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Weak* properties of weighted convolution algebras II

by

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Abstract. We show that if ϕ is a continuous homomorphism between weighted convolution algebras on $\mathbb{R}^+$, then its extension to the corresponding measure algebras is always weak* continuous. A key step in the proof is showing that our earlier result that normalized powers of functions in a convolution algebra on $\mathbb{R}^+$ go to zero weak* is also true for most measures in the corresponding measure algebra. For some algebras, we can determine precisely which measures have normalized powers converging to zero weak*. We also include a variety of applications of weak* results, mostly to norm results on ideals and on convergence.

1. Introduction. We study weak* properties of weighted convolution algebras on $\mathbb{R}^+=[0,\infty)$. Our main result, Theorem 1.1, says that every continuous homomorphism between such algebras is also continuous in the (relative) weak* topologies. A positive Borel function ω on $\mathbb{R}^+$ is a *weight* if both ω and $1/\omega$ are locally bounded on $[0,\infty)$. Then $L^1(\omega)$ is the Banach space of (equivalence classes of) locally integrable functions f for which $f\omega$ is integrable. We give $L^1(\omega)$ the inherited norm

$$\|f\| = \|f\|_\omega = \int_0^\infty |f(t)|\omega(t)\,dt.$$

In a similar way, $M(\omega)$ is the space of locally finite complex measures on $[0,\infty)$ which are finite in the norm

$$\|\mu\| = \|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\,d|\mu|(t).$$

We identify $L^1(\omega)$ as a subspace of $M(\omega)$ in the usual way.

We are particularly interested in the case where $L^1(\omega)$ is an algebra under convolution. Hence, we usually assume that ω is an *algebra weight* in the sense that ω is submultiplicative (that is, $\omega(x+y) \le \omega(x)\omega(y)$),

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is everywhere right continuous, and has $\omega(0) = 1$. Requiring ω to be an algebra weight is just a normalization. Whenever $L^1(\omega)$ is an algebra, we can always replace the given weight by an algebra weight without changing the space $L^1(\omega)$ or its norm topology [10, Theorem 2.1, p. 591]. We will usually implicitly assume that the weights we are considering are algebra weights.

Since an algebra weight is submultiplicative, both $M(\omega)$ and $L^1(\omega)$ are algebras, and $L^1(\omega)$ is a closed ideal in $M(\omega)$. The other conditions on an algebra weight guarantee that $M(\omega)$ is isometrically isomorphic to the multiplier algebra of $L^1(\omega)$ and the dual space of $C_0(1/\omega)$ [10, Theorem 2.2, p. 592]. Here $C_0(1/\omega)$ is the Banach space of continuous functions h on $\mathbb{R}^+$ for which h/ω is bounded and vanishes at infinity. We give $C_0(1/\omega)$ the inherited norm $\|h\| = \|h/w\|_\infty$. We often identify the measure μ in $M(\omega)$ with the multiplier $f \mapsto \mu * f$ on $L^1(\omega)$ and with the linear functional $\langle \mu, h\rangle = \int_{\mathbb{R}^+} h(t)\, d\mu(t)$ on $C_0(1/\omega)$. Thus, we can speak of the weak* and strong-operator topologies on $M(\omega)$ and on its subspace $L^1(\omega)$. In particular, we identify the semigroup of point masses $\{\delta_t\}_{t\geq 0}$ in $M(\omega)$ with the strongly continuous semigroup of right translations on $L^1(\omega)$.

Much of what we have learned about weighted convolution algebras on $\mathbb{R}^+$ in the last 30 years is proved, in part, by using weak* methods, but the first systematic study of weak* results seems to be in [6]. Not only are weak* results usually simpler and more universal than norm results, but weak* results can often be used to prove norm results and nontopological results.

Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism; then ϕ has a unique extension to a homomorphism between the corresponding measure algebras, and this extension has the same norm as the original map [10, Theorem 3.4, p. 596]. (The extension is constructed by using the fact that the image under ϕ of a bounded approximate identity in $L^1(\omega_1)$ is a weak* approximate identity.) Because of the uniqueness, we let ϕ designate both the original map and its extension. We can now state our main result more precisely.

Theorem 1.1. *If $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism, then its extension to the corresponding measure algebras is weak* continuous.*

Our proof will include verifying the necessary and sufficient condition for weak* continuity in our earlier paper [11, Theorem 4.7, p. 1681].

The main result of our earlier paper [11, Theorem 3.1, p. 1678] is that if $L^1(\omega_1) * f$ is dense in $L^1(\omega_1)$, then $L^1(\omega_2) * \phi(f)$ is weak* dense in $L^1(\omega_2)$. This result is motivated by the standard homomorphism problem first studied in [9]. A consequence of this result [11, Theorem 3.3 p. 1679] is that when ϕ is weak* continuous it is enough to assume that $L^1(\omega_1) * f$ is weak* dense. Thus we have the following consequence of Theorem 1.1.

COROLLARY 1.2. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism. If $L^1(\omega_1) * f$ is weak* dense in $L^1(\omega_1)$, then $L^1(\omega_2) * \phi(f)$ is weak* dense in $L^1(\omega_2)$.*

In [11, Section 5] we showed that if f is a nonzero element of $L^1(\omega)$, then the sequence of normalized powers $f^n/\|f^n\|$ converges to 0 in the weak* topology. In Theorem 3.3 below, we show that the same result is true for "most" measures μ in $M(\omega)$. In [11] the result was unrelated to the results on homomorphisms and was used to show that, for a class of $L^1(\omega)$, the sequence $f^{n+1}/\|f^n\|$ converged to 0 in norm [11, Cor. 5.2, p. 1683]. (For a discussion of the problem of when $x^{n+1}/\|x^n\|$ converges to zero in a Banach algebra, see [11, Section 5] and the references cited there, particularly [13].) In the present paper, the normalized power result is needed to prove our weak* continuity result. Actually, it is not precisely the normalized power result we use, but rather the following variant for semigroups.

THEOREM 1.3. *Suppose that μ_t is a weak* continuous semigroup in $M(\omega)$. If all μ_t for $t > 0$ have no point mass at 0, then*

$$\text{weak}^*\text{-}\lim_{t\to\infty} \frac{\mu_t}{\|\mu_t\|} = 0.$$

In Section 2 we prove Theorem 1.1, assuming Theorem 1.3. The main ingredient in the proof, Lemma 2.1, shows that the semigroup $\mu_t = \phi(\delta_t)$ satisfies the hypothesis of Theorem 1.3 above. More generally we show, in Theorem 2.2, that if λ in $M(\omega_1)$ has no point mass at 0, then neither does $\phi(\lambda)$.

In Section 3 we prove Theorem 1.3 above and its analogue for normalized powers, both under slightly weaker hypotheses than given above. In Section 4 we examine necessary conditions for $\mu^n/\|\mu^n\|$ to converge to 0 weak* in $M(\omega)$. We are only able to find necessary and sufficient conditions in some cases. For the finite-interval convolution algebras $M[0,a)$, the sufficient conditions we found are also necessary. At the other extreme, for $L^1(\omega)$ which admit nonzero derivations we show that, except for multiples of the identity δ_0, all μ have $\mu^n/\|\mu^n\|$ converging to zero weak*.

In Section 5, we give norm topology applications of weak* results, mostly to ideals and convergence. One application, Theorem 5.3, shows that if the principal ideal $L^1(\omega) * f$ is weak* dense, then all $L^1(\omega) * (e^{-at}f(t))$ are norm dense for $\mathrm{Re}(a) > 0$. Recently Charles Read [14] solved the main open problem about ideals in radical $L^1(\omega)$ by constructing an $L^1(\omega)$ with a function f with $\alpha(f) = \inf(\text{support } f) = 0$ and $L^1(\omega) * f$ not norm dense. He also showed [14, Section 10] that one can arrange for $L^1(\omega) * f$ to not even be weak* dense. He then conjectured [14, Section 11] that every radical

$L^1(\omega)$ with nondense principal ideal generated by some f with $\alpha(f) = 0$ also has a principal ideal which is not weak* dense. Results in the present paper relating norm dense and weak* dense ideals, such as Theorem 5.3 and Proposition 5.4, might help in studying this conjecture.

2. Weak* continuity. Throughout this section, we let ω_1 and ω_2 be algebra weights and ϕ a continuous nonzero homomorphism from $L^1(\omega_1)$ to $L^1(\omega_2)$. As always, ϕ is extended to the corresponding measure algebras and $\{\delta_t\}$ is the semigroup of point masses. In this section we show that ϕ is weak* continuous, proving Theorem 1.1. Our main tool is the following lemma, which shows that $\mu_t = \phi(\delta_t)$ satisfies the hypotheses of Theorem 1.3.

LEMMA 2.1. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism. If $\mu_t = \phi(\delta_t)$, then $\{\mu_t\}$ is a weak* continuous semigroup in $M(\omega_2)$ for which $\mu_t\{0\} = 0$ for all $t > 0$.*

Proof. That μ_t is weak* continuous [10, Theorem 3.6, p. 599] is an easy consequence of the cancellation property for weak* convergence [10, Lemma 3.2, p. 595].

For each n we define the piecewise linear function h_n as 0 on $[1/n, \infty)$ and linear from the point (0,1) to the point $(1/n, 0)$ on $[0, 1/n]$. Then $h_n(x)$ converges pointwise to the characteristic function of $\{0\}$, which we denote by χ_0. If λ is any locally finite measure on $\mathbb{R}^+ = [0, \infty)$, it follows from the Lebesgue dominated convergence theorem that $\langle \lambda, h_n \rangle$ converges to $\langle \lambda, \chi_0 \rangle = \lambda(\{0\})$.

Since μ_t is weak* continuous at 0, $\|\mu_t\|$ is bounded in a neighborhood of 0, so that $\mu_t\{0\}$ is also bounded near 0. It follows from the formulas for convolutions of measures that $\mu_{s+t}\{0\} = \mu_s * \mu_t\{0\} = (\mu_s\{0\})(\mu_t\{0\})$. So if $\mu_t\{0\}$ is not 0 for some, and hence all, $t > 0$, there is a complex number c for which $\mu_t\{0\} = e^{-ct}$ (see [10, p. 605] or [5, p. 348]). We will show that this leads to a contradiction, so that we must have all $\mu_t\{0\} = 0$ for $t > 0$.

One can represent ϕ in terms of a weak* integral [10, p. 599] so that for all f in $L^1(\omega_1)$ and all h in $C_0(1/\omega_2)$ we have

$$\langle \phi(f), h \rangle = \int_{\mathbb{R}^+} f(t) \langle \mu_t, h \rangle \, dt.$$

Since $\langle \mu_t, h_n \rangle \to \mu_t(0) = e^{-ct}$, the limit of $\langle \phi(f), h_n \rangle$ is $\hat{f}(c)$, where $\hat{f}$ is the Laplace transform of f. On the other hand, as an element of $M(\omega_2)$ the function $\phi(f)$ has no point mass at 0, so $\langle \phi(f), h_n \rangle$ has limit 0.

So we know that if our lemma were false, then we could find a complex number c for which, for all f in $L^1(\omega_1)$, the Laplace transform $\hat{f}(t)$ is defined and equals 0 at $z = c$. But one can always find an $r > 0$ for which $f = e^{-rt}$ belongs to $L^1(\omega_1)$. For this f, $\hat{f}(c) = 1/(r + c)$ cannot be 0. ∎

Thus we have proved the lemma and can now use it to prove Theorem 1.1. Our proof will rely on Theorem 1.3, which we will prove in the next section.

Proof of Theorem 1.1. By [11, Theorem 4.7, p. 1681] we just need to show that, for the semigroup $\mu_t = \phi(\delta_t)$, the quotient $\mu_t/\omega_1(t)$ goes to 0 weak* in $M(\omega_2)$ as t goes to infinity. We write

$$\frac{\mu_t}{\omega_1(t)} = \frac{\mu_t}{\|\mu_t\|}\,\frac{\|\mu_t\|}{\omega_1(t)}.$$

By Lemma 2.1 and Theorem 1.3, $\mu_t/\|\mu_t\|$ does go to 0 weak* in $M(\omega_2)$. Also, $\|\mu_t\| = \|\phi(\delta_t)\|_{\omega_2} \le \|\phi\|\,\|\delta_t\|_{\omega_1} = \|\phi\|\omega_1(t)$. Thus $\|\mu_t\|/\omega_1(t)$ is bounded, so that $\mu_t/\omega_1(t)$ does converge to 0 weak* in $M(\omega_2)$. This completes the proof. ∎

We can use Lemma 2.1 to prove the stronger result that whenever $\lambda\{0\} = 0$, then $\phi(\lambda)\{0\} = 0$. The following theorem formulates this property in a slightly different, but equivalent, way.

THEOREM 2.2. *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism. Then for all λ in $M(\omega_1)$, we have $\phi(\lambda)\{0\} = \lambda\{0\}$.*

Proof. We use the same functions h_n and χ_0 as in the proof of Lemma 2.1 and the fact that all $\langle \mu, h_n\rangle$ converge to $\mu\{0\}$. Thus $\langle \lambda, h_n\rangle \to \lambda\{0\}$ and $\langle \phi(\lambda), h_n\rangle \to \phi(\lambda)\{0\}$. So we need to show that $\langle \phi(\lambda), h_n\rangle \to \lambda\{0\}$.

Since μ_0 is just the point mass δ_0, we can rephrase the conclusion of Lemma 2.1 as saying $\mu_t\{0\} = \chi_0(t)$ for $t \ge 0$. Using the weak* integral formula for $\phi(\lambda)$, we see that $\langle \phi(\lambda), h_n\rangle = \int \langle \mu_t, h_n\rangle\, d\lambda$. By the Lebesgue dominated convergence theorem, this converges to $\int \mu_t\{0\}\, d\lambda = \int \chi_0\, d\lambda = \lambda\{0\}$ as required. ∎

3. Powers and semigroups: sufficient conditions. In this section we prove Theorem 1.3 and its analogue for powers of an element. We will actually prove slightly stronger results. In Section 4 we will examine necessary conditions for these results to hold. For a measure μ in $M_{\mathrm{loc}}(\mathbb{R}^+)$ we use the standard notation [3, Definition 4.7.18, p. 528] $\alpha(\mu)$ for the inf of the support of μ, with $\alpha(0) = \infty$. We will also need the Titchmarsh convolution theorem, which says $\alpha(\mu * \nu) = \alpha(\mu) + \alpha(\nu)$ (see [3, Theorem 4.7.22, p. 529]).

We will prove $\mu^n/\|\mu^n\|$ converges to 0 weak* in $M(\omega)$ if either $\alpha(\mu) > 0$ or $\alpha(\mu - \mu\{0\}\delta_0) = 0$ (we can also write $\mu - \mu\{0\}\delta_0$ as the restriction of μ to $(0, \infty)$). In other words, μ does not satisfy our hypotheses precisely when it is of the form $c\delta_0 + \nu$ with $c \ne 0$ and $\alpha(\nu) > 0$. As with our proof for functions [11, Section 5], we start with the result in $M[0, a)$ and obtain the result for $M(\omega)$ from the result for finite intervals. When dealing with functions, we were able to use Solovej's [15] result in the finite-interval case, but his proof does not generalize to measures. We still borrow Solovej's insights that derivations and compactness are needed in the finite interval case, but we need to construct a new proof.

In the algebra $M[0,a)$, μ is nilpotent for $\alpha(\mu) > 0$, so in this case we only consider $\mu^n/\|\mu^n\|$ when $\alpha(\mu) = 0$. We now state our results for powers and for semigroups in$M[0,a)$.

THEOREM 3.1. *Suppose that μ is a measure in $M[0,a)$ with $\alpha(\mu) = 0$. If $\alpha(\mu - \mu\{0\}\delta_0) = 0$, then $\mu^n/\|\mu^n\|$ converges to 0 weak* in $M[0,a) = C_0[0,a)^*$.*

THEOREM 3.2. *Suppose that μ_t is a weak* continuous semigroup in $M[0,a)$ with $\alpha(\mu_t) = 0$ for some (and hence all) $t > 0$. If $\alpha(\mu_t - \mu_t\{0\}\delta_0) = 0$ for some $t > 0$, then $\mu_t/\|\mu_t\|$ approaches 0 weak* in $M[0,a)$ as t goes to infinity.*

It is not hard to prove that if the hypothesis of Theorem 3.2 holds for some $t > 0$, then it holds for all $t > 0$; but it is simpler to prove the theorem as stated. Also, note that in $M[0,a)$, weak* and strong continuity of a semigroup are equivalent.

The proofs of the power theorem and the semigroup theorem are essentially the same, but the semigroup theorem has some added technicalities. We will therefore give the full proof in the semigroup case and then indicate the modifications for the case of powers of an element.

Proof of Theorem 3.2. To simplify the notation in the proof we will write $\mu(t)$ for μ_t. By rescaling if necessary, we can assume that the hypothesis holds for $t = 1$; that is, $\alpha(\mu(1) - \mu(1)\{0\}\delta_0) = 0$. Closed bounded sets in $M[0,a)$ are metrizable and sequentially compact in the weak* topology, so to prove that $\mu(t)/\|\mu(t)\|$ goes to 0 weak*, it will be enough to show that if a sequence $\{t_k\}$ of positive numbers goes to infinity and if $\mu(t_k)/\|\mu(t_k)\| \to \lambda$ weak*, then $\lambda = 0$. We assume, without loss of generality, that $t_{k+1} > t_k + 1$.

On $M[0,a)$, we define the derivation $D(\nu) = x\nu$ (see [12]). Since D is weak* continuous, as is convolution with a fixed measure, we deduce that the sequence $\mu(1) * D(\mu(t_n)/\|\mu(t_n)\|)$ is weak* convergent and therefore is norm bounded.

For each k we let $t_k = n_k + r_k$, with n_k an integer and $0 \le r_k < 1$. Using the fact that D is a derivation, together with the semigroup property of $\mu(t)$, we rewrite

$$\mu(1) * D\left(\frac{\mu(t_k)}{\|\mu(t_k)\|}\right) = \mu(1) * D\left(\frac{\mu(1)^{n_k} * \mu(r_k)}{\|\mu(t_k)\|}\right),$$

which equals

$$\left(\mu(1) * \frac{D(\mu(1)^{n_k})}{\|\mu(t_k)\|} * \mu(r_k)\right) + \left(\frac{\mu(1) * \mu(1)^{n_k}}{\|\mu(t_k)\|} * D(\mu(r_k))\right). \tag{3.1}$$

We now prove that the second term in (3.1) is bounded, which will show that the first term is bounded as well. We rewrite the second term in (3.1) as

$$\frac{\mu(n_k+1)}{\|\mu(t_k)\|} * D(\mu(r_k)) = \frac{\mu(t_k)}{\|\mu(t_k)\|} * \mu(1-r_k) * D(\mu(r_k)).$$

Now both r_k and $1-r_k$ belong to the interval $[0,1]$, on which both $\mu(t)$ and $D\mu(t)$ are bounded. Thus the right side of (3.1) can be written as a product of three bounded sequences. Hence both terms in (3.1) are bounded.

Using the semigroup and derivation properties we can rewrite the first term in (3.1) as

$$\frac{n_k(\mu(1))^{n_k} * D(\mu(1)) * \mu(r_k)}{\|\mu(t_k)\|} = n_k\biggl(\frac{\mu(t_k)}{\|\mu(t_k)\|} * D(\mu(1))\biggr).$$

Since the first term in (3.1) is bounded, this implies that the sequence $(\mu(t_k)/\|\mu(t_k)\|) * D\mu(1)$ converges to 0 in norm. But we already have this sequence converging weak* to $\lambda * D(\mu(1))$. Hence $\lambda * D(\mu(1)) = \lambda * (x\mu) = 0$. But our assumption on $\mu(1)$ is equivalent to $\alpha(x\mu) = 0$. Hence, by the Titchmarsh convolution theorem, $x\mu$ is not a divisor of 0 in $M[0,a)$. Thus $\lambda = 0$, and the proof of Theorem 3.2 is complete. ∎

Proof of Theorem 3.1. If we replace $\mu(1)$ by μ and $\mu(n)$ by μ^n when n is an integer, the proof of Theorem 3.1 is the same as that of Theorem 3.2 in the case of $t_k = n_k$ and $r_k = 0$. The proof in this case is much simpler, since $\mu(r_k) = \delta_0$ and $D(\mu(r_k)) = 0$. Hence the second term in (3.1) is zero and the first term simplifies to $\mu * (D(\mu^{n_k})/\|(\mu^{n_k})\|)$. ∎

We now give the versions of Theorems 3.1 and 3.2 which hold for $M(\omega)$ where ω is an algebra weight.

THEOREM 3.3. *Suppose that μ is a nonzero measure in $M(\omega)$. If either $\alpha(\mu) > 0$ or $\alpha(\mu - \mu\{0\}\delta_0) = 0$, then $\mu^n/\|\mu^n\|$ converges to 0 weak* in $M(\omega)$.*

THEOREM 3.4. *Suppose that μ_t is a weak* continuous semigroup in $M(\omega)$. If there is a $t > 0$ with $\alpha(\mu_t) > 0$ or $\alpha(\mu_t - \mu_t\{0\}\delta_0) = 0$, then $\mu_t/\|\mu_t\|$ converges to 0 weak* as t goes to infinity.*

Proof. The proofs in the semigroup and power cases are nearly identical, so we just give the slightly more complicated proof for the semigroup case.

When some $\alpha(\mu_t) > 0$, there is an $A > 0$ for which $\alpha(\mu_t) = At$ [5, Lemma 1, p. 344], [10, Theorem 4.3, p. 605]. Hence $\lim_{t\to\infty} \alpha(\mu_t/\|\mu_t\|) = 0$. Just as for sequences (see [11, Lemma 4.3, p. 1680]), it then follows that $\mu_t/\|\mu_t\|$ goes to 0 weak*.

We now consider the case where all $\alpha(\mu_t) = 0$. For a measure λ in $M(\omega)$ we let $\|\lambda\|_\omega$ be the norm in $M(\omega)$; and for each positive a we let $\|\lambda\|_a$ be $|\lambda|([0,a))$, the norm in $M[0,a)$ of λ (when restricted to $[0,a)$). To show that

$\mu_t/\|\mu_t\|_\omega$ goes to 0 weak*, it is enough to show that $\langle \mu_t/\|\mu_t\|_\omega, h\rangle$ goes to 0 weak* whenever h is a continuous function with compact support [7, p. 52], which is the same thing as showing that, for all a, its restriction to $[0,a)$ goes to 0 weak* in $M[0,a)$.

Fix $a>0$. Then there is a $K>0$ for which $\omega(x)\geq K$ on $[0,a)$, so that $\|\lambda\|_\omega \geq K\|\lambda\|_a$. Now write

$$\frac{\mu_t}{\|\mu_t\|_\omega} = \frac{\|\mu_t\|_a}{\|\mu_t\|_\omega}\,\frac{\mu_t}{\|\mu_t\|_a}.$$

By Theorem 3.2, we know that $\mu_t/\|\mu_t\|_a$ goes to 0 weak* in $M[0,a)$. Since $\|\mu_t\|_a/\|\mu_t\|_\omega \leq 1/K$ for all t, we therefore have $\mu_t/\|\mu_t\|_\omega$ going to 0 weak* in all $M[0,a)$, so it also goes to 0 weak* in $M(\omega)$. This completes the proof. ∎

4. Powers and semigroups: necessary conditions. In this section, we investigate necessary conditions on μ for $\mu^n/\|\mu^n\|$ to converge to 0 weak* in $M[0,a)$ or in some $M(\omega)$. We will emphasize the results for powers, and just sketch the analogous results for semigroups. First, there is a trivial case for which μ works in no $M[0,a)$ or $M(\omega)$.

PROPOSITION 4.1. *If μ is a nonzero multiple of the identity δ_0, then $\mu^n/\|\mu^n\|$ cannot converge weak* to zero in any algebra $M[0,a)$ or $M(\omega)$ containing μ. The analogous result holds for semigroups.*

Proof. Suppose that $\mu = c\delta_0$ for some $c\neq 0$. Choose a continuous function h in the predual with $h(0)=1$. Then $\langle \mu^n/\|\mu^n\|, h\rangle = c^n/|c^n|$ cannot converge to 0. If μ_t is a weak* continuous semigroup with some μ_a a multiple of δ_0, then the result on powers shows that $\mu_{an}/\|\mu_{an}\|$ cannot converge to 0 weak*. This completes the proof of the proposition. ∎

We are not able to determine in all cases exactly which μ have $\mu^n/\|\mu^n\|$ converging to 0 weak*, but we can get definitive answers in two cases. For the finite-interval case, the sufficient condition we obtained in Theorem 3.1 above is also necessary. At the other extreme, if the algebra $L^1(\omega)$ has a nonzero derivation, then, except for the trivial case in Proposition 4.1, all $\mu^n/\|\mu^n\|$ go to 0 weak*. We start with the finite-interval case.

THEOREM 4.2. *Suppose that μ is a measure in $M[0,a)$ with $\alpha(\mu)=0$. Then $\mu^n/\|\mu^n\|$ converges to 0 weak* in $M[0,a)$ if and only if $\alpha(\mu-\mu\{0\}\delta_0) = 0$.*

Proof. In Theorem 3.1 we showed that if the condition in the current theorem on μ holds, then the normalized powers go to 0 weak*. We complete the proof by assuming that the condition on μ does not hold and then showing that one cannot have $\mu^n/\|\mu^n\|$ going to 0 weak*. We know that μ is of the form $c\delta_0+\nu$ with $c\neq 0$ and $\alpha(\nu)>0$. Since 0 convergence is unchanged by multiplication by a nonzero constant, we let $c = 1$. Also, the case where ν is 0 in $M[0,a)$, that is, $\alpha(\nu) \geq a$, is covered by Proposition 4.1.

So we suppose that $\mu = \delta_0 + \nu$ with $0 < \alpha(\nu) < a$, and show that $\mu^n/\|\mu^n\|$ cannot converge to 0 weak*. Let K be the positive integer for which $K\alpha(\nu) < a \leq (K+1)\alpha(\nu)$. It then follows from the Titchmarsh convolution theorem that ν is a nilpotent element of order $K+1$ in $M[0,a)$. Then for all n, we have $\mu^n = (\delta_0+\nu)^n = \delta_0 + n\nu + \binom{n}{2}\nu^2 + \cdots + \binom{n}{K}\nu^K$. Also, $\|\mu^n\| \leq c_n$, where $c_n = 1 + n\|\nu\| + \binom{n}{2}\|\nu^2\| + \cdots + \binom{n}{K}\|\nu^K\|$. Choose a function h in the predual $C_0[0,a)$ with $\langle \nu^K, h\rangle$ strictly positive. For all n, we have $|\langle \mu^n, h\rangle|/\|\mu^n\| \geq |\langle \mu^n, h\rangle|/c_n$, so it will be enough to show that $|\langle \mu^n, h\rangle|/c_n$ does not converge to 0. Our formulas for μ^n and c_n show that $\langle \mu^n, h\rangle$ is asymptotic to $(n^K/K!)\langle \nu^K, h\rangle$ as n goes to infinity and c_n is asymptotic to $(n^K/K!)\|\nu^K\|$. Thus $|\langle \mu^n, h\rangle|/c_n$ is asymptotic to $\langle \nu^K, h\rangle/\|\nu^K\|$ and hence cannot converge to 0. This completes the proof. ∎

Necessary conditions on $\mu^n/\|\mu^n\|$ always imply the analogous conditions on semigroups. For if some $(\mu_a)^n/\|(\mu_a)^n\| = \mu_{na}/\|\mu_{na}\|$ does not go to 0 weak*, then neither does $\mu_t/\|\mu_t\|$ as t goes to infinity. Hence we have the following consequence of Theorem 4.2.

Corollary 4.3. *Suppose that μ_t is a weak* continuous semigroup in $M[0,a)$ with $\alpha(\mu_t) = 0$ for $t > 0$. Then $\mu_t/\|\mu_t\|$ goes to 0 weak* in $M[0,a)$ as t goes to infinity if and only if for all (equivalently, for some) $t > 0$, we have $\alpha(\mu_t - \mu_t\{0\}\delta_0) = 0$.*

We now look at those $L^1(\omega)$ which admit nonzero derivations. In [4, Section 2], Ghahramani has a comprehensive study of such derivations and their extensions to the corresponding $M(\omega)$. In particular, he gives a concrete condition on the algebra weight ω which is equivalent to the existence of nonzero derivations. The theorem below is a summary of many of his results.

Theorem 4.4 ([4]). *The convolution algebra $L^1(\omega)$ has a nonzero derivation if and only if there is a positive number b for which*

$$\sup_{t\in\mathbb{R}^+} \frac{t\omega(t+b)}{\omega(t)}$$

is finite. Moreover, there is a locally finite measure ν on $\mathbb{R}^+$ for which the derivation, extended to $M(\omega)$, has the form $D(\mu) = (x\mu) * \nu$.

Ghahramani also gives a necessary and sufficient condition on ν for $D(\mu) = (x\mu) * \nu$ to be a derivation on $M(\omega)$ [4, Theorem 2.5, pp. 153–154]. Since in our next theorem we will only use the existence of a single derivation, we could use the simpler result in [4, p. 155] which shows that the above condition on ω is equivalent to $D(\mu) = (x\mu) * \delta_b$ being a derivation on $M(\omega)$.

For algebras $L^1(\omega)$ with derivations, we now prove that all $\mu^n/\|\mu^n\|$ go to 0 weak*, except for the trivial case of Proposition 4.1.

THEOREM 4.5. *Suppose that the algebra $L^1(\omega)$ has a nonzero derivation. If the measure μ in $M(\omega)$ is not a multiple of the identity δ_0, then $\mu^n/\|\mu^n\|$ converges to 0 weak* in $M(\omega)$.*

Proof. As in Theorem 3.1, we just need to show that if the subsequence $\mu^{n_k}/\|\mu^{n_k}\|$ converges weak* to λ in $M(\omega)$, then $\lambda = 0$. Choose a nonzero locally finite measure ν for which $D(m) = (xm) * \nu$ defines a derivation on $M(\omega)$. As in the proofs of Theorems 3.1 and 3.2, we have $\lambda * D(\mu) = 0$. For our derivation, this means that $\lambda * ((x\mu) * \nu) = 0$. Since μ does not vanish on $(0, \infty)$, $x\mu$ is a nonzero measure. Since the convolution of nonzero measures on $\mathbb{R}^+$ is never 0, this shows that $\lambda = 0$ as required. ∎

One can make the same modifications of the proof of Theorem 3.2 to get the following semigroup analogue. In this case, if some μ_t with $t > 0$ is a multiple of δ_0, then all μ_t are multiples of δ_0, and μ_t is of the form $e^{-at}\delta_0$.

THEOREM 4.6. *Suppose that the convolution algebra $L^1(\omega)$ has nonzero derivations and that μ_t is a weak* continuous semigroup in $M(\omega)$. If μ_t is not a multiple of δ_0, then*

$$\text{weak}^*\text{-}\lim_{t\to\infty} \frac{\mu^t}{\|\mu_t\|} = 0 \qquad \textit{in } M(\omega).$$

When $L^1(\omega)$ does not have nonzero derivations, we do not know precisely which μ in $M(\omega)$ have their normalized powers converging weak* to 0. We do know that the sufficient condition of Theorem 3.3 is not necessary. For instance, the measure $\mu = \delta_0 + \delta_1$ has $\mu^n/\|\mu^n\|$ going to 0 weak* in all algebras $M(\omega)$ with $\lim_{t\to 0}\omega(t) = 0$. To show this, it is enough to show that if h is a nonzero continuous function with compact support in $[0, \infty)$, then $\langle \mu^n, h\rangle/\|\mu^n\| \to 0$ (see [7, p. 52]). If we let K be the largest nonnegative integer with $h(K) \neq 0$, then calculations like those in the proof of Theorem 4.2 show that $\langle \mu^n, h\rangle$ is asymptotic to $(n^K/K!)h(K)$, while for $n > K$, $\|\mu^n\| \geq \binom{n}{K}\omega(K)$, which is asymptotic to $(n^K/K!)\omega(K)$. Thus $\langle \mu^n, h\rangle/\|\mu^n\|$ does converge to 0 in $M(\omega)$. For arbitrary ω we can replace $\omega(t)$ with $e^{-at}\omega(t)$ for a sufficiently large $a > 0$. This is equivalent to replacing $\delta_0 + \delta_1$ with $e^{-at}(\delta_0 + \delta_1) = \delta_0 + e^{-a}\delta_1$ in $M(\omega)$.

On the other hand, we currently do not know of any μ, except multiples of δ_0, in any algebra $M(\omega)$, for which the normalized powers do not go to 0 weak*. This suggests the following open question:

Question 4.7. *For which μ and $M(\omega)$ does $\mu^n/\|\mu^n\|$ converge to* 0 *weak* ?*

In [11, Corollary 5.2, p. 1683] we used the weak* convergence of $f^n/\|f^n\|$ to prove the norm convergence to 0 of $f^{n+1}/\|f^n\|$ in $L^1(\omega)$ when the weight $\omega(t)$ is regulated in the sense of Bade and Dales [1]. The norm compactness properties of $L^1(\omega)$ for regulated weights do not hold in $M(\omega)$, so our argument does not generalize to measures. All we have is the elementary result that $\mu^n/\|\mu^n\|$ converges to 0 weak* if and only if $\mu^{n+1}/\|\mu^n\|$ converges to 0 weak*. So the following question seems natural.

Question 4.8. *For which μ and $M(\omega)$ does $\mu^{n+1}/\|\mu^n\|$ converge to* 0 *in norm?*

We do not even know the answer for functions if the weight $\omega(t)$ is not regulated. H. G. Dales pointed out to me that if $\omega(t)$ is regulated so that

$$\lim_{t\to\infty}\frac{\omega(t+a)}{\omega(t)}=0$$

for some $a>0$, then δ_a has

$$\frac{\|(\delta_a)^{n+1}\|}{\|\delta_a\|}=\frac{\omega(na+a)}{\omega(n)}$$

converging to 0. One would therefore guess that the answer to Question 4.8 should be yes for “many” μ when $\omega(t)$ is a regulated weight.

5. Applications of weak* results. In this section we apply weak* results to obtain results about ideals and convergence. We make heavy use of results from [8]. We start with a slightly modified definition of the key concept given in [8, Definition 1.2, p. 305].

Definition 5.1. Let ω be an algebra weight and η a bounded Borel function on $\mathbb{R}^+=[0,\infty)$ which is never 0. Then η is a *convergence factor* for ω if whenever $\lambda_n\to\lambda$ weak* in $M(\omega)$ and f belongs to $L^1(\omega)$, then $\lambda_n * f\to\lambda * f$ in norm in $L^1(\omega|\eta|)$. We call η a *universal convergence factor* if it is a convergence factor for all algebra weights.

In [8] we added the additional restriction that η is positive, but this makes no essential change in the theory. We also considered the case that the convergence condition was only known to hold for f in $L^1(\omega)$ with $\alpha(f)\ge a$. We called such η *convergence factors at a* for ω. The classic case of regulated weights (see [1]) is the case where $\eta=1$ is a convergence factor, or convergence factor at some $a\ge 0$. In the current paper, $\eta(t)=e^{-at}$ for $\mathrm{Re}(a)>0$ will be most useful for us. Such e^{-at} are universal convergence factors by [6, Theorem (3.2), p. 512] or [8, Corollary 4.3, p. 313]. These universal convergence factors are particularly useful because the map $f(t)\mapsto e^{-at}f(t)$ is an isometric isomorphism from the algebra $L^1(\omega(t)|e^{-at}|)$ onto the algebra $L^1(\omega)$, and similarly for the corresponding measure algebras.

We now look at a relation between dense and weak* dense principal ideals. It will be convenient to start with a simple consequence of a result in [8].

Lemma 5.2. *Suppose that ω is an algebra weight and that η is a convergence factor for ω. If f is a function in $L^1(\omega)$ for which $L^1(\omega) * f$ is (relatively) weak* dense in $L^1(\omega)$, then $L^1(\omega) * f$ is norm dense in $L^1(\omega|\eta|)$.*

Proof. Let J be the norm closure of $L^1(\omega) * f$ in $L^1(\omega|\eta|)$. Then $J \cap L^1(\omega)$ is closed in $L^1(\omega)$ in the $\omega|\eta|$ norm and contains $L^1(\omega) * f$. By [8, Theorem 5.3, p. 315], $J \cap L^1(\omega)$ must be weak* closed in $L^1(\omega)$. Since $L^1(\omega) * f$ is weak* dense, this means that J contains $L^1(\omega)$, which is a norm dense subspace of $L^1(\omega|\eta|)$. This completes the proof of the lemma. ∎

We now give our major result relating norm dense and weak* dense principal ideals.

Theorem 5.3. *Suppose that $L^1(\omega) * f$ is a weak* dense principal ideal in $L^1(\omega)$. Then $L^1(\omega) * (e^{-at} f(t))$ is norm dense if* $\mathrm{Re}(a) > 0$.

Proof. To simplify the notation, we take a to be a positive number. This is really no loss in generality since multiplication by e^{-ict} for any real number c is an isometric isomorphism of $L^1(\omega)$. By Lemma 5.2, $L^1(\omega) * f$ is norm dense in $L^1(\omega(t)e^{-at})$ and so, therefore, is the larger subspace $L^1(\omega(t)e^{-at}) * f$. Multiplication by e^{-at} is an isometric isomorphism from $L^1(\omega(t)e^{-at})$ onto $L^1(\omega)$. Hence the space

$$e^{-at}((L^1(\omega(t)e^{-at})) * f) = (e^{-at} L^1(\omega(t)e^{-at})) * (e^{-at} f(t)) = L^1(\omega) * (e^{-at} f(t))$$

is norm dense $L^1(\omega)$, as required. ∎

The proofs we gave of Lemma 5.2 and Theorem 5.3 give analogous results when the weak* closure of $L^1(\omega) * f$ is $L^1(\omega)_d = \{g \in L^1(\omega) : \alpha(g) \geq d\}$ for some $d \geq 0$. For the lemma, it is in fact enough to have η a convergence factor for ω at some $t \geq 0$. (One just uses [8, Theorem 5.5, p. 315], which generalizes [2, Proposition 1.9, p. 72], in place of [8, Theorem 5.3, p. 315]). The generalization of Theorem 5.3 is:

Proposition 5.4. *Suppose that the weak* closure of the ideal $L^1(\omega) * f$ is $L^1(\omega)_d$; then the norm closure of $L^1(\omega) * (e^{-at} f(t))$ is also $L^1(\omega)_d$ whenever a has positive real part.*

It is natural to ask if the converse of Theorem 5.3 holds.

Question 5.5. *Suppose that f belongs to the algebra $L^1(\omega)$ and that $\alpha(f) = 0$. If $L^1(\omega) * (e^{-at} f(t))$ is norm dense whenever a is positive, must $L^1(\omega) * f$ be weak* dense?*

In the special case of $L^1(\mathbb{R}^+)$, if f generates a dense principal ideal, then its Laplace transform $\hat{f}(z)$ could not equal 0 anywhere on the open half-plane where $\mathrm{Re}(z)$ is positive. Thus one could obtain Theorem 5.3 in this case from Nyman's theorem [3, Theorem 4.7.64, p. 549]. In this special case Question 5.5 reduces to a weak* analogue of Nyman's theorem. More precisely, we ask:

QUESTION 5.6. *Suppose that f is an integrable function on $\mathbb{R}^+$ with $\alpha(f) = 0$. If the Laplace transform $\hat{f}(z)$ is never 0 for $\mathrm{Re}(z) > 0$, must $L^1(\mathbb{R}^+) * f$ be weak* dense in $L^1(\mathbb{R}^+)$?*

As an application of Theorem 5.3 we extend a result of Bade and Dales for dense principal ideals to the weak* dense case.

COROLLARY 5.7. *Suppose that f belongs to the radical algebra $L^1(\omega)$ and that δ is a positive number. If $L^1(\omega) * f$ is weak* dense, then*

$$\lim_{n\to\infty}\left(\frac{\|f^n\|}{\omega(\delta n)}\right)^{1/n} = \infty.$$

Proof. When $L^1(\omega) * f$ is norm dense, the result is essentially given in [1, Theorem 3.10, p. 105] (see the final paragraph of their proof). We can therefore apply the Bade–Dales result to $g(t) = e^{-at}f(t)$, which generates a norm dense principal ideal by Theorem 5.3 above. Multiplication by e^{-t} is a norm one homomorphism of $L^1(\omega)$, so for all n we have $\|g^{*n}\| = \|e^{-t}(f^{*n})\| \le \|f^n\|$. Hence the theorem follows from the analogous formula for g. ∎

We now apply some of our weak* results to obtain convergence results. As with ideals, the strongest result involves exponentials.

THEOREM 5.8. *Suppose that the sequence (or bounded net) $\{\lambda_n\}$ converges weak* to λ in $M(\omega)$. Then, for all a with positive real part, $e^{-at}\lambda_n \to e^{-at}\lambda$ in the strong operator topology of $M(\omega)$ on $L^1(\omega)$.*

Proof. As above, we use the fact that e^{-at} is a universal convergence factor and, for simplicity, we assume that a is a positive number. Then for all g in $L^1(\omega)$, we see that $\lambda_n * g \to \lambda * g$ in the norm of $L^1(\omega(t)e^{-at})$. Hence $e^{-at}(\lambda_n * g)$ converges to $e^{-at}(\lambda * g)$ in the norm of $L^1(\omega)$. Since multiplication by e^{-at} is a homomorphism, this says that $e^{-at}\lambda_n * f$ converges in norm to $e^{-at}\lambda * f$ in $L^1(\omega)$ for all f in $e^{-at}L^1(\omega)$. Since $e^{-at}L^1(\omega)$ is dense in $L^1(\omega)$ and the set of operators $f \mapsto e^{-at}\lambda_n * f$ on $L^1(\omega)$ is norm bounded, we deduce that $e^{-at}\lambda_n * f \to e^{-at}\lambda * f$ in norm for all f in $L^1(\omega)$. This completes the proof. ∎

The next result says, essentially, that weak* convergence implies "absolute" weak* convergence.

THEOREM 5.9. *Suppose that the sequence, or bounded net, $\{\lambda_n\}$ converges weak* to λ in $M(\omega)$. Then, for all f in $L^1(\omega)$, the sequence $|\lambda_n * f(t) - \lambda * f(t)|$ also converges weak* to zero.*

Of course, it is a simple consequence of the weak* continuity of convolution by f that $\lambda_n * f - \lambda * f$ converges weak* to 0. The new result is that its absolute value also goes to 0 weak*.

Proof of Theorem 5.9. Let h be an arbitrary function in the predual $C_0(1/\omega)$. This means that h/ω is bounded and vanishes at infinity. Therefore h/ω is a universal convergence factor [8, Corollary 4.3, pp. 313–314]. Since $\lambda_n - \lambda$ goes to 0 weak* in $M(\omega)$ and h/ω is a convergence factor, we see that $\lambda_n * f - \lambda * f$ goes to 0 in the norm of $L^1(\omega|h|/\omega) = L^1(|h|)$. That is,

$$\lim_{n\to\infty} \int_0^\infty |\lambda_n * f(t) - \lambda * f(t)| h(t) = 0$$

for all h in $C_0(1/\omega)$. This completes the proof of the theorem. ∎

Theorems 5.8 and 5.9 above start with weak* convergence and use convergence factor arguments to improve the convergence. In our final result, we use convergence factors to transfer a simple property of norm convergence to weak* convergence (compare [11, Theorem 2.3, p. 1677]).

THEOREM 5.10. *Suppose that ω is an algebra weight and $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $M(\omega)$. If $\lambda_n \to 0$ weak* and $\{\mu_n\}$ is bounded, then $\lambda_n * \mu_n \to 0$ weak* in $M(\omega)$.*

Proof. Let f be a function, other than 0, in $L^1(\omega)$, and define $\omega'(t) = e^{-t}\omega(t)$. Then $\lambda_n * f$ converges to 0 in the norm of $M(\omega')$, and $\{\mu_n\}$ is bounded in this norm. Hence $(\lambda_n * \mu_n) * f \to 0$ in the norm of $L^1(\omega')$. We can thus conclude that $\lambda_n * \mu_n \to 0$ weak* in $M(\omega)$ [11, Theorem 2.1, p. 1676]. ∎

The proof above also works in the case that $\{\lambda_n\}$ and $\{\mu_n\}$ are bounded nets.

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