-action associated with $(C'_n, F'_n)_n$.

Then by construction $T = T^{(1)} \times T^{(2)}$ is an infinite measure preserving action of Γ such that $U_T(g_{k_n}) \to U_S(d)$ as $n \to \infty$.

Now let $S := \widetilde{T}$ stand for the Poisson suspension of T (see Subsection 1.4). Then S is a free probability measure preserving Γ-action. Since the mapping $\mathcal{A}_\Gamma(X,\mu) \ni T \mapsto \widetilde{T} \in \mathcal{A}_\Gamma(\widetilde{X}, \widetilde{\mu})$ is continuous, $U_S(g_{k_n}) \to U_S(d)$ as $n \to \infty$. ∎

Lemma 3.3. *For any good sequence $(g_n)_{n=1}^{\infty}$ in G the following subsets are residual in $\mathcal{A}_\Gamma$:*

$$\mathcal{I}_d := \{T \in \mathcal{A}_\Gamma \mid U_T(d) \text{ is a limit point of } \{U_T(g_n)\}_{n=1}^{\infty}\} \quad \text{for any } d \in G,$$
$$\mathcal{O} := \{T \in \mathcal{A}_\Gamma \mid 0 \text{ is a limit point of } \{U_T(g_n)\}_{n=1}^{\infty}\}.$$

Proof. $\mathcal{O}$ and $\mathcal{I}_d$, $d \in G$, are G_δ subsets in $\mathcal{A}_\Gamma$ by Lemma 1.1. We note that $\mathcal{O}$ and $\mathcal{I}_d$ are $\mathrm{Aut}(X,\mu)$-invariant. By [FW, Claim 18] the $\mathrm{Aut}(X,\mu)$-orbit of any free Γ-action is dense in $\mathcal{A}_\Gamma$. Therefore, it remains to show that $\mathcal{O}$ and $\mathcal{I}_d$, $d \in G$, each contain at least one free action. Each $\mathcal{I}_d$ is non-empty by Lemma 3.2. Each Poisson Γ-action is free and belongs to $\mathcal{O}$ [OW]. ∎

Proof of Theorem 0.1 for G is a discrete countable Abelian group. Let $\Gamma_1 := G \times J_1^{p_1} \rtimes \mathbb{Z}(p_1)$ be the auxiliary group defined above for G and p_1. Fixing a good sequence in G and applying Lemmata 3.1 and 3.3 we deduce that there is an action $\widetilde{T}_1 \in \mathcal{A}_{\Gamma_1}$ such that $T_1 := \widetilde{T}_1 \restriction G$ is weakly mixing, $\mathcal{M}(T_1) = \{p_1\}$ and $U_{T_1}(r_{n,1}) \to I$, $U_{T_1}(g_{n,1}) \to 0$, where $(r_{n,1})_{n=1}^{\infty}$, $(g_{n,1})_{n=1}^{\infty}$ are good sequences in G.

Now let $\Gamma_2 := G \times J_2^{p_2} \rtimes \mathbb{Z}(p_2)$. By Lemmata 3.1 and 3.3 for a generic $\widetilde{T}_2 \in \mathcal{A}_{\Gamma_2}$ the restriction $T_2 := \widetilde{T}_2 \restriction G$ satisfies the following conditions:

(1) T_2 is weakly mixing,
(2) $\mathcal{M}(T_2) = \{p_2\}$,
(3) $U_{T_2}(d)$ is a limit point of $\{U_{T_2}(r_{n,1})\}_{n=1}^{\infty}$ for each $d \in G$,
(4) 0 is a limit point of $\{U_{T_2}(r_{n,1})\}_{n=1}^{\infty}$,
(5) I and 0 are limit points of $\{U_{T_2}(g_{n,1})\}_{n=1}^{\infty}$.

Thus $T_1 \times T_2$ is weakly mixing with $\mathcal{M}(T_1 \times T_2) = \{p_1\} \diamond \{p_2\}$ by Lemma 1.3. Moreover, in view of (3) and (5), there are subsequences $(r_{n,2})_{n=1}^{\infty}$ and $(g_{n,2})_{n=1}^{\infty}$ of $(r_{n,1})_{n=1}^{\infty}$ and $(g_{n,1})_{n=1}^{\infty}$ such that $U_{T_1\times T_2}(r_n) \to I$, $U_{T_1\times T_2}(g_n) \to 0$ as $n \to \infty$.

Continuing, we obtain a sequence of weakly mixing G-actions T_i such that $\mathcal{M}(T_1 \times \cdots \times T_k) = \{p_1\} \diamond \cdots \diamond \{p_k\}$ for any $k > 0$. It follows that $T := T_1 \times T_2 \times \cdots$ is weakly mixing with $\mathcal{M}(T) = \{p_1\} \diamond \{p_2\} \diamond \cdots$. ∎

4. Concluding remarks. The scheme of the proof also works for the groups of the form $\mathbb{R}^m \times G$ where G is a discrete countable Abelian group and $m > 0$. For that we need to construct explicitly a 'rigid' Γ-action as in Lemmata 2.3 and 3.2 for $\Gamma = \mathbb{R}^m \times G \times J^p \rtimes \mathbb{Z}(p)$. Indeed, in both lemmata the required action was obtained as the product of two (C, F)-actions. Let us say that an element $g \in \Gamma$ is *good* if all but the first coordinate of g vanish. Then the analogs of Lemmata 2.3 and 3.2 for sequences of good elements can be easily proved by constructing separately two (C, F)-actions: an $\mathbb{R}^m$-action as in Lemma 2.3 and a $G \times J^p \rtimes \mathbb{Z}(p)$-action as in Lemma 3.2. Moreover, one may mimic the proof of Lemma 3.2 to extend it to any locally compact second countable Abelian group. Then the main result is still true for the classes of locally compact second countable Abelian groups considered in [DS].

Note that our realizations are weakly mixing but not mixing since they are rigid. The question whether there are mixing realizations of the sets considered is still open. In fact, the set of mixing G-actions is meager in $\mathcal{A}_G$ endowed with the weak topology. Therefore the weak topology is not suitable to apply the Baire category argument. In contrast, Tikhonov introduced another (stronger than the weak) topology on $\mathcal{A}_{\mathbb{Z}}$ with respect to which the subset of mixing $\mathbb{Z}$-actions is Polish [Ti1]. Using this topology he proved by a 'generic' argument the existence of mixing transformations with homogeneous spectrum [Ti2]. It looks plausible that this approach may be useful to find mixing realizations of the sets considered in the present paper.

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