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Squaring a reverse AM-GM inequality

by

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Dedicated to Harald Wimmer with affection

Abstract. Let A, B be positive operators on a Hilbert space with $0 < m \le A, B \le M$. Then for every unital positive linear map Φ,

$$\Phi^2((A+B)/2) \le K^2(h)\Phi^2(A \sharp B),$$

and

$$\Phi^2((A+B)/2) \le K^2(h)(\Phi(A) \sharp \Phi(B))^2,$$

where $A \sharp B$ is the geometric mean and $K(h) = (h+1)^2/(4h)$ with $h = M/m$.

1. Introduction. It is well known that for two general positive operators A, B,

$$A \ge B \nRightarrow A^2 \ge B^2.$$

This simple observation already reveals the subtleties of operator inequalities. It is thus interesting to know for what kind of operator inequalities, when they are squared, the inequality relation is preserved. In [L], we have shown that the operator Kantorovich inequality can be squared, more precisely:

THEOREM 1.1 ([L]). *Let $0 < m \le A \le M$. Then for every positive unital linear map Φ,*

$$\Phi^2(A^{-1}) \le K^2(h)\Phi(A)^{-2}, \tag{1.1}$$

where $K(h) = (h+1)^2/(4h)$ with $h = M/m$.

As is customary, we reserve M, m for scalars. Other capital letters are used to denote general elements of the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Also, we identify a scalar with the unit multiplied by this scalar. The quantity $K(h) = (h+1)^2/(4h)$ with $h = M/m$ is called the *Kantorovich constant*.

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In this article, the inequality between operators is in the sense of Loewner partial order, that is, $T \geq S$ (the same as $S \leq T$) means $T - S$ is positive. If $T \geq 0$, then T has a unique square root $T^{1/2}$ that is positive. The operator norm is denoted by $\|\cdot\|$. Finally, if $T, S \in \mathcal{B}(\mathcal{H})$, we use $T \oplus S$ to denote the 2×2 operator matrix $\left[\begin{smallmatrix} T & 0 \\ 0 & S \end{smallmatrix}\right]$, regarded as an operator on $\mathcal{H} \oplus \mathcal{H}$.

For $A, B > 0$, the *geometric mean* $A \sharp B$ is defined by

$$A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

It can be easily verified that the geometric mean is the unique positive solution to the Riccati equation $XA^{-1}X = B$, and so $A \sharp B = B \sharp A$. One motivation for such a notion is of course the *operator AM-GM inequality*

$$\frac{A+B}{2} \geq A \sharp B. \tag{1.2}$$

1.1. Preliminaries. Can the AM-GM inequality (1.2) be squared? The answer is no and this is the content of the next proposition.

PROPOSITION 1.2. *There are examples that*

$$\left(\frac{A+B}{2}\right)^2 \geq (A \sharp B)^2$$

fails for some $A, B > 0$.

Proof. We can easily find examples where $S \geq T > 0$, but $S^2 \geq T^2$ fails. So we wish to find $A, B > 0$ such that

$$S = \frac{A+B}{2}, \tag{1.3}$$

$$T = A \sharp B. \tag{1.4}$$

From (1.4), we get $B = TA^{-1}T$; substituting it into (1.3), we have

$$A + TA^{-1}T = 2S, \quad \text{with } A \text{ unknown.} \tag{1.5}$$

The positive solution to equation (1.5) is a much studied problem; we know from [E, Theorem 11] that (1.5) has a solution $A > 0$ if and only if the spectral radius $\rho((2S)^{-1/2}T(2S)^{-1/2})$ is at most $1/2$, i.e., $\rho(TS^{-1}) \leq 1$, which is exactly the condition $S \geq T > 0$. Thus (1.5) is consistent, so the desired $A, B > 0$ exist. ∎

The main result of this article is the assertion that a reverse version of the operator AM-GM inequality can be squared, which we will present in the next section. In Section 3, we further make use of our idea to square a recent result concerning a reverse inequality for the geometric mean of n operators. In Section 4, we formulate some open problems for future investigation.

2. Main results

THEOREM 2.1. *Let $0 < m \le A, B \le M$. Then for every positive unital linear map Φ,*

$$\Phi^2((A+B)/2) \le K^2(h)\Phi^2(A \sharp B) \tag{2.1}$$

and

$$\Phi^2((A+B)/2) \le K^2(h)(\Phi(A) \sharp \Phi(B))^2. \tag{2.2}$$

The line of proof is similar to the one presented in [L]. A key lemma is the following norm inequality.

LEMMA 2.2 ([BK]). *Let $A, B \ge 0$. Then*

$$\|AB\| \le \tfrac{1}{4}\|A+B\|^2. \tag{2.3}$$

Proof of Theorem 2.1. The inequality (2.1) is equivalent to

$$\|\Phi((A+B)/2)\Phi^{-1}(A \sharp B)\| \le K(h). \tag{2.4}$$

By Lemma 2.2, (2.1) is true if

$$\Phi((A+B)/2) + Mm\Phi^{-1}(A \sharp B) \le M + m. \tag{2.5}$$

The well known Choi inequality (see [B, p. 41]) says that

$$\Phi(T^{-1}) \ge \Phi^{-1}(T) \quad \text{for any } T > 0,$$

so (2.1) would follow if

$$\Phi((A+B)/2) + Mm\Phi((A \sharp B)^{-1}) \le M + m. \tag{2.6}$$

Indeed, we shall show a stronger inequality than (2.6). As (see [L, (2.3)])

$$\Phi(A) + Mm\Phi(A^{-1}) \le M + m,$$
$$\Phi(B) + Mm\Phi(B^{-1}) \le M + m,$$

summing up, we get

$$\Phi((A+B)/2) + Mm\Phi((A^{-1}+B^{-1})/2) \le M + m. \tag{2.7}$$

Note that (2.7) is tighter than (2.6), since $(A^{-1}+B^{-1})/2 \ge A^{-1} \sharp B^{-1} \ge (A \sharp B)^{-1}$ and Φ is order preserving (Φ is linear!). This proves (2.1). The proof of (2.2) is similar; we omit the details. ∎

It seems a direct proof of (2.6) is not apparent. My original attempt started with the scalar case, as in the proof of other operator inequalities (see for example [F, KM, T]): Suppose $0 < m \le a, b \le M$; we want to show

$$\frac{a+b}{2} + \frac{Mm}{\sqrt{ab}} \le M + m. \tag{2.8}$$

Without loss of generality, assume $m = 1$, $1 \le a \le b \le h$; then (2.8) becomes

$$\frac{a+b}{2} + \frac{h}{\sqrt{ab}} \le h + 1. \tag{2.9}$$

Observe that $(h-b)(1-(ab)^{-1/2})$ and the integral from $t=a$ to $t=b$ of $(1-(at)^{-1/2})$ are both non-negative. Then (2.9) follows from

$$(h-b)(1-(ab)^{-1/2})+\frac{1}{2}\int_a^b (1-(at)^{-1/2})\,dt \geq 0.$$

The main obstacle is that (2.6) does not reduce to (2.8) via a usual approach.

For $A, B > 0$, we know [B, p. 107] that for every positive unital linear map Φ,

$$\Phi(A) \sharp \Phi(B) \geq \Phi(A \sharp B). \tag{2.10}$$

It is tempting to consider whether (2.10) can be squared. If this were the case, then (2.1) would be stronger than (2.2). However, the following example refutes the conjecture.

Example 2.3. Consider $\widehat{A} = A \oplus B$, $\widehat{B} = B \oplus A$ with $A, B > 0$. Let the map Ψ be defined by

$$\Psi(A_1 \oplus A_2) = \left(\frac{A_1 + A_2}{2}\right) \oplus \left(\frac{A_1 + A_2}{2}\right).$$

Clearly, Ψ is a positive unital linear map. Then

$$(\Psi(\widehat{A}) \sharp \Psi(\widehat{B}))^2 \geq \Psi^2(\widehat{A} \sharp \widehat{B})$$

would imply $\left(\frac{A+B}{2}\right)^2 \geq (A \sharp B)^2$, which leads to a contradiction in view of Proposition 1.2.

To end this section, we remark that the Choi inequality cannot be squared either ([1]). This is the content of the next example.

Example 2.4. Consider the positive unital linear map Ψ defined by

$$\Psi\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = A_1.$$

We can easily find examples where $S > T > 0$, but $S^2 \geq T^2$ (and hence $S^{-2} \leq T^{-2}$) fails. Let $C = S - T$ (so $C > 0$). Given any $B > 0$, it is easy to check $\left[\begin{smallmatrix} S & B\sharp C \\ B\sharp C & B \end{smallmatrix}\right] > 0$. Now note that

$$\Psi^{-1}\left(\begin{bmatrix} S & B \sharp C \\ B \sharp C & B \end{bmatrix}\right) = S^{-1}$$

and

$$\Psi\left(\begin{bmatrix} S & B \sharp C \\ B \sharp C & B \end{bmatrix}^{-1}\right) = \big(S - (B \sharp C)B^{-1}(B \sharp C)\big)^{-1} = (S-C)^{-1} = T^{-1}.$$

([1]) This fact has already been observed by J. Bourin.

3. An application. In [FFNPS], J. I. Fujii et al. showed a reverse weighted AM-GM inequality for n operators. The weighted geometric mean therein is due to Lawson and Lim [LL]. When the weights are equal, Lawson–Lim's geometric mean reduces to the so called Ando–Li–Mathias geometric mean [ALM]. For notational simplicity, we consider the latter. We denote the Ando–Li–Mathias geometric mean for $A_1, \dots, A_n > 0$ by $G(A_1, \dots, A_n)$. Unfortunately, there is no explicit formula for $G(A_1, \dots, A_n)$ in terms of $A_1, \dots, A_n$ when $n \geq 3$. However, for our purpose, we only need two basic properties of such a mean:

$$G(A_1^{-1}, \dots, A_n^{-1}) = G^{-1}(A_1, \dots, A_n), \qquad G(A_1, \dots, A_n) \leq \frac{A_1 + \cdots + A_n}{n}.$$

For more details, we refer to [ALM, LL].

We start by quoting J. I. Fujii et al.'s result.

THEOREM 3.1 ([FFNPS]). *Let $0 < m \leq A_i \leq M$ for $i = 1, \dots, n$. Then*

$$\frac{A_1 + \cdots + A_n}{n} \leq K(h) G(A_1, \dots, A_n) \tag{3.1}$$

and

$$\frac{A_1 + \cdots + A_n}{n} \leq S^2(h) G(A_1, \dots, A_n), \tag{3.2}$$

where

$$S(h) = \frac{h^{1/(h-1)}}{e \log h^{1/(h-1)}}$$

(*with the understanding $S(1) = \lim_{h \to 1} S(h) = 1$*) *is the so called Specht ratio and $h = M/m$.*

Indeed, it can be shown that

$$S(h) \leq K(h) \leq S^2(h), \qquad h \geq 1. \tag{3.3}$$

We will give a heuristic proof to (3.3) in Section 4. From (3.3), we find that (3.1) is stronger than (3.2).

In this section, using the idea we presented in the previous section, we show that (3.1) can be squared as well. The reader should note that an analogous result can also be stated for Lawson–Lim's geometric mean.

THEOREM 3.2. *Let $0 < m \leq A_i \leq M$ for $i = 1, \dots, n$. Then*

$$\left(\frac{A_1 + \cdots + A_n}{n} \right)^2 \leq K^2(h) G^2(A_1, \dots, A_n). \tag{3.4}$$

Proof. Similar to the proof of Theorem 2.1, it suffices to show

$$\frac{A_1 + \cdots + A_n}{n} + Mm G^{-1}(A_1, \dots, A_n) \leq M + m.$$

As

$$A_i + MmA_i^{-1} \le M + m, \quad i = 1, \dots, n,$$

we have

$$\frac{A_1 + \cdots + A_n}{n} + Mm\frac{A_1^{-1} + \cdots + A_n^{-1}}{n} \le M + m.$$

Thus

$$\begin{aligned}\frac{A_1 + \cdots + A_n}{n} &+ MmG^{-1}(A_1, \dots, A_n) \\ &= \frac{A_1 + \cdots + A_n}{n} + MmG(A_1^{-1}, \dots, A_n^{-1}) \le M + m.\end{aligned}$$

The desired inequality follows. ∎

4. Open problems

4.1. Optimal coefficient. The reverse AM-GM inequality for scalars was first established by Specht [S]: For $0 < m \le a_i \le M$, $i = 1, \dots, n$,

$$\frac{a_1 + \cdots + a_n}{n} \le S(h)(a_1 \cdots a_n)^{1/n}, \tag{4.1}$$

and the Specht ratio $S(h)$ in (4.1) is the optimal constant.

Comparing (3.1) and (4.1), we infer that $S(h) \le K(h)$. This proves the first inequality of (3.3).

Tominaga [T] showed that in the case $n = 2$, (4.1) can be extended to the operator version.

PROPOSITION 4.1 ([T]). *Let $0 < m \le A, B \le M$. Then*

$$\frac{A + B}{2} \le S(h)(A \sharp B). \tag{4.2}$$

Thus, under the condition of Proposition 4.1 and for any $x \in \mathcal{H}$,

$$\begin{aligned} S(h)\langle (A \sharp B)x, x\rangle \ge \left\langle \frac{A + B}{2}x, x \right\rangle &= \frac{1}{2}(\langle Ax, x\rangle + \langle Bx, x\rangle) \\ &\ge \sqrt{\langle Ax, x\rangle \langle Bx, x\rangle},\end{aligned}$$

i.e.,

$$\langle Ax, x\rangle \langle Bx, x\rangle \le S^2(h)(\langle (A \sharp B)x, x\rangle)^2. \tag{4.3}$$

The classical Kantorovich inequality says that for $0 < m \le A \le M$ and any $x \in \mathcal{H}$,

$$\langle Ax, x\rangle \langle A^{-1}x, x\rangle \le K(h). \tag{4.4}$$

As we know from [FINS], the following inequality is equivalent to (4.4):

$$\langle Ax, x\rangle \langle Bx, x\rangle \le K(h)(\langle (A \sharp B)x, x\rangle)^2, \tag{4.5}$$

where $0 < m \le A, B \le M$ and $x \in \mathcal{H}$. The Kantorovich constant $K(h)$ in (4.5) is the optimal constant.

Comparing (4.3) and (4.5), we infer that $K(h) \le S^2(h)$. This proves the second inequality of (3.3). An analytical proof of (3.3) is quite tedious, though not impossible.

Considering optimality of the coefficient in these inequalities, we make the following conjecture.

CONJECTURE 4.2. The Kantorovich constant $K(h)$ can be replaced by the Specht ratio $S(h)$ in (2.1), (2.2) and (3.4).

Our study reveals an interesting phenomenon: the reverse versions of some classical operator inequalities can be squared. We are not sure whether this is a universal phenomenon.

4.2. Higher power. And what about higher powers? This is what we consider in this subsection.

Note that for $0 < m \le A \le M$, we have

$$M^2m^2A^{-2} + A^2 \le M^2 + m^2,$$

and hence

$$M^2m^2\Phi(A^{-2}) + \Phi(A^2) \le M^2 + m^2. \tag{4.6}$$

Kadison's inequality (see [B, p. 39]) says

$$\Phi^2(T) \le \Phi(T^2)$$

for every unital positive linear map Φ and $T > 0$. So it follows from (4.6) that

$$M^2m^2\Phi^2(A^{-1}) + \Phi^2(A) \le M^2 + m^2. \tag{4.7}$$

Following the line of proof of Theorem 2.1 and noting that

$$K(h^2) = \frac{(M^2 + m^2)^2}{4M^2m^2},$$

we are led to the next result.

THEOREM 4.3. *Let $0 < m \le A \le M$. Then*

$$\Phi^4(A^{-1}) \le K^2(h^2)\Phi(A)^{-4}. \tag{4.8}$$

It is clear that

$$K(h^2) \ge K^2(h).$$

Thus, it is natural to ask whether (4.8) can be strengthened, in other words, whether (1.1) can be further squared. The same question can also be raised for (2.1), (2.2) and (3.4). As I do not have a satisfactory answer to these questions for the time being, I leave them as a topic for further research.

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