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Tensor product of left n-invertible operators

by

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Abstract. A Banach space operator $T \in B(\mathcal{X})$ has a left m-inverse (resp., an essential left m-inverse) for some integer $m \geq 1$ if there exists an operator $S \in B(\mathcal{X})$ (resp., an operator $S \in B(\mathcal{X})$ and a compact operator $K \in B(\mathcal{X})$) such that $\sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} T^{m-i} = 0$ (resp., $\sum_{i=0}^{m} (-1)^i \binom{m}{i} T^{m-i} S^{m-i} = K$). If T_i is left m_i-invertible (resp., essentially left m_i-invertible), then the tensor product $T_1 \otimes T_2$ is left $(m_1 + m_2 - 1)$-invertible (resp., essentially left $(m_1 + m_2 - 1)$-invertible). Furthermore, if T_1 is strictly left m-invertible (resp., strictly essentially left m-invertible), then $T_1 \otimes T_2$ is: (i) left $(m + n - 1)$-invertible (resp., essentially left $(m + n - 1)$-invertible) if and only if T_2 is left n-invertible (resp., essentially left n-invertible); (ii) strictly left $(m + n - 1)$-invertible (resp., strictly essentially left $(m + n - 1)$-invertible) if and only if T_2 is strictly left n-invertible (resp., strictly essentially left n-invertible).

1. Introduction. Let $B(\mathcal{X})$ denote the algebra of bounded linear transformations, equivalently operators, on a Banach space $\mathcal{X}$ into itself. An operator $T \in B(\mathcal{X})$ is *left* (resp., *right*) *m-invertible*, for some integer $m \geq 1$, *by* $S \in B(\mathcal{X})$ if

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} T^{m-i} = 0 \quad \left(\text{resp. } \sum_{i=0}^{m} (-1)^i \binom{m}{i} T^{m-i} S^{m-i} = 0\right).$$

It is elementary to see that S is a left m-inverse of T if and only if (the adjoint operator) S^* is a right m-inverse of T^*. We say that $T \in B(\mathcal{X})$ is *m-invertible* if it has both a left m-inverse and a right m-inverse.

Evidently, every *left inverse* (i.e., left 1-inverse) of T is a left m-inverse of T and every right inverse of T is a right m-inverse of T, for every integer $m \geq 1$. Indeed, if T is left n-invertible for some positive integer n, then it is left m-invertible for every integer $m \geq n$. If T is left (resp., right) m-invertible then it is left (resp., right) invertible, but a left (resp., right) m-inverse of T is not necessarily a left (resp., right) inverse of T.

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Observe also that if T is left m-invertible by L and right n-invertible by R (for some integers $m, n \geq 1$), then T is invertible by the operator

$$\sum_{i=0}^{m-1} (-1)^{m+1+i} \binom{m}{i} L^{m-1-i} T^{m-1-i} = \sum_{i=0}^{n-1} (-1)^{n+1+i} \binom{n}{i} T^{n-1-i} R^{n-1-i}.$$

The study of m-left and m-right invertible operators has its roots in the work of Przeworska–Rolewicz [15, 16], and has since been carried out by a number of authors, amongst them Sid Ahmed [17]. An interesting example of a left m-invertible Hilbert space operator is that of an m-isometric operator T for which $\sum_{i=0}^{m} (-1)^i \binom{m}{i} T^{*m-i} T^{m-i} = 0$, where T^* denotes the Hilbert space adjoint of T. A study of m-isometric operators has been carried out by Agler and Stankus in a series of papers [1, 2, 3]; more recently a generalization of these operators to Banach spaces has been obtained by Bayart [4], Bermúdez et al. [6, 5] and Hoffmann et al. [13].

Let $\mathcal{K}(\mathcal{X})$ denote the two-sided ideal of compact operators in $B(\mathcal{X})$, and let $m \geq 1$ be an integer. We say that $T \in B(\mathcal{X})$ is: *essentially left m-invertible* (resp., *essentially right m-invertible*) by $S \in B(\mathcal{X})$ if there exists an operator $K_1 \in \mathcal{K}(\mathcal{X})$ (resp., $K_2 \in \mathcal{K}(\mathcal{X})$) such that $\sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} T^{m-i} = K_1$ (resp., $\sum_{i=0}^{m} (-1)^i \binom{m}{i} T^{m-i} S^{m-i} = K_2$). We call T *essentially m-invertible* if it is both essentially left m-invertible and essentially right m-invertible.

Recall from Müller [14, p. 154] that an *essentially left invertible* (i.e., essentially left 1-invertible) operator T is upper semi-Fredholm with the range $T(\mathcal{X})$ complemented, and an essentially right invertible operator is lower semi-Fredholm with $T^{-1}(0)$ complemented. Trivially: If T is essentially left (resp., right) m-invertible by S then so is $T+K$ for every compact K, an essentially left (resp., right) m-invertible operator is essentially left (resp., right) invertible, and every essentially left (resp., right) invertible operator is essentially left (resp., right) m-invertible (indeed, if S is an essential left (resp., right) m-inverse of T then S is an essential n-inverse left (resp., right) of T for all integers $n \geq m$). Observe however that S is an essential left (resp., right) m-inverse of T does not imply S is an essential left (resp., right) inverse of T.

Call an operator S a *strict left m-inverse* (resp., a *strict essential left m-inverse*) of T if S is a left m-inverse (resp., essential left m-inverse) of T but S is not a left n-inverse (resp., essential left n-inverse) of T for all integers $n < m$. Define strict right m-inverses and strict essential right m-inverses similarly.

In the following, we consider operators T_1 and T_2 such that T_1 is left (resp., right) m-invertible and T_2 is left (resp., right) n-invertible, and prove that their tensor product $T_1 \otimes T_2$ is left (resp., right) $(m+n-1)$-invertible.

Furthermore, if T_1 is strictly left (similarly, right) m-invertible, then $T_1 \otimes T_2$ is: (i) left (resp., right) $(m+n-1)$-invertible if and only if T_2 is left (resp., right) n-invertible; (ii) strictly left (resp., right) $(m+n-1)$-invertible if and only if T_2 is strictly left (resp., right) n-invertible.

These results have an essentially left (resp., right) t-invertible counterpart: If T_1 is essentially left (resp., right) m-invertible and T_2 is essentially left (resp., essentially right) n-invertible, then $T_1 \otimes T_2$ is essentially left (resp., right) $(m+n-1)$-invertible. Furthermore, if T_1 is strictly essentially left (resp., right) m-invertible, then $T_1 \otimes T_2$ is: (i) essentially left (resp., right) $(m+n-1)$-invertible if and only if T_2 is essentially left (resp., right) n-invertible; (ii) strictly essentially left (resp., right) $(m+n-1)$-invertible if and only if T_2 is strictly essentially left (resp., right) n-invertible. This generalizes some results of Botelho et al. [7, 8], Bermúdez et al. [6, 5], and those of one of the authors on the tensor product of m-isometric operators [9, 10, 11]. We remark that these results have a natural interpretation for the left-right multiplication operator $\triangle_{ST} : J \to J$, $\triangle_{ST}(A) = SAT$, where $J \subset B(\mathcal{Y}, \mathcal{X})$ is an operator ideal.

2. Results. Given two complex infinite-dimensional Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \overline{\otimes} \mathcal{Y}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$; for $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$, denote by $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ the tensor product operator defined by A and B.

Evidently, an operator $T \in B(\mathcal{X})$ is left m-invertible by $S \in B(\mathcal{X})$ if and only if $T \otimes I \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ is left m-invertible by $S \otimes I \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$. Furthermore, T is strictly left m-invertible by S if and only if $T \otimes I$ is strictly left m-invertible by $S \otimes I$. Observe also that $T_1 \otimes T_2 = (T_1 \otimes I)(I \otimes T_2) = (I \otimes T_2)(T_1 \otimes I)$. (Here and below, we shall make a slight misuse of notation and write I for the identity operator on both $\mathcal{X}$ and $\mathcal{Y}$.) Hence, given T_1 left m-invertible by S_1 and T_2 left n-invertible by S_2, in considering the left t-invertibility of $T_1 \otimes T_2$ by $S_1 \otimes S_2$ we may assume without loss of generality that the positive integer m is less than or equal to the positive integer n.

We state our theorems below for left invertibility; their analogues for right invertibility follow from a similar argument.

THEOREM 2.1. *The tensor product of a left m-invertible operator with a left n-invertible operator is a left $(m+n-1)$-invertible operator.*

A proof of the theorem may be obtained using a combinatorial argument similar to that in the papers [9, 10], or by using an argument similar to the one used to prove [11, Corollary 2.2] (see also [6]). However, we follow here an argument using double sequences satisfying certain properties. At the heart of this argument lies the following simple lemma, which along with leading to a proof of the theorem has a number of other interesting consequences. Let $\mathcal{P}_d$ denote the set of all complex polynomials of degree $\le d$.

Lemma 2.2. *Let $m \in \mathbb{N}$, and let $(a_j)_{j=0}^{\infty}$ be a sequence of complex numbers. Then the following statements are equivalent:*

(i) $\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{k+i} = 0$ *for every integer $k \ge 0$;*
(ii) *there exists a polynomial $p \in \mathcal{P}_{m-1}$ such that $a_i = p(i)$ for every $i \ge 0$.*

Proof. (ii)⇒(i). We prove the statement by induction on m. For $m = 1$, p is a constant and the statement is clear.

Suppose that $m \ge 2$ and the statement is true for $m-1$. Let $\deg p < m$. Define q by $q(t) = p(t+1) - p(t)$. Then q is a polynomial of degree $\deg q = \deg p - 1 < m - 1$. We have

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i\binom{m}{i}p(k+i) &= \sum_{i=0}^{m}(-1)^i\left(\binom{m-1}{i}+\binom{m-1}{i-1}\right)p(k+i)\\ &= \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}p(k+i) + \sum_{i=0}^{m-1}(-1)^{i+1}\binom{m-1}{i}p(k+i+1)\\ &= \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\big(p(k+i)-p(k+i+1)\big)\\ &= -\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}q(k+i) = 0\end{aligned}$$

by the induction assumption.

(i)⇒(ii). Let $\mathcal{V}$ be the vector space of all sequences (a_i) satisfying (i). Since each sequence in $\mathcal{V}$ is uniquely determined by its members a_i, $0 \le i \le m-1$, we see that $\dim \mathcal{V} \le m$. Let $\mathcal{V}_0 = \{(p(i)) : p \in \mathcal{P}_{m-1}\}$. Since $\mathcal{V}_0 \subset \mathcal{V}$ and $\dim \mathcal{V}_0 = m$, we have $\mathcal{V}_0 = \mathcal{V}$. ∎

Remark 2.3. The proof of (ii)⇒(i) above works just as well with $p(n+j)$ replaced by $p(n+rj)$ for every $r \in \mathbb{N}$. Indeed, let $p \in \mathcal{P}_{m-1}$, $r \in \mathbb{N}$ and $k \ge 0$. Then $i \mapsto p(k+ri)$ is again a polynomial of degree $\le m-1$, so we have $\sum_{i=0}^{m}(-1)^i\binom{m}{i}p(k+ri) = 0$. In particular, if $0 \le c \le m-1$, $r \in \mathbb{N}$ and $k \ge 0$, then $\sum_{i=0}^{m}\binom{m}{i}(k+ri)^c = 0$.

Lemma 2.2 leads to the following characterization of left m-invertibility. Let $\mathcal{X}^*$ denote space dual to $\mathcal{X}$.

THEOREM 2.4. *Let $S, T \in B(\mathcal{X})$, $m \in \mathbb{N}$. The following statements are equivalent:*

(i) *S is a left m-inverse of T;*
(ii) *for all $x \in \mathcal{X}$, $x^* \in \mathcal{X}^*$ and $k \geq 0$,*

$$\sum_{i=0}^{m}(-1)^i \binom{m}{i} \langle S^{i+k}T^{i+k}x, x^* \rangle = 0;$$

(iii) *for all $x \in X$ and $x^* \in X^*$ there exists a polynomial $p \in \mathcal{P}_{m-1}$ such that*

$$\langle S^i T^i x, x^* \rangle = p(i) \qquad (i \geq 0).$$

Proof. (ii)⇒(i). For all $x \in X$ and $x^* \in X^*$ we have

$$\sum_{i=0}^{m}(-1)^i \binom{m}{i} \langle S^i T^i x, x^* \rangle = 0.$$

So $\sum_{i=0}^{m}(-1)^i \binom{m}{i} S^i T^i = 0$.

(i)⇒(ii). Let $x \in X$, $x^* \in X^*$ and $k \geq 0$. We have

$$\sum_{i=0}^{m}(-1)^i \binom{m}{i} \langle S^{i+k}T^{i+k}x, x^* \rangle = \left\langle \sum_{i=0}^{m}(-1)^i \binom{m}{i} S^i T^i (T^k x), S^{*k}x^* \right\rangle = 0.$$

(ii)⇔(iii). Let $x \in X$ and $x^* \in X^*$. Write $a_i = \langle S^i T^i x, x^* \rangle$ $(i \geq 0)$. The desired equivalence then follows from the previous lemma. ■

The following two corollaries of Lemma 2.2 all but prove Theorem 2.1.

COROLLARY 2.5. *If $(a_{i,j})_{i,j=0}^{\infty}$ is a double sequence of complex numbers satisfying*

$$\sum_{i=0}^{m}(-1)^i \binom{m}{i} a_{k+i,\ell} = 0, \tag{1}$$

$$\sum_{j=0}^{n}(-1)^j \binom{n}{j} a_{k,\ell+j} = 0, \tag{2}$$

then

$$\sum_{s=0}^{m+n-1}(-1)^s \binom{m+n-1}{s} a_{s,s} = 0. \tag{3}$$

Proof. Each double sequence $(a_{i,j})$ is uniquely determined by its terms $a_{i,j}$, $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, so if we let V denote the vector space of all double sequences $(a_{i,j})$ satisfying (1) and (2) above, then $\dim V \leq mn$.

For $0 \le c \le m-1$ and $0 \le d \le n-1$, define the double sequence $(b^{(c,d)})$ by $b^{(c,d)}_{i,j} = i^c j^d$. Then

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} b^{(c,d)}_{k+i,\ell} = \ell^d \sum_{i=0}^{m} (-1)^i \binom{m}{i} (k+i)^c = 0$$

for all non-negative integers k,ℓ. Thus $(b^{(c,d)})$ satisfies (1), similarly (2). Consequently, $(b^{(c,d)}) \in V$. Since these double sequences are linearly independent, and hence form a basis of V, to prove the corollary it would now suffice to prove that $b^{(c,d)}$ satisfy (3). But this follows from the fact that $b^{(c,d)}_{s,s} = s^{c+d}$, $0 \le c+d \le m+n-2$, and

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{m+n-1}{s} b^{(c,d)}_{s,s} = \sum_{s=0}^{m+n-1} (-1)^s \binom{m+n-1}{s} s^{c+d} = 0. \blacksquare$$

For a pair of operators $A, B \in B(\mathcal{X})$, let $[A,B] = AB - BA$.

Corollary 2.6. *If $A_1 \in B(\mathcal{X})$ is left m-invertible by $B_1 \in B(\mathcal{X})$, $A_2 \in B(\mathcal{X})$ is left n-invertible by $B_2 \in B(\mathcal{X})$ and $[A_1, A_2] = 0 = [B_1, B_2]$, then A_1A_2 is left $(m+n-1)$-invertible by B_1B_2.*

Proof. Fix $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$, and let $a_{i,j} = \langle B_1^i B_2^j A_1^i A_2^j x, x^* \rangle$. Then, for all non-negative integers k and ℓ, the left m-invertibility of A_1 by B_1 implies that

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{k+i,\ell} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \langle B_1^i A_1^i (A_1^k A_2^\ell x), (B_2^{*\ell} B_1^{*k} x^*) \rangle = 0,$$

i.e., $(a_{i,j})$ satisfies (1). Similarly, $(a_{i,j})$ satisfies (2), and hence also (3). Since $a_{s,s} = \langle (B_1B_2)^s (A_1A_2)^s x, x^* \rangle$,

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{m+n-1}{s} \langle (B_1B_2)^s (A_1A_2)^s x, x^* \rangle = 0.$$

Our choice of vectors x and x^* having been arbitrary, we must have

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{m+n-1}{s} (B_1B_2)^s (A_1A_2)^s = 0. \blacksquare$$

Proof of Theorem 2.1. If we set $A_1 = (T_1 \otimes I)$ and $A_2 = (I \otimes T_2)$, then $T_1 \in B(\mathcal{X})$ is left m-invertible by $S_1 \in B(\mathcal{X})$, $T_2 \in B(\mathcal{Y})$ is left n-invertible by $S_2 \in B(\mathcal{Y})$ and $T_1 \otimes T_2$ is left $(m+n-1)$-invertible by $S_1 \otimes S_2$ if and only if A_1 is left m-invertible by $B_1 = (S_1 \otimes I)$, A_2 is left n-invertible by $B_2 = (I \otimes S_2)$ and A_1A_2 is left $(m+n-1)$-invertible by B_1B_2. Since $[A_1, A_2] = 0 = [B_1, B_2]$, the conclusion follows from Corollary 2.6. $\blacksquare$

REMARK 2.7. Suppose that $T \in B(\mathcal{X})$ is left m-invertible by $S \in B(\mathcal{X})$. Fix $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$, and let $a_n = \langle S^n T^n x, x^* \rangle$. Then it follows from Remark 2.3 that $\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{k+ri} = 0$ for all $r \in \mathbb{N}$ and integers $k \geq 0$. In particular,

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{ri} T^{ri} = 0,$$

i.e., *T^r is left m-invertible by S^r* for all $r \in \mathbb{N}$.

(m, p)**-isometries: a remark.** Recall that a Banach space operator $T \in B(\mathcal{X})$ is an (m, p)-*isometry* for some integer $m \geq 1$ and $p \in (0, \infty)$ if

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \|T^i x\|^p = 0, \qquad x \in \mathcal{X}.$$

Let T, S be commuting operators in $B(\mathcal{X})$ such that T is an (m, p)-isometry and S is an (n, p)-isometry. Define the double sequence $(a_{i,j})$ by $a_{i,j} = \|T^i S^j x\|^p$, $x \in \mathcal{X}$. Then

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{k+i,j} = 0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{i,\ell+j}$$

for integers $k, \ell \geq 0$. Applying Corollary 2.5 we conclude that

$$\sum_{s=0}^{m+n-1} (-1)^s \binom{m+n-1}{s} \|(TS)^s x\|^p = 0$$

for all $x \in \mathcal{X}$. We have proved:

COROLLARY 2.8 ([6, Theorem 3.3]). *If $T, S \in B(\mathcal{X})$ are commuting operators such that T is an (m, p)-isometry and S is an (n, p)-isometry, then TS is an $(m + n - 1, p)$-isometry.*

For an $x \in \mathcal{X}$ and an operator $T \in B(\mathcal{X})$, define the sequence (a_n) by $a_n = \|T^n x\|^p$. If T is an (m, p)-isometry, then Remark 2.3 implies that $\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{k+ri} = 0$ for all $r \in \mathbb{N}$ and integers $k \geq 0$. In particular:

COROLLARY 2.9 ([5, Theorem 3.1]). *If $T \in B(\mathcal{X})$ is an (m, p)-isometry, then so is T^r for each $r \in \mathbb{N}$.*

Strict left invertibility. If $T \in B(\mathcal{X})$ is left m-invertible and $S \in B(\mathcal{X})$ is a strict left m-inverse of T, then $\sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} T^{m-i} = 0$ and $\sum_{i=0}^{p} (-1)^i \binom{m}{i} S^{p-i} T^{p-i} \neq 0$ for all $p < m$. The proof of [8, Theorem 3.1] shows that if $S \in B(\mathcal{X})$ is a strict left m-inverse of $T \in B(\mathcal{X})$, then the set $\{I, ST, S^2T^2, \ldots, S^{m-1}T^{m-1}\}$ is linearly independent. More generally:

Theorem 2.10. *Let $S, T \in B(\mathcal{X})$, $m \in \mathbb{N}$, let S be a left m-inverse of T. The following statements are equivalent:*

(i) *S is a strict left m-inverse of T;*
(ii) *the operators $I, ST, S^2T^2, \dots, S^{m-1}T^{m-1}$ are linearly independent;*
(iii) *there exists $x \in \mathcal{X}$ such that the vectors $x, STx, \dots, S^{m-1}T^{m-1}x$ are linearly independent;*
(iv) *for every polynomial $p \in \mathcal{P}_{m-1}$ there exist $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ such that*
$$\langle S^iT^ix, x^*\rangle = p(i) \qquad (i \geq 0);$$
(v) *there exist $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ such that $\langle S^iT^ix, x^*\rangle = i^{m-1}$ $(i \geq 0)$.*

Proof. (iii)⇒(ii) is clear.

(ii)⇒(i). Suppose that S is not a strict left m-inverse of T. By definition, the operators $I, ST, \dots, S^{m-1}T^{m-1}$ are linearly dependent.

(i)⇒(iii). Suppose that for any $x \in \mathcal{X}$ the vectors $x, STx, \dots, S^{m-1}T^{m-1}x$ are linearly dependent, i.e., there exists a non-trivial linear combination $\sum_{i=0}^{m-1} \alpha_i S^iT^ix = 0$.

Since also $\sum_{i=0}^{m}(-1)^i\binom{m}{i}S^iT^ix = 0$, we can get $\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}S^iT^ix = 0$ as in [9]. So
$$\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}S^iT^i = 0,$$
a contradiction.

(iii)⇒(iv). Let $x \in \mathcal{X}$ and suppose the vectors $x, STx, \dots, S^{m-1}T^{m-1}x$ are linearly independent. Let $p \in \mathcal{P}_{m-1}$. Then there exists $x^* \in \mathcal{X}^*$ such that
$$\langle S^iT^ix, x^*\rangle = p(i) \qquad (0 \leq i \leq m-1).$$
By Theorem 2.4 this implies that $\langle S^iT^ix, x^*\rangle = p(i)$ for all $i \geq 0$.

(iv)⇒(v) is clear.

(v)⇒(i). If $x \in X$ and $x^* \in X^*$ satisfy $\langle S^iT^ix, x^*\rangle = i^{m-1}$ for all $i \geq 0$, then (by Theorem 2.4) S is not a left $(m-1)$-inverse of T, so S is a strict left m-inverse of T. ∎

The converse of Theorem 2.1, namely that if $S_1 \in B(\mathcal{X})$ is a left t-inverse of $T_1 \in B(\mathcal{X})$ and $T_1 \otimes T_2$ is left s-invertible by $S_1 \otimes S_2$ (for some $T_2, S_2 \in B(\mathcal{Y})$) then T_2 is left $(s-t+1)$-invertible by S_2, is not as straightforward. Recall that every left n_1-inverse $S \in B(\mathcal{X})$ of an operator $T \in B(\mathcal{X})$ is a left n-inverse of T for every integer $n \geq n_1$. Hence, if S_1 is a left t-inverse of T_1, then there is a least positive integer $m \leq t$ such that S_1 is a strict left m-inverse of T_1 (and then $\{I, S_1T_1, \dots, S_1^{m-1}T_1^{m-1}\}$ is an independent set).

Theorem 2.11. *Let $S_1, T_1 \in B(\mathcal{X})$, $S_2, T_2 \in B(\mathcal{Y})$, $m, n \in \mathbb{N}$. Suppose that S_1 is a strict left m-inverse of T_1 and S_2 is a strict left n-inverse of T_2. Then $S_1 \otimes S_2$ is a strict left $(m+n-1)$-inverse of $T_1 \otimes T_2$.*

Proof. By Theorem 2.1, $S_1 \otimes S_2$ is a left $(m+n-1)$-inverse of $T_1 \otimes T_2$.

By Theorem 2.10, there exist $x \in \mathcal{X}$, $x^* \in \mathcal{X}^*$, $y \in \mathcal{Y}$ and $y^* \in \mathcal{Y}^*$ such that

$$\langle S_1^i T_1^i x, x^* \rangle = i^{m-1} \quad \text{and} \quad \langle S_2^i T_2^i y, y^* \rangle = i^{n-1} \qquad (i \geq 0).$$

So

$$\langle (S_1 \otimes S_2)^i (T_1 \otimes T_2)^i (x \otimes y), x^* \otimes y^* \rangle = \langle S_1^i T_1^i x, x^* \rangle \cdot \langle S_2^i T_2^i y, y^* \rangle = i^{m+n-2}$$

for all integers $i \geq 0$. This, again by Theorem 2.10, implies that $S_1 \otimes S_2$ is a strict left $(m+n-1)$-inverse of $T_1 \otimes T_2$. ∎

Theorem 2.12. *Let $S_1, T_1 \in B(\mathcal{X})$ and $S_2, T_2 \in B(\mathcal{Y})$. If S_1 is a strict left m-inverse of T_1, then $S_1 \otimes S_2$ is a left s-inverse of $T_1 \otimes T_2$ if and only if S_2 is a left $(s-m+1)$-inverse of T_2.*

Proof. If S_2 is a left $(s-m+1)$-inverse of T_2 then $S_1 \otimes S_2$ is a left s-inverse of $T_1 \otimes T_2$ by Theorem 2.1.

Suppose that $S_1 \otimes S_2$ is a left s-inverse of $T_1 \otimes T_2$. Let $y \in Y$ and $y^* \in Y^*$. Write $f(i) = \langle S_2^i T_2^i y, y^* \rangle$ $(i \geq 0)$.

By Theorem 2.10, for each $p \in \mathcal{P}_{m-1}$ there exist $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ such that $\langle S_1^i T_1^i x, x^* \rangle = p(i)$ for all $i \geq 0$. So

$$p(i) f(i) = \langle S_1^i T_1^i x, x^* \rangle \cdot \langle S_2^i T_2^i y, y^* \rangle = \langle (S_1 \otimes S_2)^i (T_1 \otimes T_2)^i (x \otimes y), x^* \otimes y^* \rangle.$$

Hence $i \mapsto p(i) f(i)$ is a polynomial of degree $\leq s-1$. For $p \equiv 1$ this means that f is a polynomial of degree $\leq s-1$. For $p \equiv i^{m-1}$ we get $f \in \mathcal{P}_{s-m}$.

Since $y \in Y$ and $y^* \in Y^*$ were arbitrary, Theorem 2.4 implies that S_2 is a left $(s-m+1)$-inverse of T_2. ∎

Example. If $m \geq 2$ and S is a strict left m-inverse of T, then S^2 is a left 2-inverse of T^2. Thus S^2 is not a strict left 3-inverse of T^2. Observe here that S and T do not commute.

Theorem 2.11 (and also Theorem 2.12) is not true if we assume only that S_1, S_2, T_1, T_2 are commuting operators (such that $S_1 S_2$ is a left s-inverse of $T_1 T_2$). Let $\mathcal{X}$ be the ℓ_1-space with the standard basis $e_{i,j}$ $(i, j \in \mathbb{N})$. Let the operators $T_1, T_2, S_1, S_2 \in B(X)$ be defined by

$$T_1 e_{i,j} = \frac{i+j+1}{i+j} e_{i+1,j},$$

$$T_2 e_{i,j} = \frac{i+j+1}{i+j} e_{i,j+1},$$

$$S_1 e_{i,j} = e_{i-1,j} \quad \text{if } i \geq 2, \qquad S_1 e_{1,j} = 0,$$

$$S_2 e_{i,j} = e_{i,j-1} \quad \text{if } j \geq 2, \qquad S_2 e_{i,1} = 0.$$

Clearly S_1, S_2, T_1, T_2 are mutually commuting operators. We have $S_1T_1e_{i,j} = \frac{i+j+1}{i+1}e_{i,j}$ and $S_1^2T_1^2e_{i,j} = \frac{i+j+2}{i+1}e_{i,j}$. So $(I - 2S_1T_1 + S_1^2T_1^2)e_{i,j} = 0$ for all $i, j \in \mathbb{N}$ and so S_1 is an (obviously strict) left 2-inverse of T_1. Similarly, S_2 is a strict left 2-inverse of T_2.

It is easy to verify that S_1S_2 is a left 2-inverse of T_1T_2, so it is not a strict left 3-inverse.

Evidently, S_2 may not be a strict left $(s-m+1)$-inverse of T_2 in Theorem 2.12. For S_2 to be a strict left $(s-m+1)$-inverse one requires $S_1 \otimes S_2$ to be a strict left s-inverse of $T_1 \otimes T_2$. The following result complements Theorem 2.11.

THEOREM 2.13. *Let $S_1, T_1 \in B(\mathcal{X})$ and $S_2, T_2 \in B(\mathcal{Y})$. Suppose that S_1 is a strict left m-inverse of T_1 and $S_1 \otimes S_2$ is a left s-inverse of $T_1 \otimes T_2$. Then S_2 is a strict left $(s-m+1)$-inverse of T_2 if and only if $S_1 \otimes S_2$ is a strict left s-inverse of $T_1 \otimes T_2$.*

Proof. It is clear from the above that if $S_1 \otimes S_2$ is a left s-inverse of $T_1 \otimes T_2$, then S_2 is a left $(s-m+1)$-inverse of T_2. To prove that S_2 is a strict $(s-m+1)$-inverse of T_2 if and only if $S_1 \otimes S_2$ is a strict left s-inverse of $T_1 \otimes T_2$, suppose (to start with) that $S_1 \otimes S_2$ is a strict left s-inverse of $T_1 \otimes T_2$ but S_2 is not a strict left $(s-m+1)$-inverse of T_2. Then there exists an integer k, $1 \le k < s-m+1$, such that S_2 is a left k-inverse of T_2, and hence $S_1 \otimes S_2$ is a left $(m+k-1)$-inverse of $T_1 \otimes T_2$ (see Theorem 2.1). Since $m+k-1 < s$, we have a contradiction. If, instead, S_2 is a strict left $(s-m+1)$-inverse of T_2, then $S_1 \otimes S_2$ is a strict left s-inverse of $T_1 \otimes T_2$ (by Theorem 2.11). ∎

Essentially left m-invertible operators. We prove next the analogues of Theorem 2.12 and 2.13 for the tensor product of essentially left m-invertible operators. To this end we start by introducing a construction, known in the literature as the Sadovskiĭ/Buoni, Harte, Wickstead construction [14, p. 159], which leads to a representation of the Calkin algebra as an algebra of operators on a suitable Banach space. Let $\ell^\infty(\mathcal{X})$ denote the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of $\mathcal{X}$ endowed with the norm $\|x\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\|$, and write T_∞, $T_\infty x := (Tx_n)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty$, for the operator induced by T on $\ell^\infty(\mathcal{X})$. The set $m(\mathcal{X})$ of all precompact sequences of elements of $\mathcal{X}$ is a closed subspace of $\ell^\infty(\mathcal{X})$ which is invariant for T_∞. Let $\mathcal{X}_q := \ell^\infty(\mathcal{X})/m(\mathcal{X})$, and denote by T_q the operator T_∞ on $\mathcal{X}_q$. The mapping $T \mapsto T_q$ is then a unital homomorphism from $B(\mathcal{X}) \to B(\mathcal{X}_q)$, with kernel the ideal $\mathcal{K}(\mathcal{X})$ of compact operators on $\mathcal{X}$, which induces a norm decreasing monomorphism from $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ to $B(\mathcal{X}_q)$ with the following properties (see [14, Section 17] for details):

(i) T is upper semi-Fredholm if and only if T_q is injective, if and only if T_q is bounded below;
(ii) $T_q = 0$ if and only if T is compact.

Furthermore, as is easily verified,

(iii) $(S \otimes T)_q = S_q \otimes T_q$ for every $S \in B(\mathcal{X})$ and $T \in B(\mathcal{Y})$.

As above, let $S_1, T_1 \in B(\mathcal{X})$ and $S_2, T_2 \in B(\mathcal{Y})$. If S_1 is an essential left m-inverse of T_1, equivalently if $\sum_{i=0}^{m} (-1)^i \binom{m}{i} {S_1}^{m-i} {T_1}^{m-i} = K$ for some $K \in \mathcal{K}(\mathcal{X})$, then $\sum_{i=0}^{m} (-1)^i \binom{m}{i} (S_1)_q^{m-i} (T_1)_q^{m-i} = 0$, i.e., $(S_1)_q \in B(\mathcal{X}_q)$ is a left m-inverse of $(T_1)_q \in B(\mathcal{X}_q)$. The converse holds, and we conclude that *$(S_1)_q \in B(\mathcal{X}_q)$ is a left m-inverse of $(T_1)_q \in B(\mathcal{X}_q)$ if and only if S_1 is an essential left m-inverse of T_1*. Again, $S_1 \otimes S_2$ is an essential left s-inverse of $T_1 \otimes T_2$ if and only if $(S_1)_q \otimes (S_2)_q$ is a left s-inverse of $(T_1)_q \otimes (T_2)_q$. Observing that the property of being "strict" transfers from an operator T to T_q (and back), we have:

THEOREM 2.14. *Let $S_1, T_1 \in B(\mathcal{X})$ and $S_2, T_2 \in B(\mathcal{Y})$.*

(i) *If S_i, $i = 1, 2$, is an essential left m_i-inverse of T_i, then $S_1 \otimes S_2$ is an essential left $(m_1 + m_2 - 1)$-inverse of $T_1 \otimes T_2$.*
(ii) *If S_1 is a strict essential left m-inverse of T_1, then $S_1 \otimes S_2$ is an essential left s-inverse of $T_1 \otimes T_2$ if and only if S_2 is an essential left $(s - m + 1)$-inverse of T_2.*
(iii) *If S_1 is a strict essential left m-inverse of T_1 and $S_1 \otimes S_2$ is an essential left s-inverse of $T_1 \otimes T_2$, then S_2 is a strict essential left $(s - m + 1)$-inverse of T_2 if and only if $S_1 \otimes S_2$ is a strict essential left s-inverse of $T_1 \otimes T_2$.*

Elementary operator $\triangle_{T_1T_2} = L_{T_1}R_{T_2}$. Given $T_1 \in B(\mathcal{X})$ and $T_2 \in B(\mathcal{Y})$, the elementary operator $\triangle_{T_1T_2} \in B(\mathcal{Y}, \mathcal{X})$ is defined by $\triangle_{T_1T_2}(A) = T_1AT_2$ for all $A \in B(\mathcal{Y}, \mathcal{X})$. Theorems 2.12–2.14 have natural analogues for the operator $\triangle_{T_1T_2}$.

Recall from [12, p. 50] that a pair $\langle \mathcal{X}, \tilde{\mathcal{X}} \rangle$ of Banach spaces is a *dual pairing* if either $\tilde{\mathcal{X}} = \mathcal{X}^*$ or $\mathcal{X} = \tilde{\mathcal{X}}^*$. Let $x \otimes y'$, $x \in \mathcal{X}$ and $y' \in \mathcal{Y}^*$, denote the rank one operator $\mathcal{Y} \to \mathcal{X}$, $y \mapsto \langle y, y' \rangle x$. An *operator ideal* J between Banach spaces $\mathcal{Y}$ and $\mathcal{X}$ is a linear subspace of $B(\mathcal{Y}, \mathcal{X})$ equipped with a Banach norm α such that

(i) $x \otimes y' \in J$ and $\alpha(x \otimes y') = \|x\| \, \|y\|$;
(ii) $\triangle_{ST}(A) = L_S R_T(A) = SAT$ and $\alpha(SAT) \leq \|S\| \alpha(A) \|T\|$

for all $x \in \mathcal{X}$, $y' \in \mathcal{Y}^*$, $A \in J$, $S \in B(\mathcal{X})$ and $T \in B(\mathcal{Y})$ [12, p. 51]. Thus defined, each J is a tensor product relative to the dual pairings $\langle \mathcal{X}, \mathcal{X}^* \rangle$ and

$\langle \mathcal{Y}^*, \mathcal{Y} \rangle$ and the bilinear mappings

$$\mathcal{X} \times \mathcal{Y}^* \to J, \quad (x, y') \mapsto x \otimes y',$$
$$B(\mathcal{X}) \times B(\mathcal{Y}^*) \to B(J), \quad (S, T^*) \mapsto S \otimes T^*,$$

where $S \otimes T^*(A) = SAT$. The following result is now evident from Theorems 2.12–2.14.

THEOREM 2.15. *Let $S_1, T_1 \in B(\mathcal{X})$ and $S_2, T_2 \in B(\mathcal{Y})$.*

(i) *If S_1 is a left m_1-inverse (resp., essential left m_1-inverse) of T_1 and S_2 is a right m_2-inverse (resp., essential right m_2-inverse) of T_2, then $\triangle_{S_1 S_2}$ is a left $(m_1 + m_2 - 1)$-inverse (resp., essential left $(m_1 + m_2 - 1)$-inverse) of $\triangle_{T_1 T_2}$.*

(ii) *If S_1 is a strict left m-inverse (resp., strict essential left m-inverse) of T_1, then $\triangle_{S_1 S_2}$ is a left s-inverse (resp., an essential left s-inverse) of $\triangle_{T_1 T_2}$ if and only if S_2 is a right $(s - m + 1)$-inverse (resp., an essential right $(s - m + 1)$-inverse) of T_2.*

(iii) *If S_1 is a strict left m-inverse (resp., strict essential left m-inverse) of T_1 and $\triangle_{S_1 S_2}$ is a left s-inverse (resp., an essential left s-inverse) of $\triangle_{T_1 T_2}$, then S_2 is a strict right $(s - m + 1)$-inverse (resp., strict essential right $(s - m + 1)$-inverse) of T_2 if and only if $\triangle_{S_1 S_2}$ is a strict left s-inverse (resp., a strict essential left s-inverse) of $\triangle_{T_1 T_2}$.*

A limited version of Theorem 2.15 has been considered by Sid Ahmed [17, Theorems 3.1 and 3.2], and versions of the theorem for m-isometric operators on the ideal $\mathcal{C}_2(\mathcal{H})$ of Hilbert–Schmidt class operators have been considered in [6, 7, 8, 9, 10].

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