

## Sharp embedding results for spaces of smooth functions with power weights

by

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**Abstract.** We consider function spaces of Besov, Triebel–Lizorkin, Bessel-potential and Sobolev type on  $\mathbb{R}^d$ , equipped with power weights  $w(x) = |x|^\gamma$ ,  $\gamma > -d$ . We prove two-weight Sobolev embeddings for these spaces. Moreover, we precisely characterize for which parameters the embeddings hold. The proofs are presented in such a way that they also hold for vector-valued functions.

**1. Introduction.** Weighted spaces of smooth functions play an important role in the context of partial differential equations (PDEs). They are widely used, for instance, to treat PDEs with degenerate coefficients or domains with a nonsmooth geometry (see e.g. [4, 22, 26, 42]). For evolution equations, power weights in time play an important role in obtaining results for rough initial data (see [11, 20, 25, 31]). In addition, here one is naturally confronted with vector-valued spaces. Our work is motivated by this and will be applied in a forthcoming paper to study weighted spaces with boundary values.

For general literature on weighted function spaces we refer to [8, 16, 22, 26, 30, 32, 41, 42] and references therein. Also vector-valued function spaces are intensively studied (see [2, 3, 36, 37, 38, 43, 45] and references therein). Less is known on vector-valued function spaces with weights (see [4, 28] and references therein). Some difficulties come from the fact that in the vector-valued case the identities  $W^{1,p} = H^{1,p}$  and  $L^p = F_{p,2}^0$  hold only under further geometric assumptions on the underlying Banach space (see below).

In this paper we characterize continuous embeddings of Sobolev type for vector-valued function spaces with weights of the form  $w(x) = |x|^\gamma$  with

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$\gamma > -d$ , where  $d$  is the dimension of the underlying Euclidian space. We consider several classes of spaces: Besov spaces, Triebel–Lizorkin spaces, Bessel-potential spaces and Sobolev spaces. In the embeddings which we study we put (possibly different) weights  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  on each of the function spaces.

These embeddings and their optimality are well-known in the unweighted case (see e.g. [37, 39, 41]). For scalar-valued Besov spaces with general weights from Muckenhoupt’s  $A_\infty$ -class (see Section 2) the embeddings were characterized in [18]. In the latter work also the compact embeddings for scalar Triebel–Lizorkin spaces are characterized. Sufficient conditions for scalar-valued Triebel–Lizorkin spaces in the case of one fixed weight  $w$  for both spaces are considered in [8]. Results for Sobolev spaces are obtained e.g. in [22, 26]. A different setting is studied [32], which we discuss in Remark 1.9 below.

The approach of [18] to the scalar Besov space case is based on discretization in terms of wavelet bases and on weighted embeddings of [23]. In the special case of power weights we can give elementary Fourier-analytic proofs for the necessary and sufficient conditions. These apply also in the general vector-valued case, and so we do not have to impose any restriction on the underlying Banach space throughout.

For a further discussion, let us describe the results in detail. Throughout, let  $X$  be a Banach space. For  $p \in (1, \infty]$  and  $q \in [1, \infty]$ , let  $B_{p,q}^s(\mathbb{R}^d, w; X)$  denote the Besov space with weight  $w(x) = |x|^\gamma$ , where  $\gamma > -d$  (see Section 3.1). The following two-weight characterization of Sobolev type embeddings for these spaces is the first main result of our paper.

**THEOREM 1.1.** *Let  $X$  be a Banach space,  $1 < p_0, p_1 \leq \infty$ ,  $q_0, q_1 \in [1, \infty]$ ,  $s_0, s_1 \in \mathbb{R}$ , and  $w_0(x) = |x|^{\gamma_0}$ ,  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . The following assertions are equivalent:*

(1) *One has the continuous embedding*

$$(1.1) \quad B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

(2) *The parameters satisfy one of the following conditions:*

$$(1.2) \quad \gamma_0 = \gamma_1, \quad p_0 = p_1 \text{ and either } s_0 > s_1 \text{ or } s_0 = s_1 \text{ and } q_0 \leq q_1;$$

$$(1.3) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0}, \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0} \text{ and } s_0 - \frac{d + \gamma_0}{p_0} > s_1 - \frac{d + \gamma_1}{p_1};$$

$$(1.4) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0}, \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0}, \quad q_0 \leq q_1 \text{ and } s_0 - \frac{d + \gamma_0}{p_0} = s_1 - \frac{d + \gamma_1}{p_1}.$$

For  $p \in (1, \infty)$ ,  $q \in [1, \infty]$  and  $w$  as above, let  $F_{p,q}^s(\mathbb{R}^d, w; X)$  denote the weighted Triebel–Lizorkin space (see Section 3.1). The following characterization is our second main result. Unlike in Theorem 1.1, the characterization

is given only for  $1 < p_0 \leq p_1 < \infty$  (see, however, Proposition 1.6 below). Important in this result is that  $q_0 \leq q_1$  is not required in the sharp case  $s_0 - (d + \gamma_0)/p_0 = s_1 - (d + \gamma_1)/p_1$  as for Besov spaces.

**THEOREM 1.2.** *Let  $X$  be a Banach space,  $1 < p_0 \leq p_1 < \infty$ ,  $q_0, q_1 \in [1, \infty]$ ,  $s_0, s_1 \in \mathbb{R}$ , and  $w_0(x) = |x|^{\gamma_0}$ ,  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . The following assertions are equivalent:*

(1) *One has the continuous embedding*

$$(1.5) \quad F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X).$$

(2) *The parameters satisfy either (1.2) or*

$$(1.6) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0}, \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1}.$$

**REMARK 1.3.**

- (i) The scalar version  $X = \mathbb{C}$  of Theorem 1.1 with general weights  $w_0, w_1$  from Muckenhoupt's  $A_\infty$ -class is proved in [18, Section 2]. For  $w_0 = w_1 \in A_\infty$  satisfying  $\inf_{x \in \mathbb{R}^d} w(B(x, t)) \geq t^\varepsilon$  with  $\varepsilon > 0$ , the implication (2) $\Rightarrow$ (1) of Theorem 1.2 can be found in [8, Theorem 2.6] in the scalar case. In our setting, this corresponds to the case  $\gamma_0 = \gamma_1 \geq 0$ .
- (ii) In the unweighted case, i.e.,  $\gamma_0 = \gamma_1 = 0$ , results such as Theorems 1.1 and 1.2 are well-known and go back to works of Jawerth, Nikol'skiĭ, Peetre and Triebel (see [41, Section 2.7.1] for a historical overview). A detailed account on these embeddings in the vector-valued setting can be found in [37].
- (iii) Theorem 1.1 gives embeddings for  $p_0 > p_1$ , which is only possible in the presence of weights. In Proposition 1.6 we obtain a partial result also for Triebel–Lizorkin spaces in this case.
- (iv) In Theorems 1.1 and 1.2, suppose that  $p_0 < p_1$ . Then the condition  $(d + \gamma_1)/p_1 < (d + \gamma_0)/p_0$  in (1.3), (1.4) and (1.6) is redundant. Similarly, if  $p_0 > p_1$  then  $\gamma_1/p_1 \leq \gamma_0/p_0$  is redundant.
- (v) It follows from  $p_0 \leq p_1$  and  $\gamma_1/p_1 \leq \gamma_0/p_0$  that  $\gamma_0 = \gamma_1 < 0$  is excluded. In this case one only has the trivial embeddings (i.e., the embeddings under the assumption (1.2)).
- (vi) It is a well-known fact that Sobolev embeddings for Triebel–Lizorkin spaces are independent of the microscopic parameters  $q_0, q_1$  (see [8, 37, 41]).

Our proof of the sufficiency of the relations stated in Theorem 1.1 is based on a direct two-weight extension of an inequality of Plancherel–Pólya–Nikol'skiĭ type (see Proposition 4.1, and [41, Section 1.3] for an overview). For  $\gamma_0, \gamma_1 \geq 0$  this inequality is obtained by extending the proof of the one-

weight version of [8] to the present situation. For negative weight exponents we use the weighted Young inequalities from [9, 19]. The necessity of these conditions follows from suitable scaling arguments (see Propositions 4.7 and 4.9). Observe that  $L^p(\mathbb{R}^d, |\cdot|^\gamma)$  scales to the power  $-(d + \gamma)/p$ , which explains the importance of this number. Moreover, the relation  $\gamma_1/p_1 \leq \gamma_0/p_0$  is in particular sufficient to apply the results of [9, 19].

Theorem 1.2 is derived from Theorem 1.1 using a weighted version of a Gagliardo–Nirenberg type inequality for  $F$ -spaces (see Proposition 5.1). Here we follow the presentation of [37].

As a consequence of Theorem 1.2 we characterize embeddings for Bessel-potential and Sobolev spaces. For  $p \in (1, \infty)$  and  $w(x) = |x|^\gamma$ , where  $\gamma \in (-d, d(p - 1))$ , let  $H^{s,p}(\mathbb{R}^d, w; X)$  denote the weighted Bessel-potential space with  $s \in \mathbb{R}$ , and let  $W^{m,p}(\mathbb{R}^d, w; X)$  denote the weighted Sobolev space with  $m \in \mathbb{N}_0$  (see Section 3.2).

**COROLLARY 1.4.** *Let  $X$  be a Banach space,  $1 < p_0 \leq p_1 < \infty$ ,  $s_0, s_1 \in \mathbb{R}$ , and  $w_0(x) = |x|^{\gamma_0}$ ,  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0 \in (-d, d(p_0 - 1))$ ,  $\gamma_1 \in (-d, d(p_1 - 1))$ . The following assertions are equivalent:*

- (1) *One has the continuous embedding*

$$H^{s_0,p_0}(\mathbb{R}^d, w_0; X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d, w_1; X).$$

- (2) *The parameters satisfy*

$$(1.7) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1}.$$

**COROLLARY 1.5.** *Let  $X$  be a Banach space,  $1 < p_0 \leq p_1 < \infty$ ,  $s_0, s_1 \in \mathbb{N}_0$ , and  $w_0(x) = |x|^{\gamma_0}$ ,  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0 \in (-d, d(p_0 - 1))$ ,  $\gamma_1 \in (-d, d(p_1 - 1))$ . The following assertions are equivalent:*

- (1) *One has the continuous embedding*

$$W^{s_0,p_0}(\mathbb{R}^d, w_0; X) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d, w_1; X).$$

- (2) *The parameters satisfy (1.7).*

The necessity of (1.7) for the embeddings in Corollaries 1.4 and 1.5 is actually valid for all  $\gamma_0, \gamma_1 > -d$ , as a consequence of Proposition 4.7. The restrictions in the sufficiency part mean that  $w_0 \in A_{p_0}$  and  $w_1 \in A_{p_1}$ , where  $A_p$  denotes Muckenhoupt’s class (see Section 2).

In the general vector-valued case,  $H$ - and integer  $W$ -spaces are not contained in the  $B$ - and  $F$ -scale, respectively. We have

$$(1.8) \quad H^{s,p}(\mathbb{R}^d; X) = F_{p,2}^s(\mathbb{R}^d; X) \quad \text{for some } s \in \mathbb{R}, p \in (1, \infty)$$

if and only if  $X$  is isomorphic to a Hilbert space (see [17] and [37, Remark 7]). Moreover,

$$(1.9) \quad H^{1,p}(\mathbb{R}^d; X) = W^{1,p}(\mathbb{R}^d; X) \quad \text{for some } p \in (1, \infty)$$

characterizes the UMD property of  $X$  (see [1, 27, 45] for details).

We obtain the sufficient conditions in the above corollaries from embeddings of the type

$$(1.10) \quad F_{p,1}^s(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^s(\mathbb{R}^d, w; X),$$

which are valid for all Banach spaces  $X$  and  $A_p$ -weights  $w$  (see Proposition 3.12), and the independence of the Sobolev embeddings for the  $F$ -spaces from the microscopic parameters. For the latter embedding in (1.10) we show in Remark 3.13 that it is necessary that  $w$  satisfies a local  $A_p$ -condition, which results in  $\gamma_0 < d(p_0 - 1)$ . Then  $\gamma_1 < d(p_1 - 1)$  is a consequence of  $\gamma_1/p_1 \leq \gamma_0/p_0$  and  $p_0 \leq p_1$ . This explains our restrictions on the weight exponents to the  $A_p$ -case. In view of Theorems 1.1 and 1.2, and the results in [32] (see Remark 1.9 below), we do not expect these restrictions to be necessary.

The result of Theorem 1.1 for Besov spaces and  $1 < p_1 < p_0 < \infty$  cannot be extended to  $F$ -,  $H$ - and  $W$ -spaces in the usual way. We prove the following in this case.

**PROPOSITION 1.6.** *Let  $X$  be a Banach space,  $1 < p_1 < p_0 < \infty$ ,  $s_0, s_1 \in \mathbb{R}$  and  $\gamma_0 \in (-d, d(p_0 - 1))$ ,  $\gamma_1 \in (-d, d(p_1 - 1))$ , and let  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$ . The following assertions are equivalent:*

(1) *One has the continuous embedding*

$$H^{s_0,p_0}(\mathbb{R}^d, w_0; X) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^d, w_1; X).$$

(2) *One has the continuous embedding*

$$F_{p_0,2}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1,2}^{s_1}(\mathbb{R}^d, w_1; X).$$

(3) *The parameters satisfy*

$$(1.11) \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} > s_1 - \frac{d + \gamma_1}{p_1}.$$

Note that the above result shows that there is no embedding in the important sharp case  $s_0 - (d + \gamma_0)/p_0 = s_1 - (d + \gamma_1)/p_1$ . Surprisingly, this is different for Besov spaces (see Theorem 1.1). The same holds if  $H$ -spaces are replaced by  $W$ -spaces since in the scalar case  $W^{m,p}(\mathbb{R}^d, w) = H^{m,p}(\mathbb{R}^d, w)$  whenever  $m \in \mathbb{N}$  and  $w \in A_p$  (see the proof of Corollary 5.5).

**REMARK 1.7.** Let  $p_0 > p_1$ ,  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . It would be interesting to characterize for which parameters one has  $F_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow F_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)$ . Proposition 1.6 only contains a partial answer to this, because of the restrictions on  $\gamma_0, \gamma_1, q_0, q_1$ .

**REMARK 1.8.** Let  $J_s = (1 - \Delta)^{s/2}$  be the Bessel potential. Corollary 1.4 gives a characterization of the boundedness of  $J_{-s_0} : L^{p_0}(\mathbb{R}^d, w_0) \rightarrow$

$L^{p_1}(\mathbb{R}^d, w_1)$  for certain power weights  $w_0$  and  $w_1$  for  $p_0 \leq p_1$  and all  $s_0 \geq 0$ . The equivalent condition of [33] can be difficult to check. Moreover, in Proposition 1.6 we provide a characterization of the boundedness of  $J_{-s_0}$  for power weights  $w_0$  and  $w_1$  for  $p_1 < p_0$  and all  $s_0 \geq 0$ . It seems that the problem for  $J_{-s}$  with  $p_1 < p_0$  has not been considered before.

In [35] necessary and sufficient conditions on  $w_0, w_1, p_0, p_1, s_0$  can be found for the boundedness of the Riesz potential  $I_{-s_0} = (-\Delta)^{-s_0/2}$  for  $p_0 \leq p_1$  and  $0 \leq s_0 < d$ . The case  $p_1 < p_0$  has been considered in [44].

Together with elementary embeddings, Theorems 1.1 and 1.2 provide certain embeddings between  $B$ - and  $F$ -spaces. These can be strengthened to so-called Jawerth–Franke embeddings (see Theorem 6.4). Here we restrict to the case of  $A_p$ -weights (see Remark 6.6 for a discussion). We also use the above results to show embeddings of weighted  $B$ -,  $F$ -,  $H$ - and  $W$ -spaces into (unweighted) spaces of continuous functions (see Proposition 7.4). Again we can follow the presentation of [37]. For the case of a single weight  $w \in A_\infty$ , in the scalar-valued case the Jawerth–Franke embeddings are shown in [8, 18].

REMARK 1.9. Recently, in [32] the classical Caffarelli–Kohn–Nirenberg inequalities (see [10]) have been extended to general power weights. It is characterized when

$$(1.12) \quad W^{1,p_0,q}(\mathbb{R}^d \setminus \{0\}, w_0, w) \hookrightarrow L^{p_1}(\mathbb{R}^d, w_1),$$

where  $W^{1,p_0,q}(\mathbb{R}^d \setminus \{0\}, w_0, w)$  is the space of functions  $f : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$  which are locally integrable and satisfy

$$\|f\|_{L^{p_0}(w_0)} + \|\nabla f\|_{L^q(w)} < \infty.$$

Here it is assumed that  $w_0(x) = |x|^{\gamma_0}$ ,  $w(x) = |x|^\gamma$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma, \gamma_1 \in \mathbb{R}$  and  $p_0, q, p_1 \in [1, \infty)$ . Our results can be compared to the ones in [32] only in a very special case, namely for  $\gamma_0 = \gamma \in (-d, d(p_0 - 1))$ ,  $\gamma_1 \in (-d, d(p_1 - 1))$ ,  $q = p_0, p_1 < \infty, s_0 = 1$  and  $s_1 = 0$ . In this special case the equivalence of (1.12) with the condition on the parameters  $\gamma_0, \gamma_1, p_0, p_1$  in [32] coincides with ours (see Corollary 1.5 if  $p_0 \leq p_1$  and Proposition 1.6 if  $p_0 > p_1$ ).

On the other hand, one of the main and novel points in the results in [32] is that the power weights are not necessarily of  $A_\infty$ -type and two different powers weights  $w_0, w$  and exponents  $p_0, q$  in the Sobolev space are considered.

REMARK 1.10. It would be interesting to extend the above results (e.g. Theorems 1.1, 1.2, etc.) to homogeneous Besov, Triebel–Lizorkin, Bessel-potential and Sobolev spaces. Some parts are easy to extend. In particular, Proposition 4.1 below can be applied in the homogeneous setting as well. The identity  $s_0 - (d + \gamma_0)/p_0 = s_1 - (d + \gamma_1)/p_1$  plays a crucial role.

**Outline.** In Section 2 we consider preliminaries for the treatment of weighted function spaces, such as maximal inequalities and multiplier theorems. These are used in Section 3 to derive basic properties of the vector-valued spaces. In Section 4 we prove the characterization for Besov spaces, and in Section 5 the case of  $F$ -,  $H$ - and  $W$  spaces is treated. In Section 6 we show embeddings of Jawerth–Franke type, and in Section 7 embeddings into unweighted function spaces.

**Notation.** Positive constants are denoted by  $C$ , and may vary from line to line. If  $X, Y$  are Banach spaces, we write  $X = Y$  if they coincide as sets and have equivalent norms. For  $p \in [1, \infty]$ , the standard sequence spaces are denoted by  $\ell^p$ .

**2. Preliminaries.** Here we collect the tools from harmonic analysis that are needed in the treatment of the weighted function spaces in the next section.

**2.1. Weights.** A function  $w : \mathbb{R}^d \rightarrow [0, \infty)$  is called a *weight* if  $w$  is locally integrable and if  $\{w = 0\}$  has Lebesgue measure zero. For  $p \in [1, \infty)$  we denote by  $A_p$  the *Muckenhoupt class* of weights, and  $A_\infty = \bigcup_{1 \leq p < \infty} A_p$ . In case  $p \in (1, \infty)$  we have  $w \in A_p$  if

$$\sup_{Q \text{ cubes in } \mathbb{R}^d} \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty.$$

We refer to [13], [15, Chapter 9] and [40, Chapter V] for the general properties of these classes.

**EXAMPLE 2.1.** Let  $w$  be a power weight, i.e.,  $w(x) = |x|^\gamma$  with  $\gamma \in \mathbb{R}$ . Then for  $p \in (1, \infty)$  we have (see [15, Example 9.1.7])

$$w \in A_p \quad \text{if and only if} \quad \gamma \in (-d, d(p-1)).$$

Let  $(X, \|\cdot\|)$  be a Banach space. For a strongly measurable function  $f : \mathbb{R}^d \rightarrow X$  and  $p \in [1, \infty)$  let

$$\|f\|_{L^p(\mathbb{R}^d, w; X)} := \left( \int_{\mathbb{R}^d} \|f(x)\|^p w(x) \, dx \right)^{1/p}.$$

For  $p \in [1, \infty)$  we consider the Banach space

$$L^p(\mathbb{R}^d, w; X) := \{f \text{ strongly measurable} : \|f\|_{L^p(\mathbb{R}^d, w; X)} < \infty\},$$

and further set  $L^\infty(\mathbb{R}^d, w; X) := L^\infty(\mathbb{R}^d; X)$ .

**2.2. Maximal functions.** For  $f \in L^1_{\text{loc}}(\mathbb{R}^d; X)$ , let the Hardy–Littlewood maximal function  $Mf$  be defined by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} \|f(y)\| \, dy, \quad x \in \mathbb{R}^d,$$

where  $B(x, r) = \{z \in \mathbb{R}^d : |z - x| < r\}$ . For any weight  $w : \mathbb{R}^d \rightarrow \mathbb{R}_+$  and  $p \in (1, \infty)$ , the maximal function  $M$  is bounded on  $L^p(\mathbb{R}^d, w)$  if and only if  $w \in A_p$  (see [40, Theorem V.3.1] and [15, Theorem 9.1.9]). The following weighted vector-valued version of the Fefferman–Stein maximal inequality holds as well.

**PROPOSITION 2.2.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$ ,  $q \in (1, \infty]$  and  $w \in A_p$ . Then there exists a constant  $C_{p,q,w}$  such that for all  $(f_k)_{k \geq 0} \subseteq L^p(\mathbb{R}^d, w; X)$  one has*

$$\|(Mf_k)_{k \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q)} \leq C_{p,q,w} \|(f_k)_{k \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}.$$

*Proof.* For  $q = \infty$  one uses the fact that

$$(2.1) \quad \|Mf_k(x)\|_{\ell^\infty(X)} \leq M\|f_k(x)\|_{\ell^\infty(X)}, \quad x \in \mathbb{R}^d, k \geq 0,$$

and applies the boundedness of  $M$  on  $L^p(\mathbb{R}^d, w)$  to the function  $f(x) = \|f_k(x)\|_{\ell^\infty(X)}$ . If  $1 < q < \infty$  and  $X = \mathbb{R}$ , the result can be found in [5, Theorem 3.1] and [21]. The vector-valued case can be obtained from the scalar case by applying it to  $(\|f_k\|)_{k \geq 0} \subseteq L^p(\mathbb{R}^d, w)$ . ■

For a given function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$  and  $t > 0$ , we define the function  $\varphi_t : \mathbb{R}^d \rightarrow \mathbb{C}$  by

$$\varphi_t(x) := t^n \varphi(tx)$$

The following lemma is well-known to experts.

**LEMMA 2.3.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$  and  $w \in A_p$ . For  $\varphi \in L^1(\mathbb{R}^d)$ , define  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  by*

$$\psi(x) := \sup\{|\varphi(y)| : |y| \geq |x|\}.$$

*If  $\psi \in L^1(\mathbb{R}^d)$ , then there is a constant  $C_{p,w}$  such that for  $f \in L^p(\mathbb{R}^d, w; X)$  one has*

$$\left\| \sup_{t>0} \|\varphi_t * f\| \right\|_{L^p(\mathbb{R}^d, w)} \leq C_{p,w} \|\psi\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d, w; X)}.$$

*Moreover, for  $q \in (1, \infty]$  there is a constant  $C_{p,q,w}$  such that for all  $(t_k)_{k \geq 0} \subseteq \mathbb{R}_+$  and  $(f_k)_{k \geq 0} \subseteq L^p(\mathbb{R}^d, w; \ell^q(X))$ ,*

$$\|(\varphi_{t_k} * f_k)_{k \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))} \leq C_{p,q,w} \|\psi\|_{L^1(\mathbb{R}^d)} \|(f_k)_{k \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}.$$

A partial converse of the lemma holds as well: if for a positive radial decreasing function  $\varphi \neq 0$  one has  $\|\varphi_t * f\|_{L^p(\mathbb{R}^d, w)} \leq C\|f\|_{L^p(\mathbb{R}^d, w)}$  with a constant  $C$  independent of  $f$  and  $t > 0$ , then  $w \in A_p$  (see [40, p. 198]).

*Proof of Lemma 2.3.* Let  $g(x) = \|f(x)\|$ . By definition of  $\psi$  and [14, Theorem 2.1.10], for  $x \in \mathbb{R}^d$  and  $t > 0$  one has

$$(2.2) \quad \begin{aligned} \|\varphi_t * f(x)\| &\leq \int_{\mathbb{R}^d} |\varphi_t(y)| \|f(x-y)\| dy \leq \int_{\mathbb{R}^d} \psi_t(y) g(x-y) dy \\ &\leq \|\psi_t\|_{L^1(\mathbb{R}^d)} Mg(x) = \|\psi\|_{L^1(\mathbb{R}^d)} Mg(x). \end{aligned}$$

Now the first asserted inequality follows from the boundedness of  $M$  on  $L^p(\mathbb{R}^d, w)$ . Further, for all  $q \in (1, \infty]$  inequality (2.2) implies that

$$\|(\varphi_{t_k} * f_k(x))_{k \geq 0}\|_{\ell^q(X)} \leq \|\psi\|_{L^1(\mathbb{R}^d)} \|(Mf_k(x))_{k \geq 0}\|_{\ell^q(X)}, \quad x \in \mathbb{R}^d.$$

Thus the second assertion for  $q \in (1, \infty)$  follows from Proposition 2.2. The case  $q = \infty$  is a consequence of (2.1) and the boundedness of  $M$  on  $L^p(\mathbb{R}^d, w)$ . ■

**2.3. A multiplier theorem.** Let  $X$  be a Banach space, and let  $\mathcal{S}(\mathbb{R}^d; X)$  be the space of  $X$ -valued Schwartz functions. We write  $\mathcal{S}(\mathbb{R}^d)$  in the scalar case  $X = \mathbb{C}$ . The Fourier transform of a function  $f \in \mathcal{S}(\mathbb{R}^d; X)$  is given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

Recall that  $\mathcal{F}$  is a continuous isomorphism on  $\mathcal{S}(\mathbb{R}^d; X)$ . A linear map  $f : \mathcal{S}(\mathbb{R}^d) \rightarrow X$  is called an  $X$ -valued tempered distribution if for all  $\phi \in \mathcal{S}(\mathbb{R}^d)$  there are a constant  $C$  and  $k, N \in \mathbb{N}$  such that

$$\|f(\phi)\| \leq C \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \leq N} (1 + |x|)^k |D^\alpha \phi(x)|.$$

The space of all  $X$ -valued tempered distributions is denoted by  $\mathcal{S}'(\mathbb{R}^d; X)$ . Standard operators (Fourier transform, convolution, etc.) on  $\mathcal{S}'(\mathbb{R}^d; X)$  can be defined as in the scalar case (cf. [1, Section III.4]).

The following result is an extension of [41, Theorem 1.6.3] and [43, formula 15.3(iv)] to the weighted vector-valued setting. For a compact set  $K \subseteq \mathbb{R}^d$ , let  $L^p_K(\mathbb{R}^d, w)$  be the space of functions  $f \in L^p(\mathbb{R}^d, w)$  with  $\text{supp}(\widehat{f}) \subseteq K$  in the sense of distributions.

**PROPOSITION 2.4.** *Let  $X$  be a Banach space,  $p \in [1, \infty)$ ,  $q \in [1, \infty]$ ,  $r \in (0, \min\{p, q\})$  and  $w \in A_{p/r}$ . Let  $K_0, K_1, \dots \subseteq \mathbb{R}^d$  be compact sets with  $\theta_n = \text{diam}(K_n) > 0$  for all  $n$ . Then there is a constant  $C_{p,q,r,w}$  such that for all  $(m_n)_{n \geq 0} \subseteq L^\infty(\mathbb{R}^d)$  and  $(f_n)_{n \geq 0} \subseteq L^p_{K_n}(\mathbb{R}^d, w; X)$  one has*

$$\begin{aligned} &\|(\mathcal{F}^{-1}[m_n \widehat{f}_n])_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))} \\ &\leq C_{p,q,d,r,w} \sup_{k \geq 0} \|(1 + |\cdot|^{d/r}) \mathcal{F}^{-1}[m_k(\theta_k \cdot)]\|_{L^1(\mathbb{R}^d)} \|(f_n)_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))} \\ &\leq C_{p,q,d,r,w,\lambda} \sup_{k \geq 0} \|m_k(\theta_k \cdot)\|_{H^{\lambda,2}(\mathbb{R}^d)} \|(f_n)_{n \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))}, \end{aligned}$$

where  $\lambda > d/2 + d/r$ .

By considering only  $m_1 \neq 0$  and  $f_1 \neq 0$ , one obtains a multiplier theorem on  $L^p_{K_1}(\mathbb{R}^d)$ .

*Proof.* This can be proved as in [41, Theorem 1.6.3] (see also [43, Section 15.3]). Indeed, using Proposition 2.2 one can extend [41, Theorem 1.6.2] to the vector-valued setting with weights in  $A_{p/r}$ . Now the argument in [41, Theorem 1.6.3] can be repeated to obtain the result. ■

**3. Definitions and properties of weighted function spaces.** In this section we define and investigate the basic properties of function spaces with general  $A_\infty$ -weights. The definitions extend those of [8] to the vector-valued setting. In [34] it is shown that one actually only needs a local version of this condition, called  $A^\text{loc}_\infty$ , to obtain reasonable spaces. Since we are mainly interested in power weights  $w(x) = |x|^\gamma$ , for which one easily sees that  $w \in A_\infty$  if and only if  $w \in A^\text{loc}_\infty$ , we restrict ourselves to the  $A_\infty$ -case.

The arguments employed for the basic properties of the spaces are well-known and completely analogous to the unweighted, scalar-valued case (see e.g. [37, 41]). We thus refrain from giving too many details and rather refer to the literature at most of the points.

**3.1. Besov and Triebel–Lizorkin spaces**

DEFINITION 3.1. Let  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  be such that

$$(3.1) \quad 0 \leq \widehat{\varphi}(\xi) \leq 1, \quad \xi \in \mathbb{R}^d, \quad \widehat{\varphi}(\xi) = 1 \text{ if } |\xi| \leq 1, \quad \widehat{\varphi}(\xi) = 0 \text{ if } |\xi| \geq 3/2.$$

Let  $\widehat{\varphi}_0 = \widehat{\varphi}$ ,  $\widehat{\varphi}_1(\xi) = \widehat{\varphi}(\xi/2) - \widehat{\varphi}(\xi)$  and

$$\widehat{\varphi}_k(\xi) = \widehat{\varphi}_1(2^{-k+1}\xi) = \widehat{\varphi}(2^{-k}\xi) - \widehat{\varphi}(2^{-k+1}\xi), \quad \xi \in \mathbb{R}^d, \quad k \geq 1.$$

Let  $\Phi$  be the set of all sequences  $(\varphi_n)_{n \geq 0}$  constructed in the above way from a function  $\varphi$  that satisfies (3.1).

For  $\varphi$  as in the definition and  $f \in \mathcal{S}'(\mathbb{R}^d; X)$  one sets

$$S_k f := \varphi_k * f = \mathcal{F}^{-1}[\widehat{\varphi}_k \widehat{f}],$$

which belongs to  $C^\infty(\mathbb{R}^d; X) \cap \mathcal{S}'(\mathbb{R}^d; X)$  (see [36, Remark 4.3.3]). Since  $\sum_{k \geq 0} \widehat{\varphi}_k(\xi) = 1$  for all  $\xi \in \mathbb{R}^d$ , we have  $\sum_{k \geq 0} S_k f = f$  in the sense of distributions.

DEFINITION 3.2. Let  $X$  be a Banach space,  $p, q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $w \in A_\infty$ . The *Besov space*  $B^s_{p,q}(\mathbb{R}^d; w; X)$  is defined as the space of all  $f \in \mathcal{S}'(\mathbb{R}^d; X)$  for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d; w; X)} := \|(2^{ks} S_k f)_{k \geq 0}\|_{\ell^q(L^p(\mathbb{R}^d, w; X))} < \infty.$$

Moreover, if  $s \in \mathbb{R}_+ \setminus \mathbb{N}$ , then we set

$$W^{s,p}(\mathbb{R}^d, w; X) := B^s_{p,p}(\mathbb{R}^d, w; X).$$

DEFINITION 3.3. Let  $X$  be a Banach space,  $p \in [1, \infty)$ ,  $q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $w \in A_\infty$ . The *Triebel–Lizorkin space*  $F_{p,q}^s(\mathbb{R}^d, w; X)$  is defined as the space of all  $f \in \mathcal{S}'(\mathbb{R}^d; X)$  for which

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} := \|(2^{ks} S_k f)_{k \geq 0}\|_{L^p(\mathbb{R}^d, w; \ell^q(X))} < \infty.$$

If  $w \equiv 1$ , we write  $B_{p,q}^s(\mathbb{R}^d; X)$  for  $B_{p,q}^s(\mathbb{R}^d, w; X)$  and  $F_{p,q}^s(\mathbb{R}^d; X)$  for  $F_{p,q}^s(\mathbb{R}^d, w; X)$ . As in the scalar case, one can show that these are Banach spaces. Note that

$$\begin{aligned} \|f\|_{B_{p,q}^s(\mathbb{R}^d, w; X)} &= \left( \sum_{k \geq 0} 2^{ks} \|S_k f\|_{L^p(\mathbb{R}^d, w; X)}^q \right)^{1/q}, \\ \|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} &= \left\| \left( \sum_{k \geq 0} 2^{ks} \|S_k f\|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d, w)}, \end{aligned}$$

with the usual modifications for  $q = \infty$ .

The following is a consequence of Proposition 2.4.

PROPOSITION 3.4 (Independence from  $\varphi$ ). *Let  $X$  be a Banach space and  $w \in A_\infty$ .*

- (1) *For all  $s \in \mathbb{R}$  and  $p, q \in [1, \infty]$ , the space  $B_{p,q}^s(\mathbb{R}^d, w; X)$  is independent of the choice  $(\varphi_n)_{n \geq 0} \in \Phi$ . Any  $(\psi_n)_{n \geq 0} \in \Phi$  leads to an equivalent norm in  $B_{p,q}^s(\mathbb{R}^d, w; X)$ .*
- (2) *For all  $s \in \mathbb{R}$ ,  $p \in [1, \infty)$  and  $q \in [1, \infty]$ , the space  $F_{p,q}^s(\mathbb{R}^d, w; X)$  is independent of the choice  $(\varphi_n)_{n \geq 0} \in \Phi$ . Any  $(\psi_n)_{n \geq 0} \in \Phi$  leads to an equivalent norm in  $F_{p,q}^s(\mathbb{R}^d, w; X)$ .*

*Proof.* Choose  $r \in (0, \min\{p, q\})$  such that  $w \in A_{p/r}$ . Then one can apply Proposition 2.4 in the same way as in the unweighted case in [41, Section 2.3.2] and [43, Section 15.5]. ■

REMARK 3.5.

- (i) At a technical point in the proofs,  $w \in A_\infty$  is required to have the boundedness of the Hardy–Littlewood maximal function and of the Fefferman–Stein maximal function in some  $L^r$ -space and some  $L^r(\ell^q)$ -space with  $r \in (1, \infty)$ , respectively.
- (ii) Definition 3.3 can be extended to  $p = \infty$  and  $q \in [1, \infty]$ , but Proposition 3.4 is not true in this setting (see [41, Remark 2.3.1/4]).
- (iii) Rychkov [34] considers scalar  $B$ - and  $F$ -spaces with more general weights of class  $A_\infty^{\text{loc}}$ , i.e., satisfying only a local  $A_\infty$  condition.

### 3.2. Sobolev and Bessel-potential spaces

DEFINITION 3.6. Let  $X$  be a Banach space,  $m \in \mathbb{N}$ ,  $p \in [1, \infty]$ , and let  $w$  be a weight. The *Sobolev space*  $W^{m,p}(\mathbb{R}^d, w; X)$  is defined as the space of

functions  $f \in L^p(\mathbb{R}^d; w, X)$  for which  $D^\alpha f$ , taken in the distributional sense, is in  $L^p(\mathbb{R}^d, w; X)$  for all multiindices  $\alpha$  with  $|\alpha| \leq m$ . Let

$$\|f\|_{W^{m,p}(\mathbb{R}^d,w;X)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\mathbb{R}^d,w;X)}.$$

For  $s \in \mathbb{R}$  and  $f \in \mathcal{S}'(\mathbb{R}^d; X)$ , the Bessel potential  $J_s f \in \mathcal{S}'(\mathbb{R}^d; X)$  is defined by

$$J_s f := \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2} \widehat{f}].$$

Obviously,  $J_{s_1} J_{s_2} = J_{s_1+s_2}$  for  $s_1, s_2 \in \mathbb{R}$ , and  $J_0$  is the identity mapping on  $\mathcal{S}'(\mathbb{R}^d; X)$ .

DEFINITION 3.7. Let  $X$  be a Banach space,  $s \in \mathbb{R}$ ,  $p \in [1, \infty)$ , and let  $w$  be a weight. The *Bessel-potential space*  $H^{s,p}(\mathbb{R}^d, w; X)$  is defined as the space of all  $f \in \mathcal{S}'(\mathbb{R}^d; X)$  for which  $J_s f \in L^p(\mathbb{R}^d, w; X)$ . Let

$$\|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} := \|J_s f\|_{L^p(\mathbb{R}^d,w;X)}.$$

It is immediate from the definition that

$$(3.2) \quad J_\sigma : H^{s,p}(\mathbb{R}^d, w; X) \rightarrow H^{s-\sigma,p}(\mathbb{R}^d, w; X) \quad \text{isomorphically.}$$

Moreover,  $W^{0,p}(\mathbb{R}^d, w; X) = H^{0,p}(\mathbb{R}^d, w; X) = L^p(\mathbb{R}^d, w; X)$ . Certain embeddings and identities between these spaces and Triebel–Lizorkin spaces hold under geometric assumptions on  $X$  (see (1.8) and (1.9)).

**3.3. Density, lifting property, equivalent norms.** The elementary properties of the  $A_\infty$ -weighted spaces are the same as in the unweighted case. Proposition 2.4 allows us to carry over the proofs of [41, Section 2.3] and [43, Section 15] to the weighted setting.

LEMMA 3.8. *Let  $X$  be a Banach space,  $p \in (1, \infty)$ ,  $q \in [1, \infty)$ ,  $s \in \mathbb{R}$ . Let  $w \in A_\infty$ . The set  $\mathcal{S}(\mathbb{R}^d; X)$  is dense in  $B_{p,q}^s(\mathbb{R}^d, w; X)$ ,  $F_{p,q}^s(\mathbb{R}^d, w; X)$  and  $H^{s,p}(\mathbb{R}^d, w; X)$ .*

*Proof.* Let us consider  $F_{p,q}^s(\mathbb{R}^d, w; X)$ . Using Proposition 2.4, the same arguments as in Step 5 of the proof of [41, Theorem 2.3.3] show that  $f_N = \sum_{k=0}^N S_k f$  converges to  $f$  in  $F_{p,q}^s(\mathbb{R}^d, w; X)$  as  $N \rightarrow \infty$ . Still following [41], let  $\eta \in \mathcal{S}(\mathbb{R}^d)$  with  $\eta(0) = 1$  and  $\text{supp}(\hat{\eta}) \subseteq B(0, 1)$ . Since  $\mathcal{F}\eta(\delta \cdot) = \delta^{-d} \hat{\eta}(\delta^{-1} \cdot)$ , the support of  $\mathcal{F}(\eta(\delta \cdot) f_N)$  is for all  $\delta \in (0, 1)$  contained in a ball that only depends on  $N$ . Applying again Proposition 2.4, we deduce that there is  $C > 0$ , independent of  $\delta$ , such that

$$\|f_N - \eta(\delta \cdot) f_N\|_{F_{p,q}^s(\mathbb{R}^d,w;X)} \leq C \|f_N - \eta(\delta \cdot) f_N\|_{L^p(\mathbb{R}^d,w;X)},$$

and the right-hand side tends to zero as  $\delta \rightarrow 0$ . Since  $\eta(\delta \cdot) f_N \in \mathcal{S}(\mathbb{R}^d; X)$ , the assertion for the Triebel–Lizorkin spaces follows. Similiar arguments show the assertion for the Besov spaces.

For the density of  $\mathcal{S}(\mathbb{R}^d)$  in  $L^p(\mathbb{R}^d, w)$  see [15, Exercise 9.4.1]. The invariance of  $\mathcal{S}(\mathbb{R}^d; X)$  under the Bessel potential  $J_s$  gives the density in  $H^{s,p}(\mathbb{R}^d, w; X)$ . ■

PROPOSITION 3.9. *Let  $X$  be a Banach space,  $p \in [1, \infty)$ ,  $q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $w \in A_\infty$ . Then for all  $\sigma \in \mathbb{R}$ ,*

$$(3.3) \quad J_\sigma : B_{p,q}^s(\mathbb{R}^d, w; X) \rightarrow B_{p,q}^{s-\sigma}(\mathbb{R}^d, w; X) \quad \text{isomorphically,}$$

$$(3.4) \quad J_\sigma : F_{p,q}^s(\mathbb{R}^d, w; X) \rightarrow F_{p,q}^{s-\sigma}(\mathbb{R}^d, w; X) \quad \text{isomorphically.}$$

*Proof.* Choose  $r \in (0, \min\{p, q\})$  such that  $w \in A_{p/r}$ . Using Proposition 2.4, the same proof as in the unweighted case gives the assertions (see [41, Theorem 2.3.8]). ■

PROPOSITION 3.10. *Let  $X$  be a Banach space,  $p \in [1, \infty)$ ,  $q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $w \in A_\infty$ . Then for all  $k \in \mathbb{N}$ ,*

$$(3.5) \quad \sum_{|\alpha| \leq k} \|D^\alpha f\|_{B_{p,q}^{s-k}(\mathbb{R}^d, w; X)} \text{ defines an equivalent norm on } B_{p,q}^s(\mathbb{R}^d, w; X)$$

and

$$(3.6) \quad \sum_{|\alpha| \leq k} \|D^\alpha f\|_{F_{p,q}^{s-k}(\mathbb{R}^d, w; X)} \text{ defines an equivalent norm on } F_{p,q}^s(\mathbb{R}^d, w; X).$$

*Proof.* In the unweighted scalar case, these results are shown in [41, Theorem 2.3.8]. The proofs are essentially based on a multiplier theorem of Mihlin–Hörmander type in Besov and Triebel–Lizorkin spaces; see [41, Theorem 2.3.7] for the scalar and [43, Section 15.6] for the vector-valued case. Using Proposition 2.4, the proof given in [43, Section 15.6] carries over to the weighted setting, for all  $w \in A_\infty$ . ■

**3.4. Elementary embeddings.** The elementary embedding properties and their proofs for the above vector-valued function spaces are the same as in the unweighted case (see [41, Section 2.3.2]):

PROPOSITION 3.11. *Let  $X$  be a Banach space and  $w \in A_\infty$ .*

(1) *For all  $1 \leq q_0 \leq q_1 \leq \infty$  and  $s \in \mathbb{R}$ ,*

$$B_{p,q_0}^s(\mathbb{R}^d, w; X) \hookrightarrow B_{p,q_1}^s(\mathbb{R}^d, w; X), \quad p \in [1, \infty],$$

$$F_{p,q_0}^s(\mathbb{R}^d, w; X) \hookrightarrow F_{p,q_1}^s(\mathbb{R}^d, w; X), \quad p \in [1, \infty).$$

(2) *For all  $q_0, q_1 \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $\varepsilon > 0$ ,*

$$B_{p,q_0}^{s+\varepsilon}(\mathbb{R}^d, w; X) \hookrightarrow B_{p,q_1}^s(\mathbb{R}^d, w; X), \quad p \in [1, \infty],$$

$$F_{p,q_0}^{s+\varepsilon}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,q_1}^s(\mathbb{R}^d, w; X), \quad p \in [1, \infty).$$

For all  $q \in [1, \infty]$ ,  $s \in \mathbb{R}$  and  $p \in [1, \infty)$ ,

$$B_{p,\min\{p,q\}}^s(\mathbb{R}^d, w; X) \hookrightarrow F_{p,q}^s(\mathbb{R}^d, w; X) \hookrightarrow B_{p,\max\{p,q\}}^s(\mathbb{R}^d, w; X).$$

$H$ -spaces are related to  $B$ - and  $F$ -spaces as follows.

PROPOSITION 3.12. *Let  $X$  be a Banach space,  $s \in \mathbb{R}$ ,  $p \in (1, \infty)$ , and  $w \in A_p$ . Then*

$$(3.7) \quad B_{p,1}^s(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow B_{p,\infty}^s(\mathbb{R}^d, w; X),$$

$$(3.8) \quad F_{p,1}^s(\mathbb{R}^d, w; X) \hookrightarrow H^{s,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^s(\mathbb{R}^d, w; X).$$

Moreover, if  $m \in \mathbb{N}$ , then

$$(3.9) \quad B_{p,1}^m(\mathbb{R}^d, w; X) \hookrightarrow W^{m,p}(\mathbb{R}^d, w; X) \hookrightarrow B_{p,\infty}^m(\mathbb{R}^d, w; X),$$

$$(3.10) \quad F_{p,1}^m(\mathbb{R}^d, w; X) \hookrightarrow W^{m,p}(\mathbb{R}^d, w; X) \hookrightarrow F_{p,\infty}^m(\mathbb{R}^d, w; X).$$

*Proof.* Using Propositions 3.9 and 3.10, it suffices to consider the cases  $s = m = 0$ . But then (3.9) and (3.10) are the same as (3.7) and (3.8). Further, (3.7) follows from (3.8) and Proposition 3.11. To prove (3.8), we extend the argument in [38, Proposition 2].

For  $f \in \mathcal{S}(\mathbb{R}^d; X)$  we have  $f = \sum_{n \geq 0} S_n f$  almost everywhere on  $\mathbb{R}^d$ . Therefore

$$\|f\|_X \leq \sum_{n \geq 0} \|S_n f\|_X,$$

and the first part of (3.8) follows by taking  $L^p(\mathbb{R}^d, w)$ -norms and appealing to Lemma 3.8.

Next let  $f \in L^p(\mathbb{R}^d, w; X)$ . By the definition of  $S_n$  and  $\varphi_n$  for each  $x \in \mathbb{R}^d$ , one has

$$\begin{aligned} \|S_n f(x)\|_X &\leq \int_{\mathbb{R}^d} |\varphi_n(y)| \|f(x-y)\|_X dy \\ &\leq 2 \sup_{n \geq 0} \int_{\mathbb{R}^d} |2^{nd} \varphi_0(2^n y)| \|f(x-y)\|_X dy. \end{aligned}$$

Let  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  be defined by  $\psi(y) = \sup\{\varphi_0(z) : |z| \geq |y|\}$ . It follows from Lemma 2.3 that

$$\|f\|_{F_{p,\infty}^0(\mathbb{R}^d, w; X)} = \left\| \sup_{n \geq 0} \|S_n f\|_X \right\|_{L^p(\mathbb{R}^d, w)} \leq 2 \|\psi\|_{L^1(\mathbb{R}^d)} C_{p,w} \|f\|_{L^p(\mathbb{R}^d, w; X)},$$

which proves the second part of (3.8). ■

REMARK 3.13. The proof shows that the embeddings on the left-hand sides in the above proposition are also true for  $p = 1$  and  $w \in A_\infty$ . For the embeddings on the right-hand side this is different:

For  $p \in (1, \infty)$ , let  $A_p^{\text{loc}}$  be the class of weights defined in [34], i.e.,  $w \in A_p^{\text{loc}}$  if

$$\sup_{|Q| \leq 1} \frac{1}{|Q|^p} \left( \int_Q w \, dx \right) \left( \int_Q w' \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes  $Q$  in  $\mathbb{R}^d$  with sides parallel to the coordinate axes. For weights of the form  $w(x) = |x|^\gamma$  one can check that  $w \in A_p^{\text{loc}}$  if and only if  $w \in A_p$ . Now we claim that

$$(3.11) \quad L^p(\mathbb{R}^d, w) \hookrightarrow F_{p, \infty}^0(\mathbb{R}^d, w)$$

if and only if  $w \in A_p^{\text{loc}}$ .

*Proof.* If  $w \in A_p^{\text{loc}}$ , then (3.11) follows from  $L^p(\mathbb{R}^d, w) = F_{p, 2}^0(\mathbb{R}^d, w)$  (see [34, p. 178]) and the embedding  $\ell^2 \hookrightarrow \ell^\infty$ .

Conversely, assume that (3.11) holds for a weight  $w$ . Let  $(\varphi_j)_{j \geq 0} \in \Phi$ . Using the continuity of  $\varphi_1$  and  $\varphi_1(0) > 0$ , we can find  $c > 0$  and  $N \in \mathbb{N}$  such that  $\text{Re}(\varphi(x)) \geq c$  for all  $|x| \leq d2^{-N+2}$ . Let  $Q$  be a cube with  $|Q| \leq 2^{-Nd}$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a function which satisfies  $f \geq 0$  on  $Q$  and  $f = 0$  on  $\mathbb{R}^d \setminus Q$ . Let  $j \in \mathbb{N}$  be such that  $2^{-(j+N)d} \leq |Q| < 2^{-(j+N-1)d}$ . Denoting by  $\ell(Q)$  the maximal axis length of  $Q$ , we have  $\ell(Q) < 2^{-(j+N-1)}$ . Now for every  $x \in Q$ ,

$$|\varphi_{j+1} * f(x)| = 2^{jd} \left| \int_Q \varphi_1(2^j(x-y)) f(y) \, dy \right| \geq 2^{jd} c \int_Q f(y) \, dy \geq \frac{C}{|Q|} \int_Q f(y) \, dy,$$

where we used the fact that

$$|2^j(x-y)| \leq 2^j 2d\ell(Q) < d2^{-N+2} \quad \text{for all } x, y \in Q.$$

Let  $\lambda = \frac{C}{|Q|} \int_Q f(y) \, dy$ . The above estimate and (3.11) yield

$$\int_Q w(x) \, dx \leq \lambda^{-p} \int_Q |\varphi_{j+1} * f(x)|^p w(x) \, dx \leq C \lambda^{-p} \|f\|_{L^p(\mathbb{R}^d, w)}^p.$$

Rewriting this gives

$$\left( \int_Q w(x) \, dx \right) \left( \int_Q f(y) \, dy \right)^p \leq C |Q|^p \int_Q |f(x)|^p w(x) \, dx.$$

As in [24, equation (3.12)] (basically by taking  $f = \mathbf{1}_Q w'$ ) this implies

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w' \, dy \right)^{p-1} \leq C.$$

Since the definition of  $A_p^{\text{loc}}$  is independent of the upper bound for the cube size (see [34, Remark 1.5]) we obtain  $w \in A_p^{\text{loc}}$ . ■

**4. Embeddings for Besov spaces—proof of Theorem 1.1.** From now on we specialize to power weights, i.e.,  $w(x) = |x|^\gamma$  with  $\gamma > -d$ . We first consider sufficient conditions for the embedding, and in the next subsection show their optimality.

**4.1. Sufficient conditions.** In this subsection we prove the sufficiency part for the embedding for Besov spaces, i.e. Theorem 1.1(2) $\Rightarrow$ (1). The main ingredient of the proof is the following two-weight version of an inequality of Plancherel–Pólya–Nikol’skiĭ type. As already mentioned in the introduction, a completely different proof for the scalar version of Theorem 1.1 is given in [18].

PROPOSITION 4.1. *Let  $X$  be a Banach space and let  $1 < p_0, p_1 \leq \infty$ . Let  $\gamma_0, \gamma_1 > -d$  and  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$ . Suppose*

$$(4.1) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0}.$$

*Let  $f : \mathbb{R}^d \rightarrow X$  be a function with  $\text{supp}(\widehat{f}) \subseteq \{x \in \mathbb{R}^d : |x| < t\}$ , where  $t > 0$  is fixed. Then for any multiindex  $\alpha$  there is a constant  $C_\alpha$ , independent of  $f$  and  $t$ , such that*

$$(4.2) \quad \|D^\alpha f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq C_\alpha t^{|\alpha| + \delta} \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)},$$

where  $\delta = (d + \gamma_0)/p_0 - (d + \gamma_1)/p_1 > 0$ .

REMARK 4.2.

- (i) Suppose that (4.2) holds true for  $\alpha = 0$  and all  $f$  as in the proposition. Then it follows from the proof of Theorem 1.1 given below that the embedding (1.1) for Besov spaces holds true (with suitably chosen  $s_0, s_1$ ). By the necessary conditions for this embedding obtained in Propositions 4.7 and 4.9 below, we conclude that (4.2) holds true if and only if either  $p_0 = p_1$  and  $\gamma_0 = \gamma_1$  or (4.1) are satisfied.
- (ii) The case where  $w_0 = w_1$  is an  $A_\infty$ -weight with the property that  $\inf_{x \in \mathbb{R}^d} w(B(x, t)) \geq t^\varepsilon$  with  $\varepsilon > 0$  is considered in [8, Lemma 2.5]. A part of the argument in [8, Lemma 2.5] will be repeated in (4.4) below, because the details are needed again at a later point.
- (iii) It would be interesting to find a two-weight characterization for (4.2) for general weights  $w_0$  and  $w_1$  in the case of  $t = 1$  or more general  $t$ . There might be a connection to [18, Proposition 2.1].
- (iv) Certain weighted versions of inequalities of Nikol’skiĭ type can also be found in [41, Sections 1.3.4 and 6.2.3]. However, the power weights we consider are not covered by those results.
- (v) It follows from the proof below that for  $\gamma_0, \gamma_1 \geq 0$ , Proposition 4.1 holds for all  $p_0, p_1 \in (0, \infty)$  which satisfy (4.1).

For the proof of the proposition we make use of the following weighted version of Young’s inequality (see [19, Theorem 3.4, (3.7)] or [9, Theorem 2.2(ii)]). The proof is based on the Stein–Weiss result on fractional integration (see [6] for a short proof).

LEMMA 4.3. *Let  $1 < q \leq r < \infty$  and  $a, b, c \in \mathbb{R}$  be such that*

$$\frac{1}{r} = \frac{1}{q} - 1 + \frac{a + b + c}{d}, \quad b + c \geq 0,$$

$$0 < a < d, \quad b < d\left(1 - \frac{1}{q}\right), \quad c < \frac{d}{r}.$$

*Then there is a constant  $C$  such that for all measurable functions  $f$  and  $g$ ,*

$$(4.3) \quad \| |x \mapsto f * g(x)|x|^{-c} \|_{L^r(\mathbb{R}^d)} \leq C \| |x \mapsto f(x)|x|^a \|_{L^\infty(\mathbb{R}^d)} \| |x \mapsto g(x)|x|^b \|_{L^q(\mathbb{R}^d)}.$$

REMARK 4.4. In [9, Theorem 2.1] several necessary conditions for weighted Young’s inequalities are obtained. In our situation we obtain another necessary condition which also appears in the sufficient conditions for (4.3) in [9, condition (14)]. In fact, it follows from the proof below that if (4.3) holds for some  $b$  and  $c$  with  $b + c < 0$ , then one obtains (1.1) with  $\gamma_1/p_1 > \gamma_0/p_0$ . This is impossible according to Theorem 1.1. Therefore,  $b + c \geq 0$  from Lemma 4.3 is also necessary for (4.3). With some additional arguments one can derive the same necessary condition if the  $L^\infty$ -norm in (4.3) is replaced by an  $L^r$ -norm. In a similar way one can see that  $a < d$  from Lemma 4.3 is necessary for (4.3).

The next lemma is stated without proof in [8, condition  $(B_p)$ ]. We include a proof for convenience.

LEMMA 4.5. *Let  $p \in (1, \infty)$  and  $w \in A_p$ . Then there is a constant  $C$  such that for all  $x \in \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} w(y)(1 + |x - y|)^{-dp} dy \leq C \int_{B(x,1)} w(y) dy.$$

*Proof.* If we let  $g_x(z) = \mathbf{1}_{B(x,1)}(z)$  and  $s_0 = |x - y| + 1$ , then its maximal function  $Mg_x$  satisfies

$$(Mg_x)(y) = \sup_{s>0} \frac{|B(y, s) \cap B(x, 1)|}{|B(y, s)|} \geq \frac{|B(y, s_0) \cap B(x, 1)|}{|B(y, s_0)|}$$

$$= \frac{|B(x, 1)|}{|B(y, s_0)|} = s_0^{-d} = (|x - y| + 1)^{-d}.$$

Therefore, since  $w \in A_p$ , we see that

$$\begin{aligned} \int_{\mathbb{R}^d} w(y)(1 + |x - y|)^{-dp} dy &\leq \|Mg_x\|_{L^p(\mathbb{R}^d, w)}^p \\ &\leq C \|g_x\|_{L^p(\mathbb{R}^d, w)}^p = C \int_{B(x,1)} w(y) dy. \blacksquare \end{aligned}$$

*Proof of Proposition 4.1.* By a scaling argument it suffices to consider the case  $t = 1$ . Let  $B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$ . For  $\alpha \neq 0$ , the same arguments as in the proof of [41, Proposition 1.3.2] show that

$$\|D^\alpha f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq C_\alpha \|f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)}$$

for all  $p_1 \in (1, \infty]$  and  $\gamma_1 > -d$ . We may thus restrict to the case  $\alpha = 0$ .

The proof of (4.2) for  $\alpha = 0$  is split into several cases of which some are overlapping. In Case 1 we treat  $p_1 = \infty$ , in Cases 2–6 we consider  $p_0 \leq p_1 < \infty$ , and in Case 7 we derive the estimate for  $p_0 > p_1$ .

CASE 1:  $p_0 < \infty$  and  $p_1 = \infty$ . Then  $\gamma_0 \geq 0$  by assumption, and (4.2) is independent of  $\gamma_1$  due to  $L^\infty(\mathbb{R}^d, w_1; X) = L^\infty(\mathbb{R}^d; X)$ . We follow the arguments given in [8, Lemma 2.5]. First assume that  $f \in L^\infty(\mathbb{R}^d; X)$ . Let  $\eta \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\text{supp}(\hat{\eta}) \subseteq B_2$  and  $\hat{\eta} = 1$  on  $B_1$ . Then one has  $f = f * \eta$ . Let  $q \in (0, 1)$  be so small that  $\gamma_0/(r - 1) < d$ , where  $r = p_0/q$ . Then  $w'_0 = w_0^{-1/(r-1)} \in A_{r'}$ . For any  $x \in \mathbb{R}^d$ , Hölder’s inequality with  $1/r + 1/r' = 1$  yields

$$\begin{aligned} (4.4) \quad \|f(x)\| &\leq \int_{\mathbb{R}^d} \|f(y)\| |\eta(x - y)| dy \\ &\leq \|f\|_{L^\infty(\mathbb{R}^d; X)}^{1-q} \int_{\mathbb{R}^d} (\|f(y)\| |y|^{\gamma_0/p_0})^q |y|^{-q\gamma_0/p_0} |\eta(x - y)| dy \\ &\leq \|f\|_{L^\infty(\mathbb{R}^d; X)}^{1-q} \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}^q \left( \int_{\mathbb{R}^d} w'_0(y) |\eta(x - y)|^{r'} dy \right)^{1/r'}. \end{aligned}$$

Now since  $\eta$  is a Schwartz function, there is a constant  $C$  such that  $|\eta(y)|^{r'} \leq C(1 + |y|)^{-dr'}$  for all  $y \in \mathbb{R}^d$ . Therefore, it follows from Lemma 4.5 and  $-\gamma/(r - 1) > -d$  that for all  $x \in \mathbb{R}^d$  we have

$$\begin{aligned} \left( \int_{\mathbb{R}^d} w'_0(y) |\eta(x - y)|^{r'} dy \right)^{1/r'} &\leq C \int_{B(x,1)} w'_0(y) dy \\ &= C \int_{B(x,1)} |y|^{-\gamma_0/(r-1)} dy \leq C_{p_0, \gamma_0, d}. \end{aligned}$$

Combining this with (4.4) we obtain

$$(4.5) \quad \|f\|_{L^\infty(\mathbb{R}^d; X)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}.$$

If  $f \notin L^\infty(\mathbb{R}^d; X)$ , then as in [8] we take a function  $\phi \in \mathcal{S}(\mathbb{R}^d)$  with  $\phi(0) = 1$  and  $\text{supp}(\widehat{\phi}) \subseteq B_1$ . Since  $f$  is smooth, we have  $\phi(r \cdot) f \in L^\infty(\mathbb{R}^d; X)$  for all  $r > 0$ . For  $x \in \mathbb{R}^d$ , the embedding (4.5) and the dominated convergence theorem imply that

$$\begin{aligned} \|f(x)\| &= \lim_{r \rightarrow 0} \|\phi(rx)f(x)\| \leq C \lim_{r \rightarrow 0} \|\phi(r \cdot) f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)} \\ &= C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}. \end{aligned}$$

CASE 2:  $p_0 \leq p_1 < \infty$  and  $\gamma := \gamma_0 = \gamma_1 \geq 0$ . Let  $w(x) = |x|^\gamma$ . We can assume  $p_0 < p_1$ , since the other case is trivial. By Case 1 we know that  $f \in L^\infty(\mathbb{R}^d; X)$ , and from (4.5) we obtain

$$\|f\|_{L^{p_1}(\mathbb{R}^d, w; X)} \leq \|f\|_{L^\infty(\mathbb{R}^d; X)}^{1-p_0/p_1} \|f\|_{L^{p_0}(\mathbb{R}^d, w; X)}^{p_0/p_1} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w; X)}.$$

CASE 3:  $p_0 \leq p_1 < \infty$  and  $0 \leq \gamma_1 < \gamma_0$ . First consider the integral over  $\mathbb{R}^d \setminus B_1$ . Since  $|x|^{\gamma_1} \leq |x|^{\gamma_0}$  for  $|x| \geq 1$ , one has

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_1; X)} &\leq \|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_0; X)} \leq \|f\|_{L^{p_1}(\mathbb{R}^d, w_0; X)} \\ &\leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}, \end{aligned}$$

where in the last step we applied Case 2. For the integral over  $B_1$ , let  $p = p_1 \gamma_0 / \gamma_1 > p_1$ . We apply Hölder’s inequality with  $\gamma_1 / \gamma_0 + 1/r = 1$ , to obtain

$$\|f\|_{L^{p_1}(B_1, w_1; X)} \leq \|f\|_{L^p(\mathbb{R}^d, w_0; X)} |B_1|^{1/p_1 r} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)},$$

where in the last step we used Case 2.

CASE 4:  $p_0 \leq p_1 < \infty$  and  $0 \leq \gamma_0 < \gamma_1$ . Then necessarily  $p_0 < p_1$ . First consider the integral over  $B_1$ . Since  $\gamma_1 > \gamma_0$ , we have

$$\|f\|_{L^{p_1}(B_1, w_1; X)} \leq \|f\|_{L^{p_1}(B_1, w_0; X)} \leq \|f\|_{L^{p_1}(\mathbb{R}^d, w_0; X)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)},$$

where in the last step we applied Case 2. Next consider  $\mathbb{R}^d \setminus B_1$ . We have

$$\begin{aligned} (4.6) \quad \|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_1; X)} &= \left( \int_{\mathbb{R}^d \setminus B_1} \|f(x)\|^{p_0} |x|^{\gamma_0} \|f(x)\|^{p_1 - p_0} |x|^{\gamma_1 - \gamma_0} dx \right)^{1/p_1} \\ &\leq \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}^{p_0/p_1} \left( \sup_{x \in \mathbb{R}^d \setminus B_1} \|f(x)\| |x|^{(\gamma_1 - \gamma_0)/(p_1 - p_0)} \right)^{1 - p_0/p_1}. \end{aligned}$$

Now fix  $x \in \mathbb{R}^d$  with  $|x| \geq 1$ . Since  $f \in L^\infty(\mathbb{R}^d; X)$ , inequality (4.4) implies

$$\|f(x)\| \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)} \left( \int_{\mathbb{R}^d} w'(y) |\eta(x - y)|^{r'} dy \right)^{1/(r'q)},$$

where  $r = p_0/q$  as in Case 1 and  $w'(y) = |y|^{-\gamma_0/(r-1)}$ . By Lemma 4.5, for

$|x| \geq 1$  one can estimate

$$\int_{\mathbb{R}^d} w'(y) |\eta(x-y)|^{r'} dy \leq C \int_{B(x,1)} w'(y) dy \leq C|x|^{-\gamma_0/(r-1)},$$

and thus  $(r-1)r'q = p_0$  yields

$$(4.7) \quad \|f(x)\| \leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)} |x|^{-\gamma_0/p_0}, \quad x \in \mathbb{R}^d \setminus B_1.$$

Substituting (4.7) into (4.6), we obtain

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_1; X)} &\leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)} \sup_{x \in \mathbb{R}^d \setminus B_1} |x|^{-\gamma_0/p_0 + (\gamma_1 - \gamma_0)/(p_1 - p_0)} \\ &\leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}, \end{aligned}$$

where the last estimate is a consequence of  $|x| \geq 1$  and  $\gamma_1/p_1 \leq \gamma_0/p_0$ .

CASE 5:  $p_0 \leq p_1 < \infty$ ,  $-d < \gamma_0 < d(p_0 - 1)$  and  $-d < \gamma_1$ . Then  $\gamma_1 < d(p_1 - 1)$  by assumption. Let  $\eta \in \mathcal{S}(\mathbb{R}^d)$  be as in Case 1. For all  $x \in \mathbb{R}^d$  we then have

$$\|f(x)\| \leq g * |\eta|(x)$$

where  $g(x) = \|f(x)\|$ . Set

$$r = p_1, \quad q = p_0, \quad a = d - \frac{d + \gamma_0}{p_0} + \frac{d + \gamma_1}{p_1}, \quad b = \frac{\gamma_0}{p_0}, \quad c = -\frac{\gamma_1}{p_1}.$$

We claim the conditions of Lemma 4.3 hold. Indeed, note that  $b + c \geq 0$  is equivalent to  $\gamma_1/p_1 \leq \gamma_0/p_0$ ,  $a > 0$  follows from  $-\gamma_0 > -d(p_0 - 1)$  and  $\gamma_1 > -d$ , and  $a < d$  follows from  $-(d + \gamma_0)/p_0 + (d + \gamma_0)/p_0 < 0$ . The estimate  $b < d(1 - 1/q)$  is equivalent to  $\gamma_0 < d(p_0 - 1)$ , and  $c < d/r$  is equivalent to  $\gamma_1 > -d$ . Therefore, Lemma 4.3 yields

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} &\leq \|g * |\eta|\|_{L^{p_1}(\mathbb{R}^d, w_1)} \\ &\leq C\|x \mapsto |x|^a \eta(x)\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^{p_0}(\mathbb{R}^d, w_0)} = C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}. \end{aligned}$$

CASE 6:  $p_0 \leq p_1 < \infty$ ,  $d(p_0 - 1) \leq \gamma_0$  and  $-d < \gamma_1 < 0$ . Take an arbitrary  $\gamma \in (0, d(p_0 - 1))$ , and set  $w(x) = |x|^\gamma$ . Then  $\gamma_1/p_1 \leq 0 < \gamma/p_0 < \gamma_0/p_0$ . Therefore, by Cases 5 and 3 we have

$$\|f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w; X)} \leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}.$$

CASE 7:  $p_1 < p_0 < \infty$  and  $\gamma_0, \gamma_1 > -d$ . For  $\varepsilon > 0$  we use Hölder's inequality with respect to the finite measure  $|x|^{-d-\varepsilon} dx$  on  $\mathbb{R}^d \setminus B_1$ , to obtain

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_1)} &= \left( \int_{\mathbb{R}^d \setminus B_1} \|f(x)\|^{p_1} |x|^{\gamma_1 + d + \varepsilon} |x|^{-d - \varepsilon} dx \right)^{1/p_1} \\ &\leq C \left( \int_{\mathbb{R}^d \setminus B_1} \|f(x)\|^{p_0} |x|^{\frac{p_0}{p_1}(\gamma_1 + d + \varepsilon) - d - \varepsilon} dx \right)^{1/p_0}. \end{aligned}$$

The assumption  $(\gamma_1 + d)/p_1 < (\gamma_0 + d)/p_0$  implies that  $(p_0/p_1)(\gamma_1 + d + \varepsilon) - d - \varepsilon \leq \gamma_0$  for sufficiently small  $\varepsilon$ . Hence

$$\|f\|_{L^{p_1}(\mathbb{R}^d \setminus B_1, w_1)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

To estimate the weighted  $L^{p_1}$ -norms over  $B_1$ , for  $\varepsilon > 0$  we use Hölder's inequality with respect to the finite measure  $|x|^{-d+\varepsilon} dx$  on  $B_1$ , which gives

$$\begin{aligned} \|f\|_{L^{p_1}(B_1, w_1; X)} &= \left( \int_{B_1} \|f(x)\|^{p_1} |x|^{\gamma_1+d-\varepsilon} |x|^{-d+\varepsilon} dx \right)^{1/p_1} \\ &\leq C \left( \int_{B_1} \|f(x)\|^{p_0} |x|^{\tilde{\gamma}_0} dx \right)^{1/p_0} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, \tilde{w}_0; X)}. \end{aligned}$$

In the last line we have set  $\tilde{w}_0(x) = |x|^{\tilde{\gamma}_0}$  with  $\tilde{\gamma}_0 := (p_0/p_1)(\gamma_1 + d - \varepsilon) - d + \varepsilon$ . Observe that if  $\varepsilon$  is sufficiently small, then the assumptions  $\gamma_1 + d > 0$  and  $(d + \gamma_1)/p_1 < (d + \gamma_0)/p_0$  imply  $-d < \tilde{\gamma}_0$  and  $\tilde{\gamma}_0/p_0 < \gamma_0/p_0$ , respectively. Therefore, by Cases 2–6 we obtain

$$\|f\|_{L^{p_0}(\mathbb{R}^d, \tilde{w}_0; X)} \leq \|f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}.$$

Combining these estimates yields (4.2). ■

*Proof of Theorem 1.1(2)⇒(1).* If (1.2) holds, then the embedding (1.1) follows from Proposition 3.11. Moreover, if (1.4) implies (1.1), then (1.1) also follows from (1.3) by Proposition 3.11.

Assume that (1.4) holds. For (1.1) it suffices to consider the case  $q := q_0 = q_1$ . Let  $\delta = (\gamma_0 + d)/p_0 - (\gamma_1 + d)/p_1$ . Since  $\widehat{S_n f} = \widehat{\varphi_n f}$  is supported in  $\{x \in \mathbb{R}^d : |x| < 3 \cdot 2^{n-1}\}$ , Proposition 4.1 gives

$$\|S_n f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)} \leq C 2^{\delta n} \|S_n f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)}, \quad n \geq 0.$$

Therefore, using  $s_1 + \delta = s_0$ , we find that

$$\begin{aligned} \|f\|_{B_{p_1, q}^{s_1}(\mathbb{R}^d, w_1; X)} &= \|(2^{s_1 n} \|S_n f\|_{L^{p_1}(\mathbb{R}^d, w_1; X)})_{n \geq 0}\|_{\ell_q} \\ &\leq C \|(2^{s_0 n} \|S_n f\|_{L^{p_0}(\mathbb{R}^d, w_0; X)})_{n \geq 0}\|_{\ell_q} \\ &= C \|f\|_{B_{p_0, q}^{s_0}(\mathbb{R}^d, w_0; X)}. \quad \blacksquare \end{aligned}$$

**4.2. Necessary conditions.** In this subsection we prove Theorem 1.1(1)⇒(2). We start with an elementary lemma.

LEMMA 4.6. *Let  $p \in [1, \infty]$  and  $w(x) = |x|^\gamma$  with  $\gamma > -d$ . Let  $(\varphi_n)_{n \geq 0} \in \Phi$  and  $j \in \{-1, 0, 1\}$ . Then there is a constant  $C_{\varphi, p, \gamma, j}$  such that for every  $n \geq 2$ ,*

$$\|\varphi_n * \varphi_{n+j}\|_{L^p(\mathbb{R}^d, w)} = C_{\varphi, p, \gamma, j} 2^{nd} 2^{-n(d+\gamma)/p}.$$

*Proof.* Since  $\widehat{\varphi_n} = \widehat{\varphi_1}(2^{-n+1} \cdot)$ , this follows from a straightforward substitution argument. ■

We give necessary conditions for an embedding for general  $p_0$  and  $p_1$ .

PROPOSITION 4.7. *Let  $1 < p_0, p_1 \leq \infty$  and  $s_0, s_1 \in \mathbb{R}$ . Let further  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . Suppose*

$$(4.8) \quad B_{p_0,1}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow B_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1).$$

Then

$$(4.9) \quad s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1}, \quad \frac{\gamma_1 + d}{p_1} \leq \frac{\gamma_0 + d}{p_0} \quad \text{and} \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0}.$$

*Proof.* Let  $(\varphi_n)_{n \geq 0} \in \Phi$ . By (4.8), for every  $n \geq 0$  one has

$$\|\varphi_n\|_{B_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1)} \leq C \|\varphi_n\|_{B_{p_0,1}^{s_0}(\mathbb{R}^d, w_0)}.$$

By the assumption on the support of  $(\hat{\varphi}_n)_{n \geq 0}$  one gets

$$2^{ns_1} \|\varphi_n * \varphi_n\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C 2^{ns_0} \sum_{j=-1}^1 \|\varphi_n * \varphi_{n+j}\|_{L^{p_0}(\mathbb{R}^d, w_0)}, \quad n \geq 0.$$

Using Lemma 4.6, this implies that there is a constant  $\tilde{C}$  such that

$$2^{ns_1} 2^{nd} 2^{-n(d+\gamma_1)/p_1} \leq \tilde{C} 2^{ns_0} 2^{nd} 2^{-n(d+\gamma_0)/p_0}.$$

Letting  $n$  tend to infinity gives  $s_1 - (d + \gamma_1)/p_1 \leq s_0 - (d + \gamma_0)/p_0$ .

We next show that  $\gamma_1/p_1 \leq \gamma_0/p_0$ . Let  $f \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\hat{f}$  has support in  $\{x \in \mathbb{R}^d : |x| < 1\}$ . Let  $\lambda \geq 1$  and  $e_1 = (1, 0, \dots, 0)$ . Then  $\text{supp}(\mathcal{F}(f(\cdot - \lambda e_1))) = \text{supp}(\hat{f})$ . By (4.8) one has

$$\|f(\cdot - \lambda e_1)\|_{B_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1)} \leq C \|f(\cdot - \lambda e_1)\|_{B_{p_0,1}^{s_0}(\mathbb{R}^d, w_0)},$$

and as before this yields

$$(4.10) \quad \|f(\cdot - \lambda e_1)\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C \|f(\cdot - \lambda e_1)\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

Let  $p \in (1, \infty]$ ,  $\gamma > -d$  and  $w(x) = |x|^\gamma$ . We claim that there are constants  $c, C > 0$ , depending on  $p, \gamma, d, f$ , such that for all  $\lambda \geq 1$  one has

$$(4.11) \quad c\lambda^{\gamma/p} \leq \|f(\cdot - \lambda e_1)\|_{L^p(\mathbb{R}^d, w)} \leq C(1 + \lambda^\gamma)^{1/p}.$$

From (4.10) and (4.11) it would follow that for all  $\lambda \geq 1$  we have  $c\lambda^{\gamma_1/p_1} \leq C(1 + \lambda^{\gamma_0})^{1/p_0}$ . Letting  $\lambda$  tend to infinity then gives  $\gamma_1/p_1 \leq \gamma_0/p_0$ .

Estimate (4.11) is trivial for  $p = \infty$ . To prove (4.11) for  $p < \infty$ , we first consider the case  $\gamma \geq 0$ . We estimate

$$\begin{aligned} \|f(\cdot - \lambda e_1)\|_{L^p(\mathbb{R}^d, w)}^p &= \int_{\mathbb{R}^d} |f(x - \lambda e_1)|^p |x|^\gamma dx = \int_{\mathbb{R}^d} |f(x)|^p |x + \lambda e_1|^\gamma dx \\ &\leq C_\gamma \int_{\mathbb{R}^d} |f(x)|^p (|x|^\gamma + \lambda^\gamma) dx = C_{p,\gamma,d,f} (1 + \lambda^\gamma). \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f(\cdot - \lambda e_1)\|_{L^p(\mathbb{R}^d, w)}^p &= \int_{\mathbb{R}^d} |f(x)|^p |x + \lambda e_1|^\gamma dx \geq \int_{[0,1]^d} |f(x)|^p |x + \lambda e_1|^\gamma dx \\ &\geq \int_{[0,1]^d} |f(x)|^p \lambda^\gamma dx = C_{d,p,f} \lambda^\gamma. \end{aligned}$$

Next we consider the case  $\gamma < 0$ . Since  $f \in \mathcal{S}(\mathbb{R}^d)$ , there is a  $C$  such that one has  $|f(x)|^p \leq C(1 + |x|)^{-dp}$  for  $x \in \mathbb{R}^d$ . By Lemma 4.5 we can estimate

$$\begin{aligned} \|f(\cdot - \lambda e_1)\|_{L^p(\mathbb{R}^d, w)}^p &\leq C \int_{\mathbb{R}^d} (1 + |x|)^{-dp} |x + \lambda e_1|^\gamma dx \\ &= C \int_{\mathbb{R}^d} (1 + |y - \lambda e_1|)^{-dp} |y|^\gamma dx \\ &\leq C \int_{B(\lambda e_1, 1)} |y|^\gamma dy \leq C(1 + \lambda)^\gamma \end{aligned}$$

for  $\lambda \geq 1$ . For the lower estimate we have

$$\begin{aligned} \|f(\cdot - \lambda e_1)\|_{L^p(\mathbb{R}^d, w)}^p &\geq \int_{[0,1]^d} |f(x)|^p |x + \lambda e_1|^\gamma dx \\ &\geq \int_{[0,1]^d} |f(x)|^p (1 + \lambda)^\gamma dx \geq C_{d,p,f,\gamma} \lambda^\gamma. \end{aligned}$$

This completes the proof of (4.11) and therefore the proof of  $\gamma_1/p_1 \leq \gamma_0/p_0$ .

Next let  $\varphi$  be as in Definition 3.1 and for each  $t > 0$  define  $f_t : \mathbb{R}^d \rightarrow \mathbb{C}$  by  $f_t(x) := t^n \varphi(tx)$ . By (4.8) one has  $\|f_t\|_{B_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1)} \leq C \|f_t\|_{B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0)}$ . Taking  $t > 0$  small enough shows that

$$\|f_t\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C \|f_t\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

Rescaling gives

$$t^{-(\gamma_1+d)/p_1} \|\varphi\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C t^{-(\gamma_0+d)/p_0} \|\varphi\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

Letting  $t \downarrow 0$  implies that  $(\gamma_1 + d)/p_1 \leq (\gamma_0 + d)/p_0$ . ■

REMARK 4.8. In the above proof, the assumption  $p_0, p_1 > 1$  was only employed to show  $\gamma_1/p_1 \leq \gamma_0/p_0$  in case  $\gamma_0 < 0$ .

For  $p_1 < p_0$  we can sharpen the necessary condition (4.9) for an embedding.

PROPOSITION 4.9. *Let  $1 < p_1 < p_0 < \infty$  and  $s_0, s_1 \in \mathbb{R}$ . Let further  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . Suppose*

$$(4.12) \quad B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow B_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1).$$

Then

$$(4.13) \quad s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1} \quad \text{and} \quad \frac{\gamma_1 + d}{p_1} < \frac{\gamma_0 + d}{p_0}.$$

REMARK 4.10. Observe that  $\gamma_1/p_1 < \gamma_0/p_0$  is already a consequence of  $(\gamma_1 + d)/p_1 < (\gamma_0 + d)/p_0$  and  $p_1 < p_0$ .

To prove the proposition, we need the following density result. Observe that the proof heavily depends on the fact that the weight is of power type.

LEMMA 4.11. *Let  $p \in [1, \infty)$ ,  $\gamma > -d$  and let  $w(x) = |x|^\gamma$ . The set*

$$\mathcal{FC}_c^\infty(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) : \widehat{f} \text{ has compact support}\}$$

*is dense in  $L^p(\mathbb{R}^d, w)$ .*

*Proof.* By Lemma 3.8, for the assertion it suffices to consider  $f \in \mathcal{S}(\mathbb{R}^d)$ . We construct a sequence  $(f_n)_{n \geq 0}$  of functions in  $\mathcal{FC}_c^\infty(\mathbb{R}^d)$  such that  $f = \lim_{n \rightarrow \infty} f_n$  in  $L^p(\mathbb{R}^d, w)$ . We proceed as in [29, Section 6], where a much stronger result has been obtained for the one-dimensional setting. Let  $\zeta \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\widehat{\zeta}(\xi) = 1$  if  $|\xi| \leq 1$ , and  $\widehat{\zeta}(\xi) = 0$  if  $|\xi| \geq 2$ . Let  $\zeta_n = n^d \zeta(nx)$  and  $f_n = \zeta_n * f$ . Choose an integer  $k \geq 0$  so large that  $-2kp + \gamma < -d$ . Observe that

$$\begin{aligned} \|f - f_n\|_{L^p(\mathbb{R}^d, w)} &\leq \sup_{x \in \mathbb{R}^d} (|x|^{2k} + 1) |f(x) - f_n(x)| \left( \int_{\mathbb{R}^d} \frac{|x|^\gamma}{(|x|^{2k} + 1)^p} dx \right)^{1/p} \\ &\leq C \sup_{x \in \mathbb{R}^d} (|x|^{2k} + 1) |f(x) - f_n(x)| \\ &\leq C \| [(-\Delta)^k + 1] (\widehat{f} - \widehat{f}_n) \|_{L^1(\mathbb{R}^d)} \\ &\leq C \|\Delta^k (\widehat{f} - \widehat{f}_n)\|_{L^1(\mathbb{R}^d)} + \|(\widehat{f} - \widehat{f}_n)\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

where we used  $\mathcal{F}[(|\cdot|^{2k} + 1)(f - f_n)] = ((-\Delta)^k + 1)(\widehat{f} - \widehat{f}_n)$ . It suffices to show that for any multiindex  $a$  one has

$$\lim_{n \rightarrow \infty} \|D^a(\widehat{f} - \widehat{f}_n)\|_{L^1(\mathbb{R}^d)} = 0.$$

Note that  $D^a[\widehat{f} - \widehat{f}_n] = D^a[\widehat{f}(1 - \widehat{\zeta}_n)]$ . From the Leibniz rule we see that  $D^a(\widehat{f} - \widehat{f}_n)$  consists of finitely many terms of the form  $D^b \widehat{f} D^c(1 - \widehat{\zeta}_n)$ , where  $b, c$  are multiindices. One has

$$\|D^c(1 - \widehat{\zeta}_n)\|_{L^1(\mathbb{R}^d)} \leq \|D^b \widehat{f}\|_{L^\infty(\mathbb{R}^d)} \|D^c(1 - \widehat{\zeta}_n)\|_{L^1(\mathbb{R}^d)},$$

which converges to zero as  $n$  tends to infinity, by the dominated convergence theorem. ■

*Proof of Proposition 4.9.* In Proposition 4.7 we have already seen that

$$s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1} \quad \text{and} \quad \frac{\gamma_1 + d}{p_1} \leq \frac{\gamma_0 + d}{p_0}$$

Assume that  $(\gamma_1 + d)/p_1 = (\gamma_0 + d)/p_0$ . We show that this leads to a contradiction. Let  $f \in L^{p_0}(\mathbb{R}^d; w_0)$  be such that  $\text{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$ . Then it follows from (4.12) that (4.2) holds with  $t = 1$ . By scaling we see that for all  $f \in L^{p_0}(\mathbb{R}^d, w_0)$  such that  $\widehat{f}$  has compact support one has

$$(4.14) \quad \|f\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

From Lemma 4.11 we see that (4.14) extends to all  $f \in L^{p_0}(\mathbb{R}^d, w_0)$ . Now define  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $f(x) = |x|^{-d/p_0} \log(1/|x|)^{-1/p_1} \mathbf{1}_{[0,1/2]}(|x|)$ . Then using polar coordinates one easily checks that  $f \in L^{p_0}(\mathbb{R}^d, w_0)$ , but  $f \notin L^{p_1}(\mathbb{R}^d, w_1)$ , which contradicts (4.14). ■

We can finish the proof of the necessary conditions for the embeddings of Besov spaces.

*Proof of Theorem 1.1(1)⇒(2).* It suffices to consider  $X = \mathbb{C}$ . It follows from (1.1) and Proposition 3.11 that

$$B_{p_0,1}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow B_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1).$$

From Proposition 4.7 we see that (4.9) holds. Now there are two possibilities: either (i)  $(d + \gamma_1)/p_1 < (d + \gamma_0)/p_0$ , or (ii)  $(d + \gamma_1)/p_1 = (d + \gamma_0)/p_0$ .

Suppose that (i) holds. If  $s_0 - (d + \gamma_0)/p_0 > s_1 - (d + \gamma_1)/p_1$ , then (1.3) follows. If  $s_0 - (d + \gamma_0)/p_0 = s_1 - (d + \gamma_1)/p_1$ , then to obtain (1.4) we have to show that  $q_0 \leq q_1$ . Let  $(\varphi_n)_{n \geq 0}$  be as in Definition 3.1. For a sequence  $(a_j)$  and  $N \in \mathbb{N}$ , define the function  $f = \sum_{j=1}^N 2^{-3j(d+s_0-(d+\gamma_0)/p_0)} a_j \varphi_{3j}$ . We have  $\varphi_n * \varphi_{3j} \neq 0$  only for  $n = 3j + l$  with  $l \in \{-1, 0, 1\}$ , and by Lemma 4.6,  $\|\varphi_{3j+l} * \varphi_{3j}\|_{L^p(\mathbb{R}^d, w)} = C 2^{3jd} 2^{-3j(d+\gamma)/p}$ . It follows that

$$\|(a_j)_{j \leq N}\|_{\ell^{q_1}} \leq C \|(a_j)_{j \leq N}\|_{\ell^{q_0}},$$

with a constant  $C$  independent of  $N$  and  $a_j$ . But this is only possible for  $q_0 \leq q_1$ .

Now suppose that (ii) holds. Then  $\gamma_1/p_1 \leq \gamma_0/p_0$  yields  $p_0 \geq p_1$ . If  $p_0 > p_1$ , then Proposition 4.9 yields  $(d + \gamma_1)/p_1 < (d + \gamma_0)/p_0$  and this contradicts (ii). If  $p_0 = p_1$ , then  $\gamma_0 = \gamma_1$  follows from (ii) and therefore,  $s_0 \geq s_1$  by (4.9). If  $s_0 = s_1$ , then it follows as above that  $q_0 \leq q_1$ . Hence (1.2) is valid. ■

## 5. Embeddings for Triebel–Lizorkin spaces —proof of Theorem 1.2

**5.1. Sufficient conditions.** In the proof of the sufficiency part of Theorem 1.2 we employ ideas from [7] and [38]. One has the following Gagliardo–Nirenberg type inequality for spaces with weights.

PROPOSITION 5.1. *Let  $X$  be a Banach space,  $q, q_0, q_1 \in [1, \infty]$  and  $\theta \in (0, 1)$ . Let  $p, p_0, p_1 \in (1, \infty)$  and  $-\infty < s_0 < s_1 < \infty$  satisfy*

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad s = (1 - \theta)s_0 + \theta s_1.$$

*Let further  $w, w_0, w_1 \in A_\infty$  be such that  $w = w_0^{(1-\theta)p/p_0} w_1^{\theta p/p_1}$ . Then there exists a constant  $C$  such that for all  $f \in \mathcal{S}'(\mathbb{R}^d; X)$ ,*

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X)}^{1-\theta} \|f\|_{F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1; X)}^\theta.$$

*In particular, one can take  $w = w_0 = w_1$ .*

*Proof.* Due to [7, Lemma 3.7], for any sequence of scalars  $(a_j)_{j \geq 0}$  one has

$$\|(2^{sj} a_j)_{j \geq 0}\|_{\ell^q} \leq \|(2^{s_0 j} a_j)_{j \geq 0}\|_{\ell^\infty}^{1-\theta} \|(2^{s_1 j} a_j)_{j \geq 0}\|_{\ell^\infty}^\theta.$$

Taking  $a_j(x) = \|S_j f(x)\|$  with  $x \in \mathbb{R}^d$ , one obtains

$$\begin{aligned} & \| (2^{sj} a_j(x))_{j \geq 0} \|_{\ell^q}^p w(x) \\ & \leq \| (2^{s_0 j} a_j(x))_{j \geq 0} \|_{\ell^\infty}^{(1-\theta)p} w_0(x)^{(1-\theta)p/p_0} \| (2^{s_1 j} a_j(x))_{j \geq 0} \|_{\ell^\infty}^{\theta p} w_1(x)^{\theta p/p_1}. \end{aligned}$$

Thus Hölder’s inequality gives

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d, w; X)} \leq C \|f\|_{F_{p_0, \infty}^{s_0}(\mathbb{R}^d, w_0; X)}^{1-\theta} \|f\|_{F_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1; X)}^\theta,$$

and the assertion follows from Proposition 3.11. ■

We turn to the proof of sufficiency.

*Proof of Theorem 1.2(2) ⇒ (1).* By the elementary embeddings of Proposition 3.11, one can assume that  $s_0 - (d + \gamma_0)/p_0 = s_1 - (d + \gamma_1)/p_1$  and  $q_1 = 1$ . The trivial cases in (2) are also covered Proposition 3.11. We thus have to show that (1.3) implies the embedding (1.5).

Let  $\theta_0 \in [0, 1)$  be such that  $1/p_1 - (1 - \theta_0)/p_0 = 0$ . Consider the function  $g : (\theta_0, 1] \rightarrow \mathbb{R}$  given by

$$g(\theta) = \frac{\gamma_1/p_1 - (1 - \theta)\gamma_0/p_0}{1/p_1 - (1 - \theta)/p_0}.$$

Obviously,  $g$  is continuous, and  $\lim_{\theta \uparrow 1} g(\theta) = \gamma_1$ . Since  $\gamma_1 > -d$  we can choose a  $\theta \in (\theta_0, 1)$  such that  $\gamma := g(\theta) > -d$ . Let  $v(x) = |x|^\gamma$ , and let  $r$  be defined by  $1/p_1 = (1 - \theta)/p_0 + \theta/r$ . Note that  $p_0 \leq p_1 < \infty$  implies  $r \in [p_1, \infty)$ . Let further  $t$  be defined by  $t - (d + \gamma)/r = s_1 - (d + \gamma_1)/p_1$ . Observe that  $t < s_0$ ,  $s_1 = \theta t + (1 - \theta)s_0$  and  $v^{p_1 \theta/r} w_0^{(1-\theta)p_1/p_0} = w_1$ . Therefore, by Proposition 5.1,

$$(5.1) \quad \|f\|_{F_{p_1, 1}^{s_1}(\mathbb{R}^d, w_1; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X)}^{1-\theta} \|f\|_{F_{r, r}^t(\mathbb{R}^d, v; X)}^\theta.$$

Now one can check that

$$\frac{\gamma_1}{p_1} - \frac{\gamma}{r} = \frac{1 - \theta}{\theta} \left( \frac{\gamma_0}{p_0} - \frac{\gamma_1}{p_1} \right) \geq 0.$$

From Proposition 3.11 and Theorem 1.1 (using  $r \geq p_1$ ) one obtains

$$\|f\|_{F_{r,r}^t(\mathbb{R}^d, v; X)} = \|f\|_{B_{r,r}^t(\mathbb{R}^d, v; X)} \leq C \|f\|_{B_{p_1, p_1}^{s_1}(\mathbb{R}^d, w_1; X)} \leq C \|f\|_{F_{p_1, 1}^{s_1}(\mathbb{R}^d, w_1; X)}.$$

Substituting the latter estimate into (5.1), one deduces that

$$\|f\|_{F_{p_1, 1}^{s_1}(\mathbb{R}^d, w_1; X)} \leq C \|f\|_{F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X)},$$

using the density of  $\mathcal{S}(\mathbb{R}^d; X)$  for  $q_0 < \infty$  and a standard Fatou argument for  $q_0 = \infty$ . ■

**5.2. Necessary conditions.** The necessary conditions for  $F$ -spaces are a direct consequence of the result for  $B$ -spaces.

*Proof of Theorem 1.2(1)⇒(2).* Assume (1). It suffices to consider  $X = \mathbb{C}$ . By Proposition 3.11 one has

$$B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1) \hookrightarrow B_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1).$$

Therefore, (2) follows from the Propositions 4.7 and 3.11. ■

Now we can prove the characterization for  $H$ - and  $W$ -spaces in the case  $p_0 \leq p_1$ .

*Proof of Corollaries 1.4 and 1.5.* These follow from Theorem 1.2 and Proposition 3.12. ■

REMARK 5.2. It is unclear to us whether Corollaries 1.4 and 1.5 hold for all  $\gamma_0, \gamma_1 > -d$ . This is contained in [32] for  $s_0 = 0, s_1 = 0$  in the case of  $W$ -spaces.

In Proposition 1.6 we give a characterization for the embedding of  $H$ - and  $F$ -spaces in case  $p_1 < p_0$ . Its proof will be based on the following result.

PROPOSITION 5.3. *Let  $1 < p_1 < p_0 < \infty$  and  $s_0, s_1 \in \mathbb{R}$ . Let further  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $-d < \gamma_0 < d(p_0 - 1)$  and  $-d < \gamma_1 < d(p_1 - 1)$ . Suppose*

$$(5.2) \quad H^{s_0, p_0}(\mathbb{R}^d, w_0) \hookrightarrow H^{s_1, p_1}(\mathbb{R}^d, w_1).$$

Then

$$(5.3) \quad \frac{\gamma_1 + d}{p_1} < \frac{\gamma_0 + d}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} > s_1 - \frac{d + \gamma_1}{p_1}.$$

*Proof.* It suffices to consider the case  $X = \mathbb{C}$  and  $s_1 = 0$ . Propositions 3.12, 4.7 and 4.9 imply that  $s_0 - (d + \gamma_0)/p_0 \geq -(d + \gamma_1)/p_1$  and  $(\gamma_1 + d)/p_1 < (\gamma_0 + d)/p_0$ . In particular, we have  $s_0 > 0$ .

We suppose that

$$s_0 - \frac{d + \gamma_0}{p_0} = -\frac{d + \gamma_1}{p_1}, \quad \frac{\gamma_1 + d}{p_1} < \frac{\gamma_0 + d}{p_0},$$

and show that this gives a contradiction. It follows from [40, Proposition VI.4.4/2, Corollary V.4.2] that the operator  $(1 - \Delta)^{s_0/2}(1 + (-\Delta)^{s_0/2})^{-1}$  is bounded on  $L^{p_0}(\mathbb{R}^d, w_0)$ . Thus for  $f \in \mathcal{S}(\mathbb{R}^d)$ , the embedding (5.2) implies that

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d, w_1)} &\leq C\|(1 - \Delta)^{s_0/2}f\|_{L^{p_0}(\mathbb{R}^d, w_0)} \\ &\leq C\|f\|_{L^{p_0}(\mathbb{R}^d, w_0)} + \|(-\Delta)^{s_0/2}f\|_{L^{p_0}(\mathbb{R}^d, w_0)}. \end{aligned}$$

By applying the above estimate to  $f(\lambda \cdot)$  for  $\lambda > 0$ , scaling shows that

$$\|f\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq C\lambda^{-s_0}\|f\|_{L^{p_0}(\mathbb{R}^d, w_0)} + C\|(-\Delta)^{s_0/2}f\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

Letting  $\lambda \rightarrow \infty$  gives

$$(5.4) \quad \|f\|_{L^{p_1}(\mathbb{R}^d, w_1)} \leq \tilde{C}\|(-\Delta)^{s_0/2}f\|_{L^{p_0}(\mathbb{R}^d, w_0)}.$$

By density it follows that for all  $f \in \mathcal{S}'(\mathbb{R}^d)$  for which  $(-\Delta)^{s_0/2}f \in L^{p_0}(\mathbb{R}^d, w_0)$ , estimate (5.4) holds.

Now define  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  as  $g(x) = |x|^{-a} \log(1/|x|)^{-b} \mathbf{1}_{[0,1/2]}(|x|)$ , where

$$a = s_0 + \frac{\gamma_1 + d}{p_1} = \frac{\gamma_0 + d}{p_0}, \quad b = 1/p_1.$$

One has  $g \in L^{p_0}(\mathbb{R}^d, w_0)$ . Indeed, using polar coordinates one sees that

$$\begin{aligned} \|g\|_{L^{p_0}(\mathbb{R}^d, w_0)}^{p_0} &= c \int_0^{1/2} r^{d-1} r^{-ap_0} \log(1/r)^{-bp_0} r^{\gamma_0} dr \\ &= c \int_0^{1/2} r^{-1} \log(1/r)^{-p_0/p_1} dr < \infty, \end{aligned}$$

because  $p_0/p_1 > 1$ . Let  $f \in \mathcal{S}'(\mathbb{R}^d)$  be defined by  $f = (-\Delta)^{-s_0/2}g$ . To show that (5.4) cannot hold, and therefore to prove (5.3), it suffices to show that  $f \notin L^{p_1}(\mathbb{R}^d, w_1)$ . This will be checked for  $d \geq 2$  and  $d = 1$  separately.

First assume  $d \geq 2$ . One has the following representation of the Riesz potential for radial symmetric functions  $v : \mathbb{R}^d \rightarrow \mathbb{R}$  (see [12]):

$$(-\Delta)^{-s_0/2}v(x) = c \int_0^\infty v(r)r^{s_0-1}I_{j,k}(\rho/r) dr.$$

Here  $j = d - s_0$ ,  $k = (d - 3)/2$ ,  $\rho = |x|$ ,

$$I_{j,k}(z) = \int_{-1}^1 \frac{(1 - t^2)^k}{(1 - 2zt + z^2)^{j/2}} dt, \quad z \geq 0.$$

The function  $I_{j,k} : [0, \infty) \rightarrow [0, \infty)$  is continuous on  $[0, \infty) \setminus \{1\}$  and its singularity at 1 is well-understood (see [12, Lemma 4.2]). For  $\rho = |x| \leq 1/2$ ,  $\rho \neq 0$ , we obtain

$$\begin{aligned} f(x) &= (-\Delta)^{-s_0/2} g(x) = c \int_0^{1/2} r^{s_0-a-1} \log(1/r)^{-b} I_{j,k}(\rho/r) dr \\ &\geq c \log(2/\rho)^{-b} \int_{\rho/2}^{\rho} r^{s_0-a-1} I_{j,k}(\rho/r) dr \\ &= c \rho^{s_0-a} \log(2/\rho)^{-b} \int_{1/2}^1 u^{s_0-a-1} I_{j,k}(1/u) du \\ &\geq C \rho^{s_0-a} \log(2/\rho)^{-b}. \end{aligned}$$

It follows that

$$\begin{aligned} \|f\|_{L^{p_1}(\mathbb{R}^d, w_1)}^{p_1} &\geq \int_{|x| \leq 1/2} |f(x)|^{p_1} |x|^{\gamma_1} dx \\ &\geq C \int_0^{1/2} \rho^{d-1} \rho^{(s_0-a)p_1} \log(2/\rho)^{-bp_1} \rho^{\gamma_1} d\rho \\ &= C \int_0^{1/2} \rho^{-1} \log(2/\rho)^{-1} d\rho = \infty. \end{aligned}$$

Hence  $f \notin L^{p_1}(\mathbb{R}^d, w_1)$ .

If  $d = 1$ , then for  $x \in [0, 1/2]$  one has

$$\begin{aligned} f(x) &= (-\Delta)^{-s_0/2} g(x) = c \int_{-1/2}^{1/2} |x - y|^{s_0-1} |y|^{-a} \log(1/|y|)^{-b} dy \\ &\geq \log(2/x)^{-b} \int_{x/2}^x |x - y|^{s_0-1} |y|^{-a} dy = cx^{s_0-a} \log(2/x)^{-b}. \end{aligned}$$

Now the proof can be finished as before. ■

We obtain the following consequences for  $F$ - and  $W$ -spaces.

**COROLLARY 5.4.** *Let  $1 < p_1 < p_0 < \infty$  and  $s_0, s_1 \in \mathbb{R}$ . Let further  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $-d < \gamma_0 < d(p_0 - 1)$  and  $-d < \gamma_1 < d(p_1 - 1)$ . Suppose that for some  $q_0 \in [2, \infty]$  and  $q_1 \in [1, 2]$ ,*

$$(5.5) \quad F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0) \hookrightarrow F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1).$$

Then (5.3) holds.

*Proof.* By Proposition 3.9 and the results in [8, Section 4] we have

$$(5.6) \quad F_{p_i,2}^{s_i}(w_i) = H^{s_i,p_i}(w_i) \quad \text{for } i = 1, 2.$$

Therefore, by Proposition 3.11, (5.5) implies (5.2). Now the result follows from Proposition 5.3. ■

**COROLLARY 5.5.** *Let  $1 < p_1 < p_0 < \infty$  and  $s_0, s_1 \in \mathbb{N}_0$ . Let further  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $-d < \gamma_0 < d(p_0 - 1)$  and  $-d < \gamma_1 < d(p_1 - 1)$ . Suppose*

$$(5.7) \quad W^{s_0,p_0}(\mathbb{R}^d, w_0) \hookrightarrow W^{s_1,p_1}(\mathbb{R}^d, w_1).$$

*Then (5.3) holds.*

*Proof.* Since  $L^{p_i}(\mathbb{R}^d, w_i) = F_{p_i,2}^0(\mathbb{R}^d, w_i)$  by [34, Theorem 1.10], Proposition 3.10 implies that  $W^{s_i,p_i}(\mathbb{R}^d, w_i) = F_{p_i,2}^{s_i}(\mathbb{R}^d, w_i)$  for  $i = 1, 2$ . Now the result follows from Corollary 5.4. ■

We end this section with the characterization of embeddings for  $H$ -spaces in case  $p_0 > p_1$ .

*Proof of Proposition 1.6.* To prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3), it suffices to consider  $X = \mathbb{C}$ . Note that because of (5.6) it suffices to prove (1) $\Rightarrow$ (3), since (1) and (2) coincide for  $X = \mathbb{C}$ . Now, (3) follows from Proposition 5.3.

We prove the sufficiency part. Assume (3). By Theorem 1.1 it follows that  $B_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1,\infty}^{t_1}(\mathbb{R}^d, w_1; X)$ , where  $t_1$  is defined by  $s_0 - (\gamma_0 + d)/p_0 = t_1 - (\gamma_1 + d)/p_1$ . Note that it follows from (3) that  $t_1 > s_1$ . Therefore, combining the above embedding with Proposition 3.11 yields

$$B_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1,1}^{s_1}(\mathbb{R}^d, w_1; X).$$

Now (1) and (2) are a consequence of the Propositions 3.11 and 3.12. ■

**6. Embeddings of Jawerth–Franke type.** In this section we only treat power weights of  $A_p$ -type. We first consider real interpolation of the weighted function spaces.

**PROPOSITION 6.1.** *Let  $X$  be a Banach space,  $p \in (1, \infty)$ ,  $q_0, q_1, q \in [1, \infty]$ ,  $s_0 \neq s_1 \in \mathbb{R}$ . Then for  $\theta \in [0, 1]$ ,  $s = (1 - \theta)s_0 + \theta s_1$  and  $w \in A_p$ ,*

$$(6.1) \quad (B_{p,q_0}^{s_0}(\mathbb{R}^d, w; X), B_{p,q_1}^{s_1}(\mathbb{R}^d, w; X))_{\theta,q} = B_{p,q}^s(\mathbb{R}^d, w; X),$$

$$(6.2) \quad (F_{p,q_0}^{s_0}(\mathbb{R}^d, w; X), F_{p,q_1}^{s_1}(\mathbb{R}^d, w; X))_{\theta,q} = B_{p,q}^s(\mathbb{R}^d, w; X),$$

$$(6.3) \quad (H^{s_0,p}(\mathbb{R}^d, w; X), H^{s_1,p}(\mathbb{R}^d, w; X))_{\theta,q} = B_{p,q}^s(\mathbb{R}^d, w; X),$$

*and if additionally  $s_0, s_1 \geq 0$  are integers, then*

$$(6.4) \quad (W^{s_0,p}(\mathbb{R}^d, w; X), W^{s_1,p}(\mathbb{R}^d, w; X))_{\theta,q} = B_{p,q}^s(\mathbb{R}^d, w; X).$$

Moreover, for  $p_0, p_1 \in (1, \infty)$ ,  $s \in \mathbb{R}$ ,  $\theta \in (0, 1)$ ,  $1/p = (1 - \theta)/p_0 + \theta/p_1$ ,  $q \in (1, \infty]$ , and  $w_0 \in A_{p_0}$ ,  $w_1 \in A_{p_1}$ ,  $w \in A_p$  with  $w^{1/p} = w_0^{(1-\theta)/p_0} w_1^{\theta/p_1}$ ,

$$(6.5) \quad (F_{p_0,q}^s(\mathbb{R}^d, w_0; X), F_{p_1,q}^s(\mathbb{R}^d, w_1; X))_{\theta,p} = F_{p,q}^s(\mathbb{R}^d, w; X),$$

$$(6.6) \quad (F_{p_0,1}^s(\mathbb{R}^d, w_0; X), F_{p_1,1}^s(\mathbb{R}^d, w_1; X))_{\theta,p} \hookrightarrow F_{p,1}^s(\mathbb{R}^d, w; X).$$

*Proof.* Since all the weights under consideration are of class  $A_p$ , the proofs are straightforward generalizations of the unweighted case, as presented in [42, Section 2.4] and [37, Proposition 12], for instance.

We nevertheless provide the details for (6.5) and (6.6). By Proposition 3.9 it suffices to consider  $s = 0$ . Let  $(\varphi_n)_{n \geq 0} \in \Phi$ . For  $p_* \in (1, \infty)$ ,  $q_* \in [1, \infty]$  and  $w_* \in A_{p_*}$  it follows from the definitions that the map

$$S : F_{p_*,q_*}^0(\mathbb{R}^d, w_*; X) \rightarrow L^{p_*}(\mathbb{R}^d, w_*; \ell^{q_*}(X)), \quad Sf = (\varphi_n * f)_{n \geq 0},$$

is continuous. If  $(\psi_n)_{n \geq 0}$  is as in the proof of Proposition 3.9, then Lemma 2.3 implies that

$$R : L^{p_*}(\mathbb{R}^d, w_*; \ell^{q_*}(X)) \rightarrow F_{p_*,q_*}^0(\mathbb{R}^d, w_*; X), \quad R(g_k)_{k \geq 0} = \sum_{k=0}^{\infty} \psi_k * g_k,$$

is continuous for  $q \in (1, \infty]$ . Now if  $p_0, p_1, p$  and  $w_0, w_1, w$  are as in (6.5) and (6.6), then [42, Theorem 1.18.5] gives

$$(6.7) \quad (L^{p_0}(\mathbb{R}^d, w_0; \ell^q(X)), L^{p_1}(\mathbb{R}^d, w_1; \ell^q(X)))_{\theta,p} = L^p(\mathbb{R}^d, w; \ell^q(X))$$

where  $q \in [1, \infty]$ . Since  $R$  is a right-inverse for  $S$ , (6.5) is a consequence of the well-known retraction-coretraction method (see [42, Theorem 1.2.4]). Moreover, (6.7) implies that

$$S : (F_{p_0,1}^0(\mathbb{R}^d, w_0; X), F_{p_1,1}^0(\mathbb{R}^d, w_1; X))_{\theta,p} \rightarrow L^p(\mathbb{R}^d, w; \ell^1(X))$$

is continuous as well. Thus (6.6) follows from

$$(6.8) \quad \begin{aligned} \|f\|_{F_{p,1}^0(\mathbb{R}^d, w; X)} &= \|Sf\|_{L^p(\mathbb{R}^d, w; \ell^1(X))} \\ &\leq C \|f\|_{(F_{p_0,1}^0(\mathbb{R}^d, w_0; X), F_{p_1,1}^0(\mathbb{R}^d, w_1; X))_{\theta,p}}. \end{aligned}$$

Employing Lemma 2.3 and interpolation theory for a class of weighted  $\ell^q$ -spaces (see [42, Theorem 1.18.2]), one can show (6.1) in a similar way. Finally, the identities (6.2), (6.3) and (6.4) follow from the independence of (6.1) from the microscopic parameters  $q_0, q_1 \in [1, \infty]$ , and from Propositions 3.11 and 3.12. ■

REMARK 6.2. The operator  $S$  in the proof above is continuous for all  $w \in A_\infty$ . It thus follows from (6.8) that the embeddings from the left to the right in (6.5) and (6.6) are true for  $w \in A_\infty$ .

REMARK 6.3. In [8], interpolation results for scalar  $B$ - and  $F$ -spaces are shown for a single weight  $w \in A_\infty$ . Even more general results and different proofs are given in [34, Theorem 2.14].

After these preparations we can show embeddings of Jawerth–Frankel type, which is an improvement of the embeddings in (1.1) and (1.5). We argue similarly to [37, Theorem 6].

THEOREM 6.4. *Let  $X$  be a Banach space,  $1 < p_0 < p_1 < \infty$ ,  $s_0, s_1 \in \mathbb{R}$ ,  $q \in [1, \infty]$ , and  $w_0(x) = |x|^{\gamma_0}$ ,  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0 \in (-d, d(p_0 - 1))$  and  $\gamma_1 \in (-d, d(p_1 - 1))$ . Suppose*

$$(6.9) \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d + \gamma_0}{p_0} \geq s_1 - \frac{d + \gamma_1}{p_1}.$$

Then

$$(6.10) \quad B_{p_0, p_1}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1, q}^{s_1}(\mathbb{R}^d, w_1; X),$$

$$(6.11) \quad F_{p_0, q}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1, p_0}^{s_1}(\mathbb{R}^d, w_1; X).$$

*Proof.* For (6.10), by Proposition 3.11 it suffices to consider the case  $q = 1$  and  $s_0 - (d + \gamma)/p_0 = s_1 - (d + \gamma)/p_1$ . Let  $r_0, r_1 \in (1, \infty)$  be such that  $p_0 < r_0 < p_1 < r_1$ , and let  $\mu_0 \in (-d, d(r_0 - 1))$  and  $\mu_1 \in (-d, d(r_1 - 1))$  satisfy  $\mu_0/r_0 = \mu_1/r_1 = \gamma_1/p_1$ . Let further  $\theta \in (0, 1)$  and  $t_0, t_1 \in \mathbb{R}$  be such that

$$\frac{1}{p_1} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad t_0 - \frac{d + \gamma_0}{p_0} = s_1 - \frac{d + \mu_0}{r_0}, \quad t_1 - \frac{d + \gamma_0}{p_0} = s_1 - \frac{d + \mu_1}{r_1}.$$

Since  $(1 - \theta)t_0 + \theta t_1 = s_0$ , we deduce from (6.1) that

$$B_{p_0, p_1}^{s_0}(\mathbb{R}^d, w_0; X) = (F_{p_0, 1}^{t_0}(\mathbb{R}^d, w_0; X), F_{p_0, 1}^{t_1}(\mathbb{R}^d, w_0; X))_{\theta, p_1}.$$

Set  $v_0(x) = |x|^{\mu_0}$  and  $v_1(x) = |x|^{\mu_1}$ . Theorem 1.2 gives the embeddings

$$F_{p_0, 1}^{t_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{r_0, 1}^{s_1}(\mathbb{R}^d, v_0; X), \quad F_{p_0, 1}^{t_1}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{r_1, 1}^{s_1}(\mathbb{R}^d, v_1; X),$$

due to the definition of  $t_0, t_1$  and  $\mu_0/r_0 = \mu_1/r_1 = \gamma_1/p_1 \leq \gamma_0/p_0$ . Therefore

$$\begin{aligned} B_{p_0, p_1}^{s_0}(\mathbb{R}^d, w_0; X) &\hookrightarrow (F_{r_0, 1}^{s_1}(\mathbb{R}^d, v_0; X), F_{r_1, 1}^{s_1}(\mathbb{R}^d, v_1; X))_{\theta, p_1} \\ &\hookrightarrow F_{p_1, 1}^{s_1}(\mathbb{R}^d, w_1; X), \end{aligned}$$

where the latter embedding follows from (6.6), as  $1/p_1 = (1 - \theta)/r_0 + \theta/r_1$  and  $\gamma_1/p_1 = (1 - \theta)\mu_0/r_0 + \theta\mu_1/r_1$ .

To show (6.11), as above it suffices to consider  $q = \infty$  and  $s_0 - (d + \gamma)/p_0 = s_1 - (d + \gamma)/p_1$ . Let  $r_0, r_1 \in (1, \infty)$ ,  $\theta \in (0, 1)$ ,  $\mu_0 \in (-d, d(r_0 - 1))$  and  $\mu_1 \in (-d, d(r_1 - 1))$  be such that

$$r_0 < p_0 < r_1 < p_1, \quad \frac{1}{p_0} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{\mu_0}{r_0} = \frac{\mu_1}{r_1} = \frac{\gamma_0}{p_0}.$$

Setting again  $v_0(x) = |x|^{\mu_0}$  and  $v_1(x) = |x|^{\mu_1}$ , we find from (6.5) that

$$F_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0; X) = (F_{r_0,\infty}^{s_0}(\mathbb{R}^d, v_0; X), F_{r_1,\infty}^{s_0}(\mathbb{R}^d, v_1; X))_{\theta,p_0}.$$

Let the numbers  $t_0, t_1 \in \mathbb{R}$  be defined by

$$s_0 - \frac{d + \mu_0}{r_0} = t_0 - \frac{d + \gamma_1}{p_1}, \quad s_0 - \frac{d + \mu_1}{r_1} = t_1 - \frac{d + \gamma_1}{p_1}.$$

Using  $\gamma_1/p_1 \leq \gamma_0/p_0 = \mu_0/r_0 = \mu_1/r_1$ , Proposition 3.11 and Theorem 1.1 yield

$$\begin{aligned} F_{r_0,\infty}^{s_0}(\mathbb{R}^d, v_0; X) &\hookrightarrow B_{r_0,\infty}^{s_0}(\mathbb{R}^d, v_0; X) \hookrightarrow B_{p_1,\infty}^{t_0}(\mathbb{R}^d, w_1; X), \\ F_{r_1,\infty}^{s_0}(\mathbb{R}^d, v_1; X) &\hookrightarrow B_{r_1,\infty}^{s_0}(\mathbb{R}^d, v_1; X) \hookrightarrow B_{p_1,\infty}^{t_1}(\mathbb{R}^d, w_1; X). \end{aligned}$$

We thus have

$$\begin{aligned} F_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0; X) &\hookrightarrow (B_{p_1,\infty}^{t_0}(\mathbb{R}^d, w_1; X), B_{p_1,\infty}^{t_1}(\mathbb{R}^d, w_1; X))_{\theta,p_0} \\ &= B_{p_1,p_0}^{s_1}(\mathbb{R}^d, w_1; X), \end{aligned}$$

where the latter identity follows from  $(1 - \theta)t_0 + \theta t_1 = s_1$  and (6.1). ■

REMARK 6.5. For  $w_0 = w_1 \in A_\infty$ , the scalar versions of (6.10) and (6.11) are shown in [8, Theorem 2.6] and [18, Proposition 1.8].

REMARK 6.6. In the scalar case, the interpolation identities (6.1) and (6.2) are shown in [8] for  $w \in A_\infty$ . Inspecting the proof for (6.10), we see that only the interpolation embedding (6.6) was used, which is also true for  $w \in A_\infty$  by Remark 6.2. Hence (6.9) implies (6.10) for all weight exponents  $\gamma_0, \gamma_1 > -d$ .

Also for (6.11) we expect that the restrictions  $\gamma_0 < d(p_0 - 1)$  and  $\gamma_1 < d(p_1 - 1)$  are not necessary. In case  $\gamma_1/p_1 < \gamma_0/p_0$  one can give a more direct proof which only makes use of (6.5) for a single weight. The scalar version of (6.5) is shown in [8, Theorem 3.5]. Hence (6.9) implies (6.10) for all  $\gamma_0, \gamma_1 > -d$  under these assumptions.

However, the sharp case  $\gamma_1/p_1 = \gamma_0/p_0$  for (6.11) remains open. One needs (6.5) also for  $A_\infty$ -weights, which seems to be an open problem (see [8, Remark 3.4]).

**7. Embeddings into  $L^p$ - and Hölder spaces.** In this section we discuss conditions under which weighted spaces of smooth functions embed into function spaces such as  $L^p$ -spaces and spaces of Hölder continuous functions. The results are consequences of our main results.

PROPOSITION 7.1. *Let  $X$  be a Banach space, let  $1 < p_0, p_1 \leq \infty$ ,  $q_0 \in [1, p_0]$ . Let  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . Assume*

$$s_0 - \frac{d + \gamma_0}{p_0} \geq -\frac{d + \gamma_1}{p_1}, \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0} \quad \text{and} \quad \frac{d + \gamma_1}{p_1} < \frac{d + \gamma_0}{p_0}.$$

If  $p_0 \leq p_1$  or  $q_0 = 1$ , then

$$B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow L^{p_1}(\mathbb{R}^d, w_1; X),$$

PROPOSITION 7.2. *Let  $X$  be a Banach space, let  $1 < p_0 \leq p_1 < \infty$ ,  $q_0 \in [1, \infty]$ . Let  $w_0(x) = |x|^{\gamma_0}$  and  $w_1(x) = |x|^{\gamma_1}$  with  $\gamma_0, \gamma_1 > -d$ . If*

$$s_0 - \frac{d + \gamma_0}{p_0} \geq -\frac{d + \gamma_1}{p_1} \quad \text{and} \quad \frac{\gamma_1}{p_1} \leq \frac{\gamma_0}{p_0},$$

then

$$F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow L^{p_1}(\mathbb{R}^d, w_1; X),$$

*Proof of Propositions 7.1 and 7.2.* Let  $p_0 \leq p_1$ . By Proposition 3.11 and  $q_0 \leq p_0$  one has the embedding  $B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_0, p_0}^{s_0}(\mathbb{R}^d, w_0; X)$ . Therefore, Proposition 7.1 follows from Proposition 7.2 in this case. The embeddings

$$F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow F_{p_1, 1}^0(\mathbb{R}^d, w_1; X) \hookrightarrow L^{p_1}(\mathbb{R}^d, w_1; X)$$

are consequences of Theorem 1.2 and Proposition 3.12 (resp. Remark 3.13).

If  $p_0 > p_1$  and  $q_0 = 1$ , then by Theorem 1.1,

$$B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{p_1, 1}^0(\mathbb{R}^d, w_0; X) \hookrightarrow L^{p_1}(\mathbb{R}^d, w_1; X),$$

where the last embedding follows again from Proposition 3.12 (resp. Remark 3.13). ■

REMARK 7.3. Many other results can be derived from Theorems 1.1 and 1.2. Moreover, employing Proposition 3.11, we see that a similar result to Proposition 7.2 above holds for  $H^{s_0, p_0}(\mathbb{R}^d, w_0; X)$  and  $W^{s_0, p_0}(\mathbb{R}^d, w_0; X)$  if  $\gamma_0 < d(p_0 - 1)$ .

For  $m \in \mathbb{N}$ , let  $\text{BUC}^m(\mathbb{R}^d; X)$  denote the space of  $m$ -times differentiable functions with bounded and uniformly continuous derivatives. For  $s = [s] + s_*$  with  $[s] \in \mathbb{N}_0$  and  $s_* \in (0, 1)$ , let further  $\text{BUC}^s(\mathbb{R}^d; X)$  be the subspace of  $\text{BUC}^{[s]}(\mathbb{R}^d; X)$  consisting of functions with  $s_*$ -Hölder continuous derivatives of order  $[s]$ .

PROPOSITION 7.4. *Let  $X$  be a Banach space, let  $1 < p_0 < \infty$ ,  $q_0 \in [1, \infty]$  and  $s_0 \in \mathbb{R}$ . Let  $w_0(x) = |x|^{\gamma_0}$  with  $\gamma_0 \geq 0$ . If  $s_1 = s_0 - (d + \gamma_0)/p_0 > 0$  is not an integer, then*

$$(7.1) \quad E^{s_0, p_0}(\mathbb{R}^d, w_0; X) \hookrightarrow \text{BUC}^{s_1}(\mathbb{R}^d; X),$$

where  $E^{s_0, p_0} \in \{F_{p_0, q_0}^{s_0}, B_{p_0, q_0}^{s_0}\}$ . If  $m = s_0 - (d + \gamma_0)/p_0 \geq 0$  is an integer, then

$$(7.2) \quad B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow \text{BUC}^m(\mathbb{R}^d; X).$$

Assuming that  $0 \leq \gamma_0 < d(p_0 - 1)$ , these embeddings are also valid for  $E^{s_0, p_0} = H^{s_0, p_0}$  and, if  $s_0 \in \mathbb{N}$ , for  $E^{s_0, p_0} = W^{s_0, p_0}$ .

*Proof.* To prove (7.1), by Propositions 3.11 and 3.12 it suffices to show that

$$B_{p_0, \infty}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow \text{BUC}^{s_1}(\mathbb{R}^d; X).$$

This embedding is a consequence of Theorem 1.1 and

$$B_{\infty, \infty}^{s_1}(\mathbb{R}^d, w_0; X) = B_{\infty, \infty}^{s_1}(\mathbb{R}^d; X) = \text{BUC}^{s_1}(\mathbb{R}^d; X),$$

where the latter identity can be proved as in the scalar case (see [41, Theorem 2.5.7]).

For (7.2), Theorem 1.1 yields

$$B_{p_0, 1}^{s_0}(\mathbb{R}^d, w_0; X) \hookrightarrow B_{\infty, 1}^m(\mathbb{R}^d; X).$$

We further have

$$B_{\infty, 1}^m(\mathbb{R}^d; X) \hookrightarrow \text{BUC}^m(\mathbb{R}^d; X),$$

due to Proposition 3.10 and (the proof of) [41, Theorem 2.5.7]. ■

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