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Segal algebras, approximate identities and norm irregularity in $C_0(X, A)$

by

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Abstract. We study three closely related concepts in the context of the Banach algebra $C_0(X, A)$. We show that, to a certain extent, Segal extensions, norm irregularity and the existence of approximate identities in $C_0(X, A)$ can be deduced from the corresponding features of A and vice versa. Extensive use is made of the multiplier norm and the tensor product representation of $C_0(X, A)$.

1. Introduction. The structure of an algebra of continuous functions on a locally compact space X with values in a Banach algebra A is well determined by the algebra A. For example, the ideal structure of the function algebra, and thus, in the commutative case, the Gelfand representation, can be directly related to that of A (see e.g. [10, 20, 1, 17]). Furthermore, properties such as spectral synthesis [14] and Arens regularity [22] carry over from A.

In this work we study Segal algebras, norm irregularity and approximate identities in the algebra $C_0(X, A)$ of continuous functions of X into A which vanish at infinity, and show that all these can be described in terms of the corresponding structures and properties of A. The first two of these concepts have not, to our knowledge, been studied in this context before. Concerning approximate identities, we obtain generalizations of earlier results (see [18, 15] as well as [13, Section I.8]).

Segal algebras were first introduced in the context of group algebras by Reiter (cf. [21]), and were later given an abstract definition by Burnham [9]. In the literature it has been customary to assume that Segal algebras have an approximate identity. The discussion has recently been generalized to a much wider class of algebras, which possess a bounded approximate identity only with respect to the multiplier norm (see Arhippainen and Kauppi [6] for the commutative case, and Kauppi and Mathieu [16] for the noncommutative case). Furthermore, though A may not have an approximate identity, it is

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possible to construct the largest ideal of A which has one. This ideal is called the *approximate ideal* of A in [6, 16].

This approach leads us to the study of a class of algebras called *norm irregular Banach algebras* [5], for which the multiplier norm is strictly weaker than the original norm and possible approximate identities are necessarily unbounded. It turns out that the concepts of Segal algebra and norm irregular algebra are equivalent (see [6, 16], also Corollary 3.5 in this article). Representing the situation in terms of norm irregular algebras has the advantage that the larger algebra (in which A is a Segal algebra) does not have to be specified. The multiplier norm also provides us with a convenient computational tool.

In Section 2, after basic definitions, we deal with norm irregularity. In Section 3 we discuss Segal algebras and establish the connection to norm irregular algebras. Section 4 is devoted to the study of approximate identities. We conclude the article by constructing the approximate ideal of $C_0(X, A)$.

2. Norm irregularity. In this section we present some basic definitions and discuss norm irregularity in the function algebra $C_0(X, A)$. We will describe the multiplier norm on $C_0(X, A)$ and show that $C_0(X, A)$ is norm irregular if and only if A is.

Throughout this paper, let X be a locally compact Hausdorff space and let A be a Banach algebra with norm $\|\cdot\|$. The algebra A is not assumed to be unital or commutative. A function $f : X \to A$ is said to *vanish at infinity* if for each $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\|f(t)\| < \epsilon$ for all $t \in X \setminus K$. The set $C_0(X, A)$ of continuous functions of X into A which vanish at infinity is a Banach algebra when equipped with pointwise defined algebraic operations and the supremum norm,

$$\|f\|_\infty := \sup_{t \in X} \|f(t)\| \qquad (f \in C_0(X, A)).$$

As a special case, $C_0(X) := C_0(X, \mathbb{C})$. The algebra $C_0(X, A)$ is unital if and only if X is compact and A is unital.

The algebra $C_0(X, A)$ is isometrically isomorphic to the injective tensor product $C_0(X) \check{\otimes} A$ [19, Proposition 1.10.22]. Elementary tensors in the algebra $C_0(X) \check{\otimes} A$ are viewed as functions of X into A via the identification $(\phi \otimes a)(t) := \phi(t)a$, where $\phi \in C_0(X)$, $a \in A$ and $t \in X$. The linear span of the elementary tensors is dense in $C_0(X, A)$, i.e. for all $f \in C_0(X, A)$ and $\epsilon > 0$ we can find $n \in \mathbb{N}$ and $\phi_j \in C_0(X)$, $a_j \in A$, $j = 1, \ldots, n$, such that

$$\Big\| f - \sum_{j=1}^{n} \phi_j \otimes a_j \Big\|_\infty < \epsilon.$$

Furthermore, $\|\phi \otimes a\|_\infty = \|\phi\|_\infty \|a\|$, $\phi \in C_0(X)$, $a \in A$.

We define the *multiplier seminorm* $\|\cdot\|_M$ on A by

$$\|a\|_M := \sup_{b\in A,\,\|b\|\le 1} \{\|ab\|, \|ba\|\} \quad (a \in A).$$

Evidently $\|a\|_M \le \|a\|$ and $\max\{\|ab\|, \|ba\|\} \le \|a\|_M\|b\|$ for all $a, b \in A$. Furthermore, $\|\cdot\|_M$ is a norm on A if and only if the *annihilator ideal* of A, defined by

$$\mathrm{ann}(A) := \{a \in A : ab = ba = 0 \text{ for all } b \in A\},$$

equals $\{0\}$.

In this paper we will assume that $\mathrm{ann}(A) = \{0\}$. We call $\|\cdot\|_M$ the *multiplier norm* on A.

The algebra $C_0(X, A)$ also satisfies $\mathrm{ann}(C_0(X, A)) = \{0\}$, so $(\|\cdot\|_\infty)_M$ is a norm on $C_0(X, A)$. By the following proposition, the multiplier norm on $C_0(X, A)$ is determined by that on A. We adopt the notation

$$\|f\|_\infty^{(M)} := \sup_{t\in X} \|f(t)\|_M \quad (f \in C_0(X, A)).$$

PROPOSITION 2.1. *The multiplier norm on* $C_0(X, A)$ *satisfies*

$$(\|f\|_\infty)_M = \|f\|_\infty^{(M)} \quad \textit{for all } f \in C_0(X, A).$$

Proof. Let $f, g \in C_0(X, A)$, $\|g\|_\infty \le 1$. Since

$$\|fg\|_\infty = \sup_{t\in X} \|f(t)g(t)\| \le \sup_{t\in X} \|f(t)\|_M\|g(t)\| \le \|f\|_\infty^{(M)}\|g\|_\infty \le \|f\|_\infty^{(M)},$$

and similarly $\|gf\|_\infty \le \|f\|_\infty^{(M)}$, we have $(\|f\|_\infty)_M \le \|f\|_\infty^{(M)}$.

On the other hand, taking $\{\phi_\alpha\}_{\alpha\in\Lambda}$ to be a family of functions in $C_0(X)$ with $\|\phi_\alpha\|_\infty \le 1$ for all $\alpha \in \Lambda$ and $\sup_{\alpha\in\Lambda} |\phi_\alpha(t)| = 1$ for all $t \in X$, we have

$$\begin{aligned}(\|f\|_\infty)_M &\ge \sup_{\alpha\in\Lambda,\,\|a\|\le 1} \|f(\phi_\alpha \otimes a)\|_\infty = \sup_{\alpha\in\Lambda,\,\|a\|\le 1} \Big(\sup_{t\in X} \|f(t)\phi_\alpha(t)a\|\Big) \\ &= \sup_{t\in X} \Big(\sup_{\alpha\in\Lambda} |\phi_\alpha(t)| \sup_{\|a\|\le 1} \|f(t)a\|\Big) = \sup_{t\in X} \sup_{\|a\|\le 1} \|f(t)a\|.\end{aligned}$$

Similarly we show that

$$(\|f\|_\infty)_M \ge \sup_{t\in X} \sup_{\|a\|\le 1} \|af(t)\|,$$

and thus

$$(\|f\|_\infty)_M \ge \sup_{t\in X} \|f(t)\|_M = \|f\|_\infty^{(M)}. \ \blacksquare$$

Although the multiplier norm $\|\cdot\|_M$ is majorized by $\|\cdot\|$, the norms $\|\cdot\|$ and $\|\cdot\|_M$ need not be equivalent. To investigate this further, we make the following definition, introduced in [5, 8].

Definition 2.2. The Banach algebra A is called

(i) *norm regular* if the norms $\|\cdot\|_M$ and $\|\cdot\|$ coincide on A;
(ii) *weakly norm regular* if the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent on A;
(iii) *norm irregular* if the norm $\|\cdot\|_M$ is strictly weaker than $\|\cdot\|$ on A.

As a computational tool to describe the equivalence of the stated norms, we introduce the following constant, the *modulus of regularity* of A (see [2]):

$$r(\|\cdot\|) := \sup_{a\in A,\, \|a\|_M\le 1} \|a\|.$$

The algebra A is weakly norm regular if and only if $r(\|\cdot\|)$ is finite. More precisely, if $r(\|\cdot\|)<\infty$, then $\|a\|_M \le \|a\| \le r(\|\cdot\|)\|a\|_M$ for all $a \in A$.

Remark 2.3. If A has a unit element, denoted by e, then A is (weakly) norm regular and $r(\|\cdot\|) = \|e\|$. This shows that the adjunction of a unit element does not preserve norm irregularity. See also Remark 3.6 for another aspect of this issue.

We will now give some examples of norm irregular Banach algebras.

Example 2.4. (i) Let Y be a locally compact Hausdorff space and let $v : Y \to \mathbb{R}$ be an upper semicontinuous function such that $\inf_{t\in Y} v(t) > 0$. We call v a *weight function* on Y and define

$$C_b^v(Y) := \{\phi \in C(Y) : v\phi \text{ is bounded on } Y\},$$
$$C_0^v(Y) := \{\phi \in C(Y) : v\phi \text{ vanishes at infinity on } Y\}.$$

Both $C_b^v(Y)$ and $C_0^v(Y)$ are Banach algebras under pointwise defined operations and the weighted supremum norm

$$\|\phi\|_v := \sup_{t\in Y} v(t)|\phi(t)| \qquad (\phi \in C_b^v(Y)).$$

If v is unbounded, $C_b^v(Y)$ and $C_0^v(Y)$ are proper ideals of $C_b(Y)$ and $C_0(Y)$, respectively. Moreover, if $v(t_\alpha)\to\infty$ whenever (t_α) is a net in X converging to the point at infinity of X, then $C_b^v(Y)$ is also a proper ideal of $C_0(Y)$.

It can be shown that $(\|\phi\|_v)_M = \|\phi\|_\infty$ for all $\phi \in C_b^v(Y)$, which implies that both $(C_b^v(Y), \|\cdot\|_v)$ and $(C_0^v(Y), \|\cdot\|_v)$ are norm irregular if and only if v is unbounded. For a more complete discussion of these algebras, see [3, 4].

(ii) Let G be an infinite, compact topological group. The set $L^p(G)$ of equivalence classes of functions $f : G \to \mathbb{C}$ which satisfy

$$\|f\|_p := \Big(\int_G |f(t)|^p \, d\lambda(t)\Big)^{1/p} < \infty$$

is a Banach algebra with respect to pointwise linear operations, convolution $f * g$ as multiplication, and norm given by $\|\cdot\|_p$. Here $1 < p < \infty$, and λ is a Haar measure on G, normalized so that $\lambda(G) = 1$. The well-known inequality $\|f * g\|_p \le \|f\|_1 \|g\|_p$ entails that $(\|f\|_p)_M \le \|f\|_1$ $(f, g \in L^p(G))$, from which it is easy to see that $(L^p(G), \|\cdot\|_p)$ is norm irregular.

The following proposition is the main result of this section.

PROPOSITION 2.5. *The equality* $r(\|\cdot\|_\infty) = r(\|\cdot\|)$ *holds.*

Proof. First of all, using Proposition 2.1,

$$r(\|\cdot\|_\infty) = \sup_{(\|f\|_\infty)_M \le 1} \|f\|_\infty = \sup_{\|f\|_\infty^{(M)} \le 1} \Big(\sup_{t\in X} \|f(t)\|\Big)$$
$$\le r(\|\cdot\|) \sup_{\|f\|_\infty^{(M)} \le 1} \Big(\sup_{t\in X} \|f(t)\|_M\Big) = r(\|\cdot\|).$$

For the converse inequality, let $a \in A$, $\|a\|_M \le 1$, and $\phi \in C_0(X)$, $\|\phi\|_\infty = 1$. We have $\|\phi \otimes a\|_\infty = \|\phi\|_\infty \|a\| = \|a\|$ and, for all $g \in C_0(X, A)$, $\|g\|_\infty \le 1$,

$$\|g(\phi \otimes a)\|_\infty = \sup_{t\in X} |\phi(t)|\, \|g(t)a\| \le \sup_{t\in X} |\phi(t)|\, \|a\|_M \le 1.$$

Similarly, $\|(\phi \otimes a)g\|_\infty \le 1$, so that $(\|\phi \otimes a\|_\infty)_M \le 1$. From this we deduce that $r(\|\cdot\|_\infty) \ge r(\|\cdot\|)$. ■

As an immediate consequence of Proposition 2.5, we find that $C_0(X, A)$ is (weakly) norm regular if and only if A is (weakly) norm regular.

EXAMPLE 2.6. Let $C_0^v(Y)$ be the weighted function algebra introduced in Example 2.4. By the example and Proposition 2.5, the algebra $C_0(X, C_0^v(Y))$ is norm irregular if and only if v is unbounded. Also recall the identity $(\|\phi\|_v)_M = \sup_{s\in Y} |\phi(s)|$ $(\phi \in C_0^v(Y))$. Let $f \in C_0(X, C_0^v(Y))$. By Proposition 2.1,

$$(\|f\|_\infty)_M = \|f\|_\infty^{(M)} = \sup_{t\in X} (\|f(t)\|_v)_M = \sup_{t\in X,\, s\in Y} |[f(t)](s)|.$$

We end this section by looking at the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_M$ in a more general setting.

REMARK 2.7. Suppose that, instead of being a complete algebra norm, $\|\cdot\|$ is assumed to be a linear space norm which makes the multiplication separately continuous but need not satisfy the condition $\|ab\| \le \|a\|\,\|b\|$ $(a, b \in A)$. Then we can investigate the equivalence of $\|\cdot\|$ and $\|\cdot\|_M$ from another point of view. Namely, defining the constant

$$m(\|\cdot\|) := \sup_{a\in A,\, \|a\|\le 1} \|a\|_M,$$

called the *modulus of m-convexity* of A (see [2]), we have

$$\frac{1}{m(\|\cdot\|)} \|a\|_M \le \|a\| \le r(\|\cdot\|) \|a\|_M$$

for all $a \in A$, if both $r(\|\cdot\|)$ and $m(\|\cdot\|)$ are finite. One can show that if $m(\|\cdot\|)$ is infinite, then the algebra A is incomplete and its completion is not even an algebra [5].

The algebra $C_0(X, A)$ satisfies $m(\|\cdot\|_\infty) = m(\|\cdot\|)$. Indeed, first of all,

$$\begin{aligned} m(\|\cdot\|_\infty) &= \sup_{\|f\|_\infty \le 1} (\|f\|_\infty)_M = \sup_{\|f\|_\infty \le 1} \Big(\sup_{t\in X} \|f(t)\|_M\Big) \\ &\le m(\|\cdot\|) \sup_{\|f\|_\infty \le 1} \Big(\sup_{t\in X} \|f(t)\|\Big) = m(\|\cdot\|), \end{aligned}$$

where we have used Proposition 2.1. For the converse, take any a and b in the closed unit ball of A, and let $\phi \in C_0(X)$, $\|\phi\|_\infty = 1$. Clearly $\|\phi \otimes a\|_\infty \le 1$ and $\|\phi \otimes b\|_\infty \le 1$. Also,

$$\|(\phi \otimes a)(\phi \otimes b)\|_\infty = \|\phi^2 \otimes ab\|_\infty = \|\phi^2\|_\infty \|ab\| = \|ab\|,$$

which implies $m(\|\cdot\|_\infty) \ge m(\|\cdot\|)$.

3. Segal algebras. Segal algebras are dense ideals in a Banach algebra which satisfy a certain norm inequality. In this section we will investigate how Segal algebras in $C_0(X, A)$ can be described in terms of those in A. We will first discuss them on a general level and establish the connection to norm irregular algebras. Throughout this section, B is a Banach algebra with norm $\|\cdot\|_B$.

DEFINITION 3.1.

(i) The Banach algebra A is a *Segal algebra* in B if A is a dense, two-sided ideal in B and there exists a constant $l > 0$ such that

$$\|a\|_B \le l\|a\| \quad \text{for all } a \in A.$$

(ii) An algebra norm $|\cdot|$ on A is called a *Segal norm* on A if there exist constants $k, l > 0$ such that

$$k\|a\|_M \le |a| \le l\|a\| \quad \text{for all } a \in A.$$

REMARK 3.2. (i) If A is a Segal algebra in B, we also say that B is a *Segal extension* of A.

(ii) If B is semisimple, the inequality $\|a\|_B \le l\|a\|$, $a \in A$, is automatically satisfied whenever A is a dense ideal in B [7, Proposition 2.2].

(iii) The multiplier norm $\|\cdot\|_M$ is a Segal norm on A. Up to equivalence, it is the weakest such norm.

The following property of Segal algebras will be very useful for our purposes: if A is a Segal algebra in B, then A is a Banach bi-module over B. For the proof, see [7, Theorem 2.3].

LEMMA 3.3. *Let A be a Segal algebra in B. Then there exists a constant $k > 0$ such that*

$$\|ax\| \leq k\|a\|\,\|x\|_B \quad \textit{and} \quad \|xa\| \leq k\|a\|\,\|x\|_B \quad \textit{for all } a \in A \textit{ and } x \in B.$$

The following two results establish the importance of norm irregular Banach algebras by relating them directly to Segal algebras. They have been noted already in [6, 16]. For the proofs, see [16, Proposition 2.6 and Corollary 2.7].

PROPOSITION 3.4. *The following conditions are equivalent:*

(i) *A is a Segal algebra in B;*
(ii) *B is the completion of A with respect to a Segal norm on A.*

At this point we adopt the following notation: We denote by A_M the algebra A equipped with the multiplier norm $\|\cdot\|_M$, i.e. $A_M = (A, \|\cdot\|_M)$. In particular, $C_0(X,A)_M = (C_0(X,A), (\|\cdot\|_\infty)_M)$. Note that the norm $\|\cdot\|$ on A is still used to determine the *set* $C_0(X,A)$. Since $\|\cdot\|_M$ is weaker than $\|\cdot\|$, we have $C_0(X,A) \subseteq C_0(X,A_M)$, where the inclusion is proper if and only if A is norm irregular. By Proposition 2.1, the embedding of $C_0(X,A)_M$ into $C_0(X,A_M)$ is isometric. The completion of A_M is denoted by $\widetilde{A}_M$, and A is identified with its canonical image in the completion.

COROLLARY 3.5. *The following conditions are equivalent:*

(i) *A is norm irregular;*
(ii) *A is a proper Segal algebra in some Banach algebra.*

REMARK 3.6. (i) Since $\|\cdot\|_M$ is the weakest Segal norm on A, the algebra $\widetilde{A}_M$ is the largest Segal extension of A, and every Segal extension of A can be embedded as a dense subalgebra of $\widetilde{A}_M$.

(ii) Since a unital algebra cannot be a proper ideal in any algebra, it cannot be a proper Segal algebra. This observation combined with Corollary 3.5 gives us another perspective on the issue of unitizing a norm irregular Banach algebra.

Throughout the rest of the paper, we will regard all Segal extensions of A as subalgebras of $\widetilde{A}_M$.

By Corollary 3.5 and Proposition 2.5, A is a proper Segal algebra in some Banach algebra if and only if the same holds for $C_0(X,A)$. The following theorem, which is the main result of this section, characterizes the Segal extensions of $C_0(X,A)$.

THEOREM 3.7. *The following conditions are equivalent:*

(i) *A is a Segal algebra in B;*
(ii) *$C_0(X,A)$ is a Segal algebra in $C_0(X,B)$.*

Furthermore,

$$C_0(X,A)\widetilde{_M} = C_0(X,\widetilde{A}_M)$$

is the largest Segal extension of $C_0(X,A)$.

Proof. We denote the norms on $C_0(X,A)$ and $C_0(X,B)$ by $\|\cdot\|_\infty^{(A)}$ and $\|\cdot\|_\infty^{(B)}$, respectively. Let A be a Segal algebra in B, and let $f \in C_0(X,A)$ and $g \in C_0(X,B)$ be arbitrary. Obviously fg and gf map X into A. To show that fg is continuous with respect to $\|\cdot\|$, take any $t \in X$ and $\epsilon > 0$. Let $k > 0$ be as in Lemma 3.3. Now choose $\delta > 0$ such that

$$k\delta(\|g(t)\|_B + \|f(t)\|) + k\delta^2 < \epsilon,$$

and neighbourhoods U and V of t such that $\|f(t) - f(s)\| < \delta$ for $s \in U$ and $\|g(t) - g(s)\|_B < \delta$ for $s \in V$. The continuity of fg at t follows, because for all $s \in U \cap V$,

$$\begin{aligned}
\|(fg)(t) - (fg)(s)\| &= \|(f(t) - f(s))g(t) + f(s)(g(t) - g(s))\| \\
&\le k\|f(t) - f(s)\|\,\|g(t)\|_B + k\|f(s)\|\,\|g(t) - g(s)\|_B \\
&< k\delta\|g(t)\|_B + k(\|f(t)\| + \|f(t) - f(s)\|)\delta \\
&< k\delta(\|g(t)\|_B + \|f(t)\|) + k\delta^2 < \epsilon.
\end{aligned}$$

Also, fg vanishes at infinity. Indeed, if $\|f(t)\|$ and $\|g(t)\|_B$ are less than a given $\sqrt{\epsilon/k} > 0$ outside compact sets K and L, respectively, then

$$\|(fg)(t)\| \le k\|f(t)\|\,\|g(t)\|_B < \epsilon$$

outside $K \cup L$. The proofs for gf are similar, so $C_0(X,A)$ is a two-sided ideal of $C_0(X,B)$.

To show the density of $C_0(X,A)$ in $C_0(X,B)$, let $g \in C_0(X,B)$ and $\epsilon > 0$ be arbitrary. Choose $n \in \mathbb{N}$ and $\phi_j \in C_0(X)$, $x_j \in B$, $j = 1,\dots,n$, such that

$$\Big\|g - \sum_{j=1}^n \phi_j \otimes x_j\Big\|_\infty^{(B)} < \frac{\epsilon}{2}.$$

Clearly we can assume that $\|\phi_j\|_\infty \le 1$ for all j. Since A is dense in B, we can choose a_j, $j = 1,\dots,n$, such that $\|x_j - a_j\|_B < \epsilon/(2n)$, $j = 1,\dots,n$. The density of $C_0(X,A)$ in $C_0(X,B)$ follows:

$$\begin{aligned}
\Big\|g - \sum_{j=1}^n \phi_j \otimes a_j\Big\|_\infty^{(B)} &= \sup_{t\in X}\Big\|g(t) - \sum_{j=1}^n \phi_j(t)a_j\Big\|_B \\
&= \sup_{t\in X}\Big\|g(t) - \sum_{j=1}^n \phi_j(t)x_j + \sum_{j=1}^n \phi_j(t)(x_j - a_j)\Big\|_B
\end{aligned}$$

$$\leq \sup_{t\in X}\Big\|g(t)-\sum_{j=1}^{n}\phi_j(t)x_j\Big\|_B+\sum_{j=1}^{n}\sup_{t\in X}|\phi_j(t)|\,\|x_j-a_j\|_B$$
$$<\frac{\epsilon}{2}+n\frac{\epsilon}{2n}=\epsilon.$$

Finally, if $l>0$ is such that $\|a\|_B\leq l\|a\|$ for all $a\in A$, then, for $f\in C_0(X,A)$,

$$\|f\|_\infty^{(B)}=\sup_{t\in X}\|f(t)\|_B\leq l\sup_{t\in X}\|f(t)\|=l\|f\|_\infty^{(A)}.$$

This concludes the part (i)⇒(ii).

Then suppose $C_0(X,A)$ is a Segal algebra in $C_0(X,B)$, and take any $a\in A$, $x\in B$. Choose $f\in C_0(X,A)$, $g\in C_0(X,B)$ and $t_0\in X$ such that $f(t_0)=a$ and $g(t_0)=x$, so $ax=(fg)(t_0)\in A$. Similarly $xa\in A$. Furthermore, for any $\epsilon>0$, there is a function $h\in C_0(X,A)$ such that $\|g-h\|_\infty^{(B)}<\epsilon$. In particular, $\|g(t_0)-h(t_0)\|_B=\|x-h(t_0)\|_B<\epsilon$, from which it follows that A is a dense two-sided ideal of B. To show that $\|\cdot\|$ majorizes $\|\cdot\|_B$ on A, pick any $a\in A$ and a nonzero $\phi\in C_0(X)$. We have $\|\phi\otimes a\|_\infty^{(B)}\leq l\|\phi\otimes a\|_\infty^{(A)}$ for some $l>0$, i.e.

$$\sup_{t\in X}|\phi(t)|\,\|a\|_B\leq l\sup_{t\in X}|\phi(t)|\,\|a\|,$$

which implies $\|a\|_B\leq l\|a\|$.

It remains to prove the final claim of the theorem. By the above, the algebra $C_0(X,\widetilde{A}_M)$ is a Segal extension of $C_0(X,A)$. By Proposition 3.4, $C_0(X,\widetilde{A}_M)$ is the completion of $C_0(X,A)$ with respect to $\|\cdot\|_\infty^{(\widetilde{A}_M)}$. However, by Proposition 2.1, the norms $\|\cdot\|_\infty^{(\widetilde{A}_M)}$ and $(\|\cdot\|_\infty)_M$ coincide on $C_0(X,A)$, so $C_0(X,\widetilde{A}_M)$ equals $C_0(X,A)\widetilde{_M}$. ■

Example 3.8. The algebra $C_0^v(Y)$ is a Segal algebra in $C_0(Y)$. In fact, we have $C_0(Y)=C_0^v(Y)\widetilde{_M}$, by the Stone–Weierstrass Theorem. Thus, Theorem 3.7 tells us that the largest Segal extension of $C_0(X,C_0^v(Y))$ is $C_0(X,C_0(Y))$, which by [19, Propositions 1.10.21–22] is isomorphic to both $C_0(X)\check{\otimes}C_0(Y)$ and $C_0(X\times Y)$.

4. Approximate identities. An *approximate identity* for the Banach algebra A is a net $(e_\alpha)_{\alpha\in\Omega}$ in A satisfying

$$\|a-e_\alpha a\|\to 0\quad\text{and}\quad\|a-ae_\alpha\|\to 0$$

for all $a\in A$. An approximate identity $(e_\alpha)_{\alpha\in\Omega}$ is said to be *bounded* if there exists a constant R such that $\sup_{\alpha\in\Omega}\|e_\alpha\|\leq R$, and *minimal* if such an R exists and $R\leq 1$. If the index set Ω equals $\mathbb{N}$, the approximate identity is said to be *sequential*.

We consider approximate identities in the algebra $C_0(X,A)$, primarily in the case where A_M has a bounded approximate identity. Note that a (bounded) approximate identity for A is automatically a (bounded) approximate identity for A_M, but the converse does not hold for norm irregular algebras. Also recall that $C_0(X)$ has a minimal approximate identity, which is sequential if and only if X is σ-compact.

Remark 4.1. The assumption that an approximate identity for the algebra A is bounded with respect to $\|\cdot\|_M$ is not very restrictive. For example, by the Uniform Boundedness Principle, any sequential approximate identity for A is automatically $\|\cdot\|_M$-bounded (see e.g. [12, p. 191]).

The following lemma summarizes current knowledge on the relation of approximate identities for $C_0(X,A)$ and for A.

Lemma 4.2.

(i) *If the algebra A has a bounded approximate identity, then $C_0(X,A)$ has a bounded approximate identity.*
(ii) *If $C_0(X,A)$ has a (bounded) approximate identity, then A has a (bounded) approximate identity.*

For the proof of Lemma 4.2, see [13, Proposition 8.1 and Theorem 8.2]. At the end of this section we will give a sufficient condition for the unbounded analogue of Lemma 4.2(i) to hold.

Lemma 4.3. *The following conditions are equivalent:*

(i) *A_M has a bounded approximate identity;*
(ii) *$\widetilde{A}_M$ has a bounded approximate identity;*
(iii) *A has a Segal extension with a bounded approximate identity.*

Furthermore, a bounded approximate identity for A_M is also a bounded approximate identity for $\widetilde{A}_M$.

The claims in Lemma 4.3 follow from Proposition 3.4 and the fact that a normed algebra has a bounded approximate identity if and only if its completion has one (see e.g. [11, Lemma 2.1]). The following proposition extends Lemma 4.2.

Proposition 4.4. *The algebra $C_0(X,A)_M$ has a bounded approximate identity if and only if A_M has a bounded approximate identity.*

Proof. Suppose $C_0(X,A)_M$ has a bounded approximate identity. By Lemma 4.3 and Theorem 3.7, $C_0(X,A)\widetilde{_M} = C_0(X,\widetilde{A}_M)$ has a bounded approximate identity. By Lemma 4.2(ii), $\widetilde{A}_M$ has a bounded approximate identity, and again by Lemma 4.3 the same is true for A_M. Obviously all the implications are reversible. ∎

Though A need not have an approximate identity, some subalgebras of A always do. Following [6, 16], we make the following definition.

DEFINITION 4.5. Let A be a Banach algebra such that A_M has a bounded approximate identity. The *approximate ideal* of A is the set

$$E_A := A\widetilde{A}_M := \{ax : a \in A,\, x \in \widetilde{A}_M\}.$$

The following lemma summarizes the key features of the approximate ideal and motivates the definition. In particular, E_A is the largest closed ideal of A which contains an approximate identity, and A has an approximate identity if and only if $A = E_A$. The results of Lemma 4.6 have already been noted in [6, 16]. For the proofs, see [16, Proposition 2.10] and the preceding discussion therein.

LEMMA 4.6. *Assume that A_M has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$. Then:*

(i) $E_A = A\widetilde{A}_M = \widetilde{A}_M A$;
(ii) E_A *is a closed, two-sided ideal of* A;
(iii) $E_A = \{a \in A : \|a - ae_\alpha\| \to 0$ *and* $\|a - e_\alpha a\| \to 0\}$;
(iv) E_A *has an approximate identity;*
(v) *if I is a closed ideal of A with an approximate identity, then $I \subseteq E_A$.*

To construct the approximate ideal of $C_0(X, A)$ we need the following lemma, which is another extension of Lemma 4.2.

LEMMA 4.7. *Assume that A_M has a bounded approximate identity. Then $C_0(X, E_A)$ has an approximate identity.*

Proof. Let $(\phi_\alpha)_{\alpha\in\Lambda}$ and $(e_\beta)_{\beta\in\Omega}$ be bounded approximate identities for the algebras $C_0(X)$ and A_M, bounded by R' and R'', respectively. Assume that $R', R'' \geq 1$, and put $R := R'R''$. Using Proposition 2.1 we see that

$$(\|\phi_\alpha \otimes e_\beta\|_\infty)_M = \|\phi_\alpha\|_\infty \|e_\beta\|_M \leq R$$

for all $(\alpha, \beta) \in \Lambda \times \Omega$. Let $n \in \mathbb{N}$ and $\psi_j \in C_0(X)$, $a_j \in E_A$, $j = 1, \ldots, n$, be arbitrary. Following the proof of [13, Proposition 8.1] we get

$$\begin{aligned}\Big\|\sum_{j=1}^n \psi_j \otimes a_j - (\phi_\alpha \otimes e_\beta)\sum_{j=1}^n \psi_j \otimes a_j\Big\|_\infty \\ \leq \sum_{j=1}^n \|\psi_j - \phi_\alpha\psi_j\|_\infty \|a_j\| + \sum_{j=1}^n \|\psi_j\|_\infty \|a_j - e_\beta a_j\| \\ + \sum_{j=1}^n \|\psi_j - \phi_\alpha\psi_j\|_\infty \|a_j - e_\beta a_j\|,\end{aligned}$$

which converges to zero by Lemma 4.6(iii). Then let $f \in C_0(X, E_A)$ and $\epsilon > 0$ be arbitrary, and choose $n \in \mathbb{N}$ and $\psi_j \in C_0(X)$, $a_j \in E_A$, $j = 1, \ldots, n$, in such a way that $g := \sum_{j=1}^{n} \psi_j \otimes a_j$ satisfies $\|f - g\|_\infty < \epsilon/(4R)$. Choose $(\alpha_\epsilon, \beta_\epsilon)$ such that $\|g - (\phi_\alpha \otimes e_\beta)g\|_\infty < \epsilon/2$ for all $(\alpha, \beta) \geq (\alpha_\epsilon, \beta_\epsilon)$ (i.e. $\alpha \geq \alpha_\epsilon$ and $\beta \geq \beta_\epsilon$). Then, recalling the general property $\|ab\| \leq \|a\|_M \|b\|$ $(a, b \in A)$ of the multiplier norm, we obtain

$$\begin{aligned}\|f - (\phi_\alpha \otimes e_\beta)f\| &\leq (\|\phi_\alpha \otimes e_\beta\|_\infty)_M \|f - g\|_\infty + \|f - g\|_\infty \\ &\quad + \|g - (\phi_\alpha \otimes e_\beta)g\|_\infty \\ &< R\frac{\epsilon}{4R} + \frac{\epsilon}{4R} + \frac{\epsilon}{2} \leq \epsilon\end{aligned}$$

for $(\alpha, \beta) \geq (\alpha_\epsilon, \beta_\epsilon)$. Similarly we show that $\|f - f(\phi_\alpha \otimes e_\beta)\|_\infty \to 0$. ■

THEOREM 4.8. *Assume that the algebra A_M has a bounded approximate identity. Then $E_{C_0(X,A)} = C_0(X, E_A)$.*

Proof. Recall that, by Proposition 4.4, $C_0(X, A)_M$ has a bounded approximate identity. The inclusion "$\subseteq$" follows at once from Theorem 3.7, since

$$E_{C_0(X,A)} = C_0(X, A)C_0(X, \widetilde{A}_M) \subseteq C_0(X, A\widetilde{A}_M) = C_0(X, E_A).$$

For the converse inclusion, note that $C_0(X, E_A)$ is a closed ideal of $C_0(X, A)$, and by Lemma 4.7 it has an approximate identity. Thus, by Lemma 4.6(v), $C_0(X, E_A) \subseteq E_{C_0(X,A)}$. ■

EXAMPLE 4.9. Let the weight function v be such that $C_b^v(Y) \subseteq C_0(Y)$. The algebra $C_0^v(Y)$ has an approximate identity (unbounded whenever v is). Thus, by the easily verified fact $C_b^v(Y)C_0(Y) \subseteq C_0^v(Y)$, we see that $E_{C_b^v(Y)} = C_0^v(Y)$, and by Theorem 4.8, $E_{C_0(X, C_b^v(Y))} = C_0(X, C_0^v(Y))$.

As immediate corollaries to Theorem 4.8 we obtain the following further extensions of Lemma 4.2. For the latter corollary, see Remark 4.1.

COROLLARY 4.10. *Assume that A_M has a bounded approximate identity. Then $C_0(X, A)$ has an approximate identity if and only if A has an approximate identity.*

COROLLARY 4.11. *Let X be σ-compact. Then $C_0(X, A)$ has a sequential approximate identity if and only if A has a sequential approximate identity.*

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