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The continuity of pseudo-differential operators on weighted local Hardy spaces

by

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Abstract. We first show that a linear operator which is bounded on L^2_w with $w \in A_1$ can be extended to a bounded operator on the weighted local Hardy space h^1_w if and only if this operator is uniformly bounded on all h^1_w-atoms. As an application, we show that every pseudo-differential operator of order zero has a bounded extension to h^1_w.

1. Introduction. Pseudo-differential operators are generalizations of differential operators and singular integrals. They are formally defined by

$$Tf(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi, \tag{1}$$

where " ˆ " denotes the Fourier transform, and σ, the *symbol* of T, is a complex-valued function defined on $\mathbb{R}^n\times\mathbb{R}^n$. Symbols are classified according to their size and the size of their derivatives. The standard symbol class of order $m \in \mathbb{Z}$, denoted by S^m, consists of the $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions σ that satisfy the differential inequalities

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

for all multi-indices α and β. If $\sigma \in S^m$, then the operator defined by (1) is called a *pseudo-differential operator of order m*.

Pseudo-differential operators given by (1) can be rewritten as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, x-y)f(y)\,dy,$$

where

$$K(x,z) = \int_{\mathbb{R}^n} \sigma(x,\xi)e^{2\pi i z\cdot\xi}\,d\xi.$$

In other words, for fixed x, $K(x,\cdot)$ is the inverse Fourier transform of $\sigma(x,\cdot)$. If $\sigma \in S^0$, then one can show that $|\partial_x^\beta \partial_y^\alpha K(x,y)| \le A_{\alpha,\beta}|y|^{-n-|\alpha|-|\beta|}$ for all

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α, β, and $y \neq 0$. By the singular integral theory, T can be extended to a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ (cf. [S, p. 250]). For the weighted case, Miller [M] showed that

Theorem A. *Suppose $1 < p < \infty$. Every pseudo-differential operator of order 0 has a bounded extension to $L^p_w(\mathbb{R}^n)$ if and only if $w \in A_p$.*

In 1979, Goldberg [G] introduced the local Hardy spaces h^1 and showed that every pseudo-differential operator of order 0 is bounded on h^1. In this article, we study the boundedness of pseudo-differential operators acting on weighted local Hardy spaces h^1_w, where $w \in A_1$. To obtain the h^1_w-boundedness of a linear operator, we reduce the problem to the L^1_w-boundedness of this linear operator acting on all h^1_w-atoms.

Theorem 1. *Let $w \in A_1$. For a linear operator P bounded on $L^2_w(\mathbb{R}^n)$, P can be extended to a bounded operator on $h^1_w(\mathbb{R}^n)$ if and only if there exists an absolute constant C such that*

$$\|Pa\|_{h^1_w} \le C \quad \text{for any } (h^1_w, 2)\text{-atom } a.$$

We apply Theorem 1 to extend Goldberg's result to the weighted case as follows.

Theorem 2. *Let $w \in A_1$. Every pseudo-differential operator of order 0 has a bounded extension to $h^1_w(\mathbb{R}^n)$.*

Throughout the article, we will use C to denote a positive constant which is independent of main parameters and not necessarily the same at each occurrence. By writing $A \approx B$, we mean that there exists a constant $C > 1$ such that $1/C \le A/B \le C$.

2. Weighted local Hardy spaces. We recall the definition and properties of A_p weights. For $1 < p < \infty$, a locally integrable nonnegative function w on $\mathbb{R}^n$ is said to belong to A_p if there exists $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C \quad \forall \text{ ball } B \subset \mathbb{R}^n.$$

For the case $p = 1$, we have $w \in A_1$ if

$$\frac{1}{|B|}\int_B w(x)\,dx \le C \operatorname*{ess\,inf}_{x\in B} w(x) \quad \forall \text{ ball } B \subset \mathbb{R}^n.$$

For $E \subset \mathbb{R}^n$, we use $w(E)$ to denote the weighted measure $\int_E w(x)\,dx$, which satisfies the doubling condition. More specifically, we have

Lemma B ([GR, p. 396]). *Let $w \in A_p$, $p \ge 1$. Then, for any ball $B(x,r)$ and $\lambda > 1$,*

$$w(B(x,\lambda r)) \le C\lambda^{np} w(B(x,r)),$$

where C does not depend on $B(x,r)$ or on λ.

LEMMA C ([GR, p. 412]). *Let $w \in A_p$, $p > 1$. Then, for all $r > 0$ and $x_0 \in \mathbb{R}^n$, there exists a constant $C > 0$ independent of r such that*

$$\int_{|x-x_0|\geq r} \frac{w(x)}{|x-x_0|^{np}}\,dx \leq Cr^{-np} \int_{|x-x_0|\leq r} w(x)\,dx.$$

The theory of local Hardy spaces was established by Goldberg [G] and extended to the weighted case by Bui [Bu]. We now recall the theory of weighted local Hardy spaces. Let φ and ψ be functions in $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. Also, let $\widetilde{\Gamma}(x)$ denote the cone $\{(y,t) : |x-y| < t,\ 0 < t < 1\}$. For $t > 0$ and $x \in \mathbb{R}^n$, set $\phi_t(x) = t^{-n}\phi(x/t)$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the local versions of the radial maximal function $\widetilde{f}^+$, the nontangential maximal function $\widetilde{f}^*$, and the Lusin integral function $\widetilde{S}(f)$ by

$$\widetilde{f}^+(x) = \sup_{0<t<1} |\varphi_t * f(x)|, \qquad \widetilde{f}^*(x) = \sup_{(y,t)\in\widetilde{\Gamma}(x)} |\varphi_t * f(y)|,$$

$$\widetilde{S}(f)(x) = \bigg(\int_{\widetilde{\Gamma}(x)} |\psi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\bigg)^{1/2}.$$

Let $w \in A_1$. The weighted local Hardy space $h^1_w(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\widetilde{f}^+ \in L^1_w(\mathbb{R}^n)$ with $\|f\|_{h^1_w} = \|\widetilde{f}^+\|_{L^1_w}$. The space $h^1_w(\mathbb{R}^n)$ can also be characterized by $\widetilde{f}^* \in L^1_w(\mathbb{R}^n)$ or $\widetilde{S}(f) \in L^1_w(\mathbb{R}^n)$, and $\|\widetilde{f}^*\|_{L^1_w} \approx \|\widetilde{f}^+\|_{L^1_w} \approx \|\widetilde{S}(f)\|_{L^1_w}$ (cf. [Bu]).

As for weighted Hardy spaces, we also have the atomic decomposition characterization of $h^1_w(\mathbb{R}^n)$.

DEFINITION. A function a is called an (h^1_w, q)-*atom centered at* x_0, $1 < q \leq \infty$, if

(i) the support of a is contained in a ball $B(x_0, r)$,
(ii) $\|a\|_{L^q_w} \leq w(B(x_0,r))^{1/q-1}$,
(iii) if $r < 1$, then $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

The condition (ii) is interpreted as $\|a\|_\infty \leq w\big(B(x_0,r)\big)^{-1}$ if $q = \infty$.

THEOREM D ([Bu]). *Let $1 < q \leq \infty$ and $w \in A_1$. A function f is in $h^1_w(\mathbb{R}^n)$ if and only if there exists a sequence $\{a_j\}$ of (h^1_w, q)-atoms and a sequence $\{\lambda_j\}$ of scalars with $\sum |\lambda_j| < \infty$ such that $f = \sum \lambda_j a_j$ in L^1_w. Furthermore,*

$$\|f\|_{h^1_w} \approx \inf\Big\{\sum |\lambda_j| : \sum \lambda_j a_j \textit{ is a decomposition of } f \textit{ into } (h^1_w, q)\textit{-atoms}\Big\}.$$

To prove Theorem 1, we need to construct an atomic decomposition of elements in $h^1_w \cap L^2_w$, which converges in L^2_w.

THEOREM 3. *Let $w \in A_1$. For $f \in h_w^1(\mathbb{R}^n) \cap L_w^2(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of $(h_w^1, 2)$-atoms and a sequence $\{\lambda_j\}$ of scalars satisfying $\sum |\lambda_j| \le C\|f\|_{h_w^1}$ such that $f = \sum \lambda_j a_j$ in $L_w^2(\mathbb{R}^n)$.*

The proof of Theorem 3 appeals to the following two lemmas about the properties of $H_w^1(\mathbb{R}^n)$. The space $H_w^1(\mathbb{R}^n)$ consists of all f's satisfying $S(f) \in L_w^1(\mathbb{R}^n)$ with $\|f\|_{H_w^1} = \|S(f)\|_{L_w^1}$, where

$$S(f)(x) = \bigg(\int_0^\infty \int_{|x-y|<t} |\psi_t * f(y)|^2 \, \frac{dy\,dt}{t^{n+1}} \bigg)^{1/2}.$$

We can characterize elements of $H_w^1(\mathbb{R}^n)$ in terms of atoms. A real-valued function $a \in L_w^2(\mathbb{R}^n)$, $w \in A_2$, is called a w-$(1,2,n)$-*atom* if (i) a is supported on a ball B, (ii) $\|a\|_{L_w^2} \le w(B)^{-1/2}$, and (iii) $\int_{\mathbb{R}^n} a(x)x^\alpha \, dx = 0$ for every multi-index α with $|\alpha| \le n$.

LEMMA E ([Bu]). *Let $w \in A_1$ and $f \in h_w^1(\mathbb{R}^n)$. If Φ is a function in $\mathcal{S}(\mathbb{R}^n)$ such that $\int \Phi(x)\,dx = 1$ and $\int x^\alpha \Phi(x)\,dx = 0$ for all $\alpha \ne 0$, then $f - \Phi * f \in H_w^1(\mathbb{R}^n)$ and $\|f - \Phi * f\|_{H_w^1} \le C\|f\|_{h_w^1}$.*

LEMMA F ([HLL]). *Let $w \in A_2$. For $f \in H_w^1(\mathbb{R}^n) \cap L_w^2(\mathbb{R}^n)$, there exist a sequence $\{a_i\}$ of w-$(1,2,n)$-atoms and a sequence $\{\lambda_i\}$ of scalars satisfying $\sum |\lambda_i| \le C\|f\|_{H_w^1}$ such that $f = \sum \lambda_i a_i$ in $L_w^2(\mathbb{R}^n)$.*

Proof of Theorem 3. Let $w \in A_1$, $f \in h_w^1(\mathbb{R}^n) \cap L_w^2(\mathbb{R}^n)$, and Φ satisfy the assumption of Lemma E. Then $f - \Phi * f \in H_w^1$. Since $f \in L_w^2$ implies $\Phi * f \in L_w^2$, it follows from Lemma F that $f - \Phi * f = \sum \eta_j b_j$ in L_w^2, where b_j's are w-$(1,2,n)$-atoms and $\sum |\eta_j| \le C\|f - \Phi * f\|_{H_w^1} \le C\|f\|_{h_w^1}$. It is clear that a w-$(1,2,n)$-atom is also an $(h_w^1, 2)$-atom.

Let $\{Q_j\}$ be the family of cubes whose vertices are the lattice points $n^{-1/2}\mathbb{Z}^n$. Then

(i) $\operatorname{diam}(Q_j) = 1$ for all j;
(ii) $\bigcup_j Q_j = \mathbb{R}^n$;
(iii) the cubes Q_j's are nonoverlapping.

Let x_j and χ_{Q_j} denote the center and the characteristic function of Q_j, respectively. Write

$$(\Phi * f)\chi_{Q_j} = \lambda_j a_j, \quad \text{where} \quad \lambda_j = w(B(x_j, 1))\|(\Phi * f)\chi_{Q_j}\|_\infty.$$

Then a_j's are $(h_w^1, 2)$-atoms and $\Phi * f = \sum \lambda_j a_j$ almost everywhere. Owing to Lemma B and $\operatorname{diam}(Q_j) = 1$,

$$\begin{aligned}\sum_j |\lambda_j| &= \sum_j w(B(x_j,1))\|(\Phi * f)\chi_{Q_j}\|_\infty \\ &\le C\sum_j w(Q_j)\|(\Phi * f)\chi_{Q_j}\|_\infty = C\sum_j \int_{Q_j} \sup_{y\in Q_j} |\Phi * f(y)| w(x)\,dx \\ &\le C\int_{\mathbb{R}^n} \sup_{(y,t)\in\widetilde{\Gamma}(x)} |\Phi_t * f(y)| w(x)\,dx \le C\|f\|_{h^1_w}.\end{aligned}$$

Since $\Phi * f \in L^2_w$, the series $\sum \lambda_j a_j$ converges to $\Phi * f$ in L^2_w. ∎

Proof of Theorem 1. If P is bounded on h^1_w, then Theorem D gives

$$\|Pa\|_{h^1_w} \le C\|a\|_{h^1_w} \le C \quad \text{for any } (h^1_w,2)\text{-atom } a.$$

Conversely, for $w \in A_1$ and $f \in h^1_w \cap L^2_w$, we have an atomic decomposition $f = \sum \lambda_j a_j$ in L^2_w and $\sum |\lambda_j| \le C\|f\|_{h^1_w}$ by Theorem 3. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. By the L^2_w-boundedness of P,

$$\psi_t * Pf = \sum_{j=1}^\infty \lambda_j \psi_t * Pa_j \quad \text{in } L^2_w,$$

which implies that there exists a subsequence (we still use the same indices) such that

$$\psi_t * Pf = \sum_{j=1}^\infty \lambda_j \psi_t * Pa_j \quad \text{almost everywhere.}$$

Fatou's lemma and Minkowski's inequality yield

$$\begin{aligned}\widetilde{S}(Pf)(x) &= \Big(\int_{\widetilde{\Gamma}(x)} |\psi_t * Pf(y)|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2} \\ &\le \liminf_{M\to\infty} \Big(\int_{\widetilde{\Gamma}(x)} \Big|\sum_{j=1}^M \lambda_j \psi_t * Pa_j(y)\Big|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2} \\ &\le \sum_{j=1}^\infty |\lambda_j| \Big(\int_{\widetilde{\Gamma}(x)} |\psi_t * Pa_j(y)|^2 \frac{dy\,dt}{t^{n+1}}\Big)^{1/2} = \sum_{j=1}^\infty |\lambda_j| \widetilde{S}(Pa_j)(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{S}(Pf)(x)w(x)\,dx &\le \sum_{j=1}^\infty |\lambda_j| \int_{\mathbb{R}^n} \widetilde{S}(Pa_j)(x)w(x)\,dx \\ &\le C\sum_{j=1}^\infty |\lambda_j| \cdot \|Pa_j\|_{h^1_w} \le C\|f\|_{h^1_w},\end{aligned}$$

which gives the h^1_w-boundedness of P on $h^1_w \cap L^2_w$. Theorem D implies that $h^1_w \cap L^2_w$ is dense in h^1_w, so P can be extended to a bounded operator on h^1_w. ∎

3. Proof of Theorem 2. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a fixed nonnegative radial decreasing function supported in the unit ball $B(0,1)$ with $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For $t > 0$, define K_t by

$$K_t(x,z) = \int_{\mathbb{R}^n} K(x-y, z-y)\phi_t(y)\,dy.$$

Goldberg [G, Lemma 6] obtained an estimate of K_t as follows, which will be used to prove Theorem 2.

LEMMA G. *Suppose $\sigma \in S^0$. Then, for all $\alpha, \beta \in (\mathbb{N} \cup \{0\})^n$,*

$$\sup_{x\in\mathbb{R}^n} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial z}\right)^\beta K_t(x,z)\right| \le \frac{C_{\alpha,\beta}}{|z|^{n+|\beta|}} \quad \text{for } z \neq 0,$$

where $C_{\alpha,\beta}$ is independent of t if $0 < t < 1$.

Proof of Theorem 2. Let T be a pseudo-differential operator given by (1). By Theorem A, T is bounded on L^2_w. We will prove that there exists a constant $C > 0$ such that $\|Ta\|_{h^1_w} \le C$ for any $(h^1_w, 2)$-atom a. Then Theorem 2 follows from Theorem 1.

Let a be an $(h^1_w, 2)$-atom centered at x_0 with $\operatorname{supp}(a) \subset B(x_0, r)$. Denote by M the Hardy–Littlewood maximal operator, we have $(\widetilde{Ta})^+(x) \le M(Ta)(x)$. By Theorem A and Lemma B,

$$\begin{aligned}(2)\qquad \int_{B(x_0,3r)} (\widetilde{Ta})^+(x)w(x)\,dx &\le \int_{B(x_0,3r)} M(Ta)(x)w(x)\,dx \\ &\le w(B(x_0,3r))^{1/2}\|M(Ta)\|_{L^2_w} \\ &\le Cw(B(x_0,r))^{1/2}\|Ta\|_{L^2_w} \\ &\le Cw(B(x_0,r))^{1/2}\|a\|_{L^2_w} \le C.\end{aligned}$$

To estimate $\int_{B(x_0,3r)^c} (\widetilde{Ta})^+(x)w(x)\,dx$, we consider the case $r < 1$ first. For $x \in B(x_0,3r)^c$, we use the fact that

$$\phi_t * (Ta)(x) = \int_{B(x_0,r)} K_t(x, x-z)a(z)\,dz.$$

Applying Taylor's theorem to the function $K_t(x, x-\cdot)$ near x_0, we have

$$K_t(x,x-z) = K_t(x,x-x_0) + R_{x_0,t}(x,z),$$

where

$$R_{x_0,t}(x,z) = \sum_{|\alpha|=1} \left[\left(\frac{\partial}{\partial z}\right)^\alpha K_t(x,z)\right]_{z=x-\xi} \cdot (z-x_0)^\alpha$$

and $\xi \in \mathbb{R}^n$ is a point lying on the line segment from x_0 to z. Note that

$|x-\xi| \approx |x-x_0|$. It follows from Lemma G that

$$(3)\qquad |R_{x_0,t}(x,z)| \le C\frac{|z-x_0|}{|x-x_0|^{n+1}} \quad \text{for } z \in B(x_0,r) \text{ and } 0<t<1.$$

Using (3) and the moment condition of a, we get

$$\begin{aligned}(4)\qquad & \int\limits_{B(x_0,3r)^c} (\widetilde{Ta})^+(x)w(x)\,dx \\ &\le \int\limits_{B(x_0,3r)^c} \sup_{0<t<1}\Big\{\int\limits_{B(x_0,r)} |K_t(x,x-z)-K_t(x,x-x_0)|\,|a(z)|\,dz\Big\}w(x)\,dx \\ &\le C\int\limits_{B(x_0,3r)^c}\Big\{\int\limits_{B(x_0,r)} \frac{|z-x_0|}{|x-x_0|^{n+1}}\,|a(z)|\,dz\Big\}w(x)\,dx \\ &\le Cr\Big(\int\limits_{B(x_0,3r)^c} \frac{w(x)\,dx}{|x-x_0|^{n+1}}\Big)\Big(\int\limits_{B(x_0,r)} |a(z)|\,dz\Big).\end{aligned}$$

By Lemmas B and C,

$$(5)\qquad \int\limits_{B(x_0,3r)^c} \frac{w(x)\,dx}{|x-x_0|^{n+1}} \le Cr^{-(n+1)}w(B(x_0,r)).$$

Since $w \in A_2$, Hölder's inequality gives

$$\begin{aligned}(6)\qquad \int\limits_{B(x_0,r)} |a(z)|\,dz &\le \Big(\int\limits_{B(x_0,r)} |a(z)|^2 w(z)\,dz\Big)^{1/2}\Big(\int\limits_{B(x_0,r)} w(z)^{-1}\,dz\Big)^{1/2} \\ &\le Cr^n w(B(x_0,r))^{-1}.\end{aligned}$$

Inequalities (4)–(6) yield

$$\int\limits_{B(x_0,3r)^c} (\widetilde{Ta})^+(x)w(x)\,dx \le C \quad \text{for } r<1.$$

For the case $r \ge 1$, we split $T = T_1 + T_2$ by decomposing its kernel

$$K(x,z) = K_1(x,z) + K_2(x,z) = \eta(z)K(x,z) + (1-\eta(z))K(x,z),$$

where $\eta \in C^\infty(\mathbb{R}^n)$ is a radial function satisfying $0 \le \eta(z) \le 1$, $\eta(z) = 1$ for $|z| < 2r$, and $\eta(z) = 0$ for $|z| \ge 4r$. If we consider the corresponding symbols $\sigma_1 = \check{\eta} * \sigma$ and $\sigma_2 = (1-\eta)^\vee * \sigma$, where "$\,\check{}\,$" denotes the inverse Fourier transform, then T_1 and T_2 are pseudo-differential operators of order 0. We note that $\operatorname{supp}(T_1 a) \subset B(x_0,5r)$. Since ϕ is supported in $B(0,1)$, by an argument similar to (2),

$$\|T_1 a\|_{h^1_w} = \int\limits_{B(x_0,6r)} (\widetilde{T_1 a})^+(x)w(x)\,dx \le C.$$

For the estimate of $\int_{B(x_0,3r)^c} (\widetilde{T_2 a})^+(x)w(x)\,dx$, $\sigma_2 \in S^0$ gives (cf. [S, p. 241])

$$|K_2(x,z)| \le C_M |z|^{-(n+M)} \quad \text{for } z \ne 0 \text{ and } M > 0.$$

Thus, for $0 < t < 1$ and $|z| \ge 2r$,

$$|(K_2)_t(x,z)| \le \int\limits_{|y|<t} \frac{C_M}{|z-y|^{n+M}}\,\phi_t(y)\,dy \le \frac{C}{|z|^{n+M}},$$

which implies, for $x \in B(x_0,3r)^c$,

$$\begin{aligned}(\widetilde{T_2 a})^+(x) &= \sup_{0<t<1} \Big| \int\limits_{B(x_0,r)} (K_2)_t(x,x-z)\,a(z)\,dz \Big| \\ &\le \frac{C}{|x-x_0|^{n+M}} \int\limits_{B(x_0,r)} |a(z)|\,dz.\end{aligned}$$

Inequality (6) and the same argument as for (5) lead to

$$\begin{aligned}&\int\limits_{B(x_0,3r)^c} (\widetilde{T_2 a})^+(x)w(x)\,dx \\ &\le \Big(\int\limits_{B(x_0,3r)^c} \frac{w(x)\,dx}{|x-x_0|^{n+M}} \Big) \Big(\int\limits_{B(x_0,r)} |a(z)|\,dz \Big) \le Cr^{-M} \le C \quad \text{for } r \ge 1.\end{aligned}$$

Thus, the proof is complete. ■

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