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Hermitian operators on Lipschitz function spaces

by

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Abstract. This paper characterizes the hermitian operators on spaces of Banach-valued Lipschitz functions.

1. Introduction. The notion of a hermitian operator on a Banach space can be traced back to the early work of Lumer [L61] and also Vidav [V]. The definition of hermitian operator proposed by Vidav requires an exponential norm condition on the operator, while Lumer considered an equivalent definition in terms of semi-inner products. A semi-inner product on a Banach space maintains some properties of an inner product and thus adds an additional geometric component to the space. Though hermitian operators have played an important role in the characterization of surjective isometries of various Banach spaces (see [FJ89] and [FJ03]), they are also interesting to be studied as a class of operators themselves. For certain Banach spaces, as for example $H^p(\Delta)$ (with $1 \leq p < \infty$, $p \neq 2$), $\mathrm{Lip}([0,1])$ and $\mathcal{C}(\Omega)$, with Ω a compact rigid space, it has been shown that hermitian operators are trivial, which means that they are real scalar multiples of the identity. Hermitian operators on $L^p(\mu)$ are known to be multipliers by real $L^\infty(\mu)$ functions. For further information we refer the reader to [FJ03, FJ08].

In the present paper we investigate the class of hermitian bounded linear operators on spaces of Lipschitz functions on a compact and 2-connected metric space and with values in a complex Banach space, following a scheme employed by Fleming and Jamison in [FJ80] in the characterization of the hermitian operators on $C(X, E)$. This approach relies on the existence of semi-inner products on such spaces which are compatible with the norm.

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We now review the definition of semi-inner product on a complex Banach space, as presented in [L61] and [L63]. Let E be a complex Banach space. A function $[\cdot,\cdot]_E\colon E\times E\to\mathbb{C}$ is called a *semi-inner product* if, for every $x,y,z\in E$ and $\lambda\in\mathbb{C}$, the following conditions hold:

(1) $[x+y,z]_E=[x,z]_E+[y,z]_E$,
(2) $[\lambda x,y]_E=\lambda[x,y]_E$,
(3) $[x,x]_E>0$ for $x\neq 0$,
(4) $|[x,y]_E|^2\le [x,x]_E[y,y]_E$.

A semi-inner product $[\cdot,\cdot]_E$ is said to be *compatible with the norm* $\|\cdot\|_E$ if $[x,x]_E=\|x\|_E^2$ for every $x\in E$. The existence of semi-inner products compatible with the norm follows from the Hahn–Banach Theorem, which guarantees the existence of duality maps $u\mapsto\varphi_u$ from E into E^* that satisfy $\|\varphi_u\|=1$ and $\varphi_u(u)=\|u\|_E$. Such a duality map yields a semi-inner product by defining $[u,v]_E=\varphi_v(u)$. Maps of this kind are not unique and so there are several semi-inner products compatible with the existing norm unless the unit ball of E is smooth. We denote the sets of bounded operators and hermitian bounded operators on E by $B(E)$ and $H(E)$, respectively. See [FJ03] for the above results.

A bounded operator T on E is *hermitian* if there exists a semi-inner product $[\cdot,\cdot]_E$ compatible with the norm such that $[Tx,x]_E\in\mathbb{R}$ for every $x\in E$. It is important to mention that if T is hermitian, then for every semi-inner product $[\cdot,\cdot]$ on E compatible with the norm, $[Tx,x]\in\mathbb{R}$ for every $x\in E$ (cf. [FJ03]).

Let (X,d) be a compact metric space and E a complex Banach space endowed with the norm $\|\cdot\|_E$. A function $f\colon X\to E$ is said to be *Lipschitz* if

$$L(f):=\sup_{x\neq y}\frac{\|f(x)-f(y)\|_E}{d(x,y)}<\infty.$$

The *Lipschitz space* $\mathrm{Lip}(X,E)$ is the Banach space of all E-valued Lipschitz functions f on X with the norm

$$\|f\|=\max\{L(f),\|f\|_\infty\},\quad\text{where}\quad \|f\|_\infty=\sup\{\|f(x)\|_E : x\in X\}.$$

A metric space is said to be 2-*connected* if it cannot be decomposed into two nonempty disjoint sets whose distance is greater than or equal to 2.

In this paper we start by embedding $\mathrm{Lip}(X,E)$ isometrically into a space of vector-valued continuous functions defined on a compact space. Then we construct a semi-inner product on $\mathrm{Lip}(X,E)$ compatible with the norm. This approach allows us to describe the bounded hermitian operators as multiplication operators via a hermitian operator on E. In particular we conclude that the space of all scalar-valued Lipschitz functions only supports trivial hermitian operators. These results yield the form of normal and adjoint abelian operators in this setting.

2. Hermitian operators on $\mathrm{Lip}(X,E)$**.** In this section we characterize the hermitian bounded operators on the spaces $\mathrm{Lip}(X,E)$ with X a compact and 2-connected metric space and E a complex Banach space with norm $\|\cdot\|_E$. We set $\widetilde{X} = (X\times X)\setminus\triangle$, with $\triangle = \{(x,x) : x\in X\}$. We also denote by E_1^* the unit ball of the dual space E^*. Then the Stone–Čech compactification $\beta(\widetilde{X}\times E_1^*)$ of $\widetilde{X}\times E_1^*$ is a compact space containing $\widetilde{X}\times E_1^*$ as a dense subspace.

For each $f\in \mathrm{Lip}(X,E)$, the bounded continuous mapping $\widetilde{f}\colon \widetilde{X}\times E_1^* \to \mathbb{C}$ given by

$$\widetilde{f}((x,y),\varphi) = \varphi\biggl(\frac{f(x)-f(y)}{d(x,y)}\biggr)$$

has a unique continuous extension $\beta(\widetilde{f})\colon \beta(\widetilde{X}\times E_1^*)\to\mathbb{C}$ such that

$$\|\beta(\widetilde{f})\|_\infty = \|\widetilde{f}\|_\infty.$$

We now consider the isometric embedding

$$\Gamma\colon \mathrm{Lip}(X,E)\to \mathrm{C}(X\cup\beta(\widetilde{X}\times E_1^*), E\oplus_\infty\mathbb{C})$$

given by

$$\begin{aligned} f \;\mapsto\; \Gamma(f)\colon X\cup\beta(\widetilde{X}\times E_1^*) \;&\mapsto\; E\oplus_\infty\mathbb{C},\\ x\in X \;&\mapsto\; (f(x),0),\\ \xi\in\beta(\widetilde{X}\times E_1^*) \;&\mapsto\; (0,\beta(\widetilde{f})(\xi)).\end{aligned}\tag{2.1}$$

Standard techniques show that Γ is a linear isometry. For each function g in $\mathrm{Lip}(X,E)$, we define

$$P_g = \{t\in X\cup\beta(\widetilde{X}\times E_1^*) : \|\Gamma(g)(t)\|_{E\oplus_\infty\mathbb{C}} = \|g\|\}.$$

We first choose on E a semi-inner product $[\cdot,\cdot]_E$ compatible with the norm. Then we define the semi-inner product $[\cdot,\cdot]_{E\oplus_\infty\mathbb{C}}$ by

$$[(u_0,\lambda_0),(u_1,\lambda_1)]_{E\oplus_\infty\mathbb{C}} = \begin{cases} [u_0,u_1]_E & \text{if } \|(u_1,\lambda_1)\|_{E\oplus_\infty\mathbb{C}} = \|u_1\|_E,\\ \lambda_0\overline{\lambda_1} & \text{if } \|(u_1,\lambda_1)\|_{E\oplus_\infty\mathbb{C}} \neq \|u_1\|_E.\end{cases}$$

It is compatible with the norm on $E\oplus_\infty\mathbb{C}$. As is easily seen, this semi-inner product induces the following semi-inner products:

$$[(u,0),(v,0)]_{E\oplus_\infty\mathbb{C}} = [u,v]_E, \qquad [(0,\lambda_0),(0,\lambda_1)]_{E\oplus_\infty\mathbb{C}} = \lambda_0\overline{\lambda_1},$$

on the component spaces $\{(u,0) : u\in E\}$ and $\{(0,\lambda) : \lambda\in\mathbb{C}\}$, respectively, compatible with the existing norms.

Let ψ be a choice function which selects, for each $g\in\mathrm{Lip}(X,E)$, an element $\psi(g)\in P_g$. We now define

$$[f,g]_\psi = [\Gamma(f)(\psi(g)),\Gamma(g)(\psi(g))]_{E\oplus_\infty\mathbb{C}} \qquad (f,g\in\mathrm{Lip}(X,E)).\tag{2.2}$$

This is a semi-inner product in $\mathrm{Lip}(X,E)$ compatible with the norm on $\mathrm{Lip}(X,E)$, since

$$[f,f]_\psi = [\Gamma(f)(\psi(f)),\Gamma(f)(\psi(f))]_{E\oplus_\infty\mathbb{C}} = \|\Gamma(f)(\psi(f))\|^2_{E\oplus_\infty\mathbb{C}} = \|f\|^2$$

for all $f\in\mathrm{Lip}(X,E)$. Given $v\in E$, the symbol $\mathbf{v}$ represents the constant function on X everywhere equal to v.

Lemma 2.1. *Let X be a compact metric space, E a complex Banach space and T a hermitian bounded operator on* $\mathrm{Lip}(X,E)$. *Then the function A from X into $B(E)$ given by*

$$A(x)(v) = T(\mathbf{v})(x) \qquad (x\in X,\, v\in E)$$

is Lipschitz on X and with values in $H(E)$.

Proof. Fix $x\in X$. Given $v\in E$, if $\psi_0\colon \mathrm{Lip}(X,E)\to X\cup\beta(\widetilde{X}\times E_1^*)$ is a choice function as in (2.2) with the added condition that $\psi(\mathbf{v})=x$, define the semi-inner product $[f,g]_{\psi_0}$ as in (2.2). Then we obtain

$$[A(x)(v),v]_E = [T(\mathbf{v})(x),\mathbf{v}(x)]_E = [T(\mathbf{v}),\mathbf{v}]_{\psi_0}\in\mathbb{R}.$$

This shows that $A(x)$ is hermitian. We now prove that A is Lipschitz on X. Given $x,y\in X$ with $x\neq y$, we have

$$\begin{aligned}\frac{\|A(x)-A(y)\|}{d(x,y)} &= \sup_{\|v\|_E=1}\frac{\|A(x)(v)-A(y)(v)\|_E}{d(x,y)}\\ &= \sup_{\|v\|_E=1}\frac{\|T(\mathbf{v})(x)-T(\mathbf{v})(y)\|_E}{d(x,y)} \le \sup_{\|v\|_E=1} L(T(\mathbf{v}))\le\|T\|. \ \blacksquare\end{aligned}$$

Given $f\in\mathrm{Lip}(X,E)$, we denote by $|f|$ the function $|f|(x)=\|f(x)\|_E$. Clearly, $|f|\in\mathrm{Lip}(X)$ and $L(|f|)\le L(f)$.

Proposition 2.2. *Let X be a compact metric space, E a complex Banach space and T a hermitian bounded operator on* $\mathrm{Lip}(X,E)$. *If $f\in$* $\mathrm{Lip}(X,E)$ *and $x_0\in X$ are such that $f(x_0)=0$, then $T(f)(x_0)=0$.*

Proof. Assume $T(f)(x_0)\neq 0$. Hence $f\neq 0$. We may suppose without loss of generality that $\|T(f)(x_0)\|_E=1$. Let h_1 and h_2 be the functions in $\mathrm{Lip}(X,E)$ defined by

$$\begin{aligned}h_1(t) &= (2\|f\|-|f|(t))T(f)(x_0)+if(t),\\ h_2(t) &= (2\|f\|-|f|(t))T(f)(x_0).\end{aligned}$$

An easy computation shows that $h_1(x_0)=h_2(x_0)=2\|f\|T(f)(x_0)$ and

$$L(h_k)\le 2L(f)\le 2\|f\| = \|h_k\|_\infty = \|h_k(x_0)\|_E$$

for $k=1,2$. Since $\|h_k(x_0)\|_E=\|h_k\|$ we have $x_0\in P_{h_k}$ for $k=1,2$.

A result in [G, Theorem 1, p. 437] states the existence of a semi-inner product $[\cdot,\cdot]_E$ on E compatible with the norm, and which satisfies

$$[u, 2\|f\|T(f)(x_0)]_E = 2\|f\|[u, T(f)(x_0)]_E$$

for all $u \in E$. We now select a choice function $\psi\colon \mathrm{Lip}(X,E) \to X \cup \beta(\widetilde{X} \times E_1^*)$ as in (2.2) with the additional condition that $\psi(h_1) = \psi(h_2) = x_0$, and we consider the semi-inner product $[\cdot,\cdot]_\psi$ on $\mathrm{Lip}(X,E)$ as defined in (2.2). Then

$$\begin{aligned}[T(h_1), h_1]_\psi &= [T(h_1)(x_0), h_1(x_0)]_E \\ &= [T(h_2)(x_0), h_2(x_0)]_E + [T(if)(x_0), h_2(x_0)]_E \\ &= [T(h_2), h_2]_\psi + 2\|f\| i [T(f)(x_0), T(f)(x_0)]_E \\ &= [T(h_2), h_2]_\psi + 2\|f\| i.\end{aligned}$$

Since T is hermitian, $[T(h_1), h_1]_\psi$ and $[T(h_2), h_2]_\psi$ are real numbers. Therefore $T(f)(x_0) = 0$. ∎

PROPOSITION 2.3. *Let X be a compact metric space, E a complex Banach space and T a bounded operator on* $\mathrm{Lip}(X,E)$. *If T is hermitian, then there exists a mapping $A \in \mathrm{Lip}(X, H(E))$ such that $T(f)(x) = A(x)(f(x))$ for every $f \in \mathrm{Lip}(X,E)$ and $x \in X$.*

Proof. Given $f \in \mathrm{Lip}(X,E)$ and $x \in X$, let $f_x = f - \mathbf{f(x)}$, where the "boldfaced" $\mathbf{f(x)}$ denotes the constant function on X everywhere equal to $f(x)$. Proposition 2.2 implies that $T(f)(x) = T(\mathbf{f(x)})(x) = A(x)(f(x))$. ∎

THEOREM 2.4. *Let X be a compact and 2-connected metric space, E a complex Banach space and $T\colon \mathrm{Lip}(X,E) \to \mathrm{Lip}(X,E)$ a bounded operator. Then T is hermitian if and only if there exists a hermitian bounded operator $A\colon E \to E$ such that $T(f)(x) = A(f(x))$ for every $f \in \mathrm{Lip}(X,E)$ and $x \in X$.*

Proof. An operator T on $\mathrm{Lip}(X,E)$ of the form described in the theorem is hermitian. Indeed, it is clear that T is linear and, for a fixed function $f \in \mathrm{Lip}(X,E)$, we have

$$[T(f), f]_\psi = [\Gamma(T(f))(\psi(f)), \Gamma(f)(\psi(f))]_{E \oplus_\infty \mathbb{C}},$$

where $\psi\colon \mathrm{Lip}(X,E) \to X \cup \beta(\widetilde{X} \times E_1^*)$ is a choice function as in (2.2). If $\|f\| = \|f\|_\infty$, we may take ψ such that also $\psi(f) = x \in X$, and then

$$[T(f), f]_\psi = [T(f)(x), f(x)]_E = [A(f(x)), f(x)]_E \in \mathbb{R}.$$

If $\|f\| \neq \|f\|_\infty$, we may find a sequence $\{(x_n, y_n)\}$ in $\widetilde{X}$ such that

$$\left\{ \frac{\|(f(x_n) - f(y_n))\|_E}{d(x_n, y_n)} \right\}$$

converges to $L(f) = \|f\|$ as $n \to \infty$. For every $n \in \mathbb{N}$, there exists $\varphi_n \in E_1^*$

such that
$$\varphi_n\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)=\frac{\|f(x_n)-f(y_n)\|_E}{d(x_n,y_n)}$$
and
$$\bigg[u,\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg]_E=\varphi_n(u)\varphi_n\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)$$
for all $u\in E$. Note that
$$\begin{aligned}&\beta(\widetilde{T(f)})((x_n,y_n),\varphi_n)\overline{\beta(\widetilde{f})((x_n,y_n),\varphi_n)}\\&\qquad=\varphi_n\bigg(\frac{T(f)(x_n)-T(f)(y_n)}{d(x_n,y_n)}\bigg)\overline{\varphi_n\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)}\\&\qquad=\varphi_n\bigg(A\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)\bigg)\overline{\varphi_n\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)}\\&\qquad=\bigg[A\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg),\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg]_E\in\mathbb{R}.\end{aligned}$$
For every $j\in\mathbb{N}$, let $F_j=\overline{\{((x_n,y_n),\varphi_n):n\geq j\}}$. Note that F_j is a closed subset of the compact set $\beta(\widetilde{X}\times E_1^*)$. Clearly, the family $\{F_j:j\in\mathbb{N}\}$ has the finite intersection property and therefore there exists $\xi\in\bigcap_{j=1}^{\infty}F_j$.

Let us assume that $\beta(\widetilde{T(f)})(\xi)\overline{\beta(\widetilde{f})(\xi)}\notin\mathbb{R}$. By the continuity of the function $\beta(\widetilde{T(f)})\overline{\beta(\widetilde{f})}$, it follows that the set $U:=(\beta(\widetilde{T(f)})\overline{\beta(\widetilde{f})})^{-1}(\mathbb{C}\backslash\mathbb{R})$ is open in $\beta(\widetilde{X}\times E_1^*)$. Since $\xi\in U$, there exists $j\in\mathbb{N}$ such that $((x_j,y_j),\varphi_j)\in U$. Hence $\beta(\widetilde{T(f)})((x_j,y_j),\varphi_j)\overline{\beta(\widetilde{f})((x_j,y_j),\varphi_j)}\in(\mathbb{C}\setminus\mathbb{R})\cap\mathbb{R}$, which is impossible, and this shows that $\beta(\widetilde{T(f)})(\xi)\overline{\beta(\widetilde{f})(\xi)}\in\mathbb{R}$.

Assume $\beta(\widetilde{f})(\xi)\neq\|f\|$ and denote $r=\big|\|f\|-\beta(\widetilde{f})(\xi)\big|/2$. Then there exists $m\in\mathbb{N}$ such that
$$\bigg|\varphi_n\bigg(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\bigg)-\|f\|\bigg|<r,\qquad\forall n\in\mathbb{N},\,n\geq m.$$
So $W:=\beta(\widetilde{f})^{-1}(\mathbb{C}\setminus\overline{D(\|f\|,r)})$ is open in $\beta(\widetilde{X}\times E_1^*)$ since the function $\beta(\widetilde{f})$ is continuous. Since $\xi\in W$, there exists $j_0\in\mathbb{N}$ with $j_0\geq m$ such that $((x_{j_0},y_{j_0}),\varphi_{j_0})\in W$. It follows that
$$r<\big|\|f\|-\beta(\widetilde{f})((x_{j_0},y_{j_0}),\varphi_{j_0})\big|=\bigg|\varphi_{j_0}\bigg(\frac{f(x_{j_0})-f(y_{j_0})}{d(x_{j_0},y_{j_0})}\bigg)-\|f\|\bigg|<r,$$
a contradiction. This proves that $\beta(\widetilde{f})(\xi)=\|f\|$. Therefore we may choose ψ such that $\psi(f)=\xi$ and so
$$[T(f),f]_\psi=\beta(\widetilde{T(f)})(\xi)\overline{\beta(\widetilde{f})(\xi)}\in\mathbb{R},$$
as we wanted. This completes the proof of the desired implication.

We now prove the reverse implication. If T is hermitian, Proposition 2.3 implies the existence of a Lipschitz mapping $B\colon X \to H(E)$ such that

$$T(f)(x) = B(x)(f(x)), \qquad \forall x \in X,\ \forall f \in \mathrm{Lip}(X, E).$$

If X reduces to a single point, the result is trivial. Otherwise, let a and x, distinct points in X, be such that $0 < d(x,a) < 2$ and $v \in E$ with $v \neq 0$. Take $\delta = d(x,a)/2$ and define $g\colon X \to E$ by

$$g(t) = \left(h_{x,\delta}(t) - h_{a,\delta}(t) + \frac{\sqrt{4-(\delta+1)^2}}{\delta+1} i\right) v \qquad (t \in X),$$

where $h_{x,\delta}$ and $h_{a,\delta}$ are the functions given by

$$h_{y,\delta}(t) = \max\left\{0, 1 - \frac{d(y,t)}{\delta}\right\}, \qquad \forall t \in X\ (y = x, a).$$

It is easy to check that $g \in \mathrm{Lip}(X,E)$ with $\|g\| = L(g) = \|v\|_E/\delta$ since

$$\|g\|_\infty = \frac{2\|v\|_E}{\delta+1} < \frac{\|v\|_E}{\delta} = L(g).$$

Take $\varphi \in E_1^*$ such that $\varphi(v) = \|v\|_E$ and observe that

$$\beta(\widetilde{g})((x,a),\varphi) = \widetilde{g}((x,a),\varphi) = \varphi\left(\frac{g(x)-g(a)}{d(x,a)}\right) = \frac{2\varphi(v)}{d(x,a)} = \frac{\|v\|_E}{\delta}.$$

Hence $((x,a),\varphi) \in P_g$. We now select a choice function $\psi\colon \mathrm{Lip}(X,E) \to X \cup \beta(\widetilde{X} \times E_1^*)$ of the form as at (2.2) such that $\psi(g) = ((x,a),\varphi)$. For each $w \in E$ there exists a functional $\varphi_w \in E_1^*$ such that $\varphi_w(w) = \|w\|_E$, and we set $\tau\colon E \to E_1^*$ to be such a choice function with $\tau(w) = \varphi_w$. We consider the semi-inner product on E given by $[u_0,u_1]_\tau = \tau(u_1)(u_0)\overline{\tau(u_1)(u_1)}$. Now we have the following relations:

$$\begin{aligned}
[Tg,g]_\psi &= \widetilde{T(g)}((x,a),\varphi)\overline{\widetilde{g}((x,a),\varphi)} \\
&= \varphi\left(\frac{T(g)(x)-T(g)(a)}{d(x,a)}\right)\overline{\varphi\left(\frac{g(x)-g(a)}{d(x,a)}\right)} \\
&= \varphi\left(\frac{T(g)(x)-T(g)(a)}{d(x,a)}\right)\cdot\overline{\frac{2\varphi(v)}{d(x,a)}} \\
&= \frac{2}{d(x,a)^2}\left(1 + \frac{\sqrt{4-(\delta+1)^2}}{\delta+1} i\right)\varphi\big((B(x)-B(a))(v)\big)\overline{\varphi(v)} \\
&\quad + \frac{4}{d(x,a)^2}\varphi(B(a)(v))\overline{\varphi(v)}.
\end{aligned}$$

Taking into account that $[Tg,g]_\psi$,

$$\varphi\big((B(x)-B(a))(v)\big)\overline{\varphi(v)} = [(B(x)-B(a))(v), v]_\tau$$

and

$$\varphi(B(a)(v))\overline{\varphi(v)} = [B(a)(v), v]_\tau$$

are real numbers, we get $\varphi((B(x) - B(a))(v))\overline{\varphi(v)} = 0$. Therefore we conclude that for every $v \in E$, $[(B(x) - B(a))(v), v]_\tau = 0$. This implies that $B(x) = B(a)$ (cf. [L61, Theorem 5, p. 33]).

Now, given $a \in X$, take $K = \{y \in X : B(y) = B(a)\}$. If $X \setminus K$ is nonempty, then given $x \in X \setminus K$ and $y \in K$, we have $d(x, y) \geq 2$. This implies that $d(X \setminus K, K) \geq 2$, contradicting the 2-connectedness assumption on X. Therefore $X = K$ and $B(x) = B(a)$ for every $x \in X$. Denoting $A = B(a)$, we conclude that $A \in H(E)$ and $T(f)(x) = A(f(x))$ for all $x \in X$ and $f \in \mathrm{Lip}(X, E)$. This completes the proof of Theorem 2.4. ∎

Remark 2.5. A shorter proof of the sufficient condition of Theorem 2.4 can be given by using the definition of hermitian operator given by Vidav [V]. Let T be an operator on $\mathrm{Lip}(X, E)$ as in the statement of the theorem. We show that $\exp(itT)$ is an isometry for every $t \in \mathbb{R}$. We have

$$\exp(itT)f = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} T^n f = \left(\sum_{n=0}^{\infty} \frac{(it)^n}{n!} A^n\right) f = \exp(itA)f$$

for all $f \in \mathrm{Lip}(X, E)$. Since A is hermitian, $\exp(itA)$ is an isometry for every $t \in \mathbb{R}$. This implies that $\|\exp(itT)f\| = \|\exp(itA)f\| = \|f\|$. Therefore T is hermitian by [BD, Corollary 13, p. 55].

Taking into account that the metric space X is compact and that the 2-connected components of X are open sets in X, the next corollary follows straightforwardly from Theorem 2.4 and Proposition 2.3.

Corollary 2.6. *Let X be a compact metric space, E a complex Banach space and $T\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ a hermitian bounded operator. Denote by $X_1, \dots, X_m$ the 2-connected components of X. Then there exist m hermitian bounded operators $A_1, \dots, A_m\colon E \to E$ such that*

$$T(f)(x) = \sum_{j=1}^{m} A_j(\chi_j(f)(x)), \qquad \forall x \in X, \forall f \in \mathrm{Lip}(X, E),$$

where, for each $j \in \{1, \dots, m\}$, $\chi_j(f)(x) = f(x)$ if $x \in X_j$ and $\chi_j(f)(x) = 0$ otherwise.

Proof. Let $j \in \{1, \dots, m\}$. For $f \in \mathrm{Lip}(X_j, E)$, define $\widehat{f} \in \mathrm{Lip}(X, E)$ by

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in X_j, \\ 0 & \text{if } x \notin X_j. \end{cases}$$

It is clear that $\|\widehat{f}\| = \|f\|$. Consider $T_j\colon \mathrm{Lip}(X_j, E) \to \mathrm{Lip}(X_j, E)$ given by

$$T_j(f) = T(\widehat{f})|_{X_j}, \qquad \forall f \in \mathrm{Lip}(X_j, E).$$

It is straightforward to see that T_j is linear and bounded. Now define in $\mathrm{Lip}(X_j, E)$ the following semi-inner product compatible with the norm:

$$[f, g]_{\mathrm{Lip}(X_j,E)} = [\widehat{f}, \widehat{g}]_\psi, \qquad \forall f, g \in \mathrm{Lip}(X_j, E).$$

Taking into account Proposition 2.3, it follows that $\widehat{T_j(f)} = \widehat{T(\widehat{f})|_{X_j}} = T(\widehat{f})$ for all $f \in \mathrm{Lip}(X_j, E)$. Then

$$[T_j(f), f]_{\mathrm{Lip}(X_j,E)} = [T(\widehat{f}), \widehat{f}]_\psi \in \mathbb{R}, \qquad \forall f \in \mathrm{Lip}(X_j, E).$$

Therefore T_j is hermitian. By Theorem 2.4 there exist m hermitian bounded operators $A_1, \ldots, A_m \colon E \to E$ such that

$$T_j(f)(x) = A_j(f(x)), \qquad \forall x \in X_j,\ \forall f \in \mathrm{Lip}(X_j, E),\ \forall j \in \{1, \ldots, m\}.$$

On the other hand, Proposition 2.3 gives us a mapping $A \in \mathrm{Lip}(X, H(E))$ such that $T(f)(x) = A(x)(f(x))$ for every $f \in \mathrm{Lip}(X, E)$ and $x \in X$. Then, for any $f \in \mathrm{Lip}(X, E)$, $j \in \{1, \ldots, m\}$ and $x \in X_j$, we have

$$\begin{aligned} T(f)(x) &= A(x)(f(x)) = A(x)(\widehat{f|_{X_j}}(x)) = T(\widehat{f|_{X_j}})(x) \\ &= T_j(f|_{X_j})(x) = A_j(f(x)) = A_j(\chi_j(f)(x)) = \sum_{k=1}^{m} A_k(\chi_k(f)(x)). \ \blacksquare \end{aligned}$$

We now state the result for the scalar case, which follows as a particular case of Theorem 2.4.

Corollary 2.7. *Let X be a compact and 2-connected metric space and T a bounded operator on* $\mathrm{Lip}(X)$. *Then T is hermitian if and only if T is a real multiple of the identity operator on* $\mathrm{Lip}(X)$.

3. Some remarks on adjoint abelian and normal operators on $\mathrm{Lip}(X, E)$. We start with the definitions of adjoint abelian and normal operators as presented in [S] and in [M].

Definition 3.1. Let E be a complex Banach space and let $T \colon E \to E$ be a bounded operator.

(i) T is *adjoint abelian* if there exists a semi-inner product $[\cdot, \cdot]$ compatible with the norm of E such that $[Tx, y] = [x, Ty]$ for all $x, y \in E$.
(ii) T is *normal* if there exist two hermitian and commuting operators T_0 and T_1 on E such that $T = T_0 + iT_1$.

The results presented in the previous section imply the following.

Theorem 3.2. *Let X be a compact and 2-connected metric space, E a complex Banach space and T a bounded operator on* $\mathrm{Lip}(X, E)$.

(i) *If T is an adjoint abelian hermitian operator, then there exist an adjoint abelian hermitian operator A on E such that $T(f) = Af$ for every $f \in$* $\mathrm{Lip}(X, E)$.

(ii) *T is normal if and only if there exist commuting hermitian operators A and B on E such that $T(f) = Af + iBf$ for all $f \in \mathrm{Lip}(X, E)$.*

Proof. The first statement follows easily from Theorem 2.4. Indeed, since T is hermitian there exists a hermitian operator $A\colon E \to E$ such that $T(f) = Af$ for all $f \in \mathrm{Lip}(X, E)$. Let $[\cdot, \cdot]$ be a semi-inner product on $\mathrm{Lip}(X, E)$ compatible with the norm of $\mathrm{Lip}(X, E)$ such that $[T(f), g] = [f, T(g)]$ for every $f, g \in \mathrm{Lip}(X, E)$. If $u, v \in E$, we define a semi-inner product on E by $[u, v]_E = [\mathbf{u}, \mathbf{v}]$. It is easy to check that it is compatible with the norm on E. Now we consider

$$[T(\mathbf{u}), \mathbf{v}] = [A\mathbf{u}, \mathbf{v}] = [\mathbf{Au}, \mathbf{v}] = [Au, v]_E,$$
$$[\mathbf{u}, T(\mathbf{v})] = [\mathbf{u}, A\mathbf{v}] = [\mathbf{u}, \mathbf{Av}] = [u, Av]_E.$$

This implies that A is adjoint abelian on E with respect to $[\cdot, \cdot]_E$.

We now prove the second statement. If T is normal, then there exist two commuting and hermitian operators T_0 and T_1 on $\mathrm{Lip}(X, E)$ such that $T = T_0 + iT_1$. An application of Theorem 2.4 implies the existence of operators A_0 and A_1 in $H(E)$ such that $T_0(f) = A_0 f$ and $T_1(f) = A_1 f$. Since $T_0 T_1 = T_1 T_0$, we see that given a constant function everywhere equal to $v \in E$, we have $A_0 A_1 v = A_1 A_0 v$. Hence A_0 and A_1 commute and this completes the proof of the implication. The converse is immediate. ∎

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