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Weyl numbers versus Z-Weyl numbers

by

BERND CARL (Jena), ANDREAS DEFANT (Oldenburg)
and DORIS PLANER (Jena)

Dedicated to the eightieth birthday of Albrecht Pietsch

Abstract. Given an infinite-dimensional Banach space Z (substituting the Hilbert space ℓ_2), the s-number sequence of Z-Weyl numbers is generated by the approximation numbers according to the pattern of the classical Weyl numbers. We compare Weyl numbers with Z-Weyl numbers—a problem originally posed by A. Pietsch. We recover a result of Hinrichs and the first author showing that the Weyl numbers are in a sense minimal. This emphasizes the outstanding role of Weyl numbers within the theory of eigenvalue distribution of operators between Banach spaces.

1. Introduction. In [16] Albrecht Pietsch developed an axiomatic approach to s-numbers of operators in Banach spaces. Since then the amount of literature dealing with inequalities between eigenvalues and s-numbers, as well as inequalities between different types of s-numbers, is constantly increasing.

The main aim of this article is to compare the s-number sequence $x = (x_n)$ of Weyl numbers with the s-number sequences of so-called Z-Weyl numbers $x(\cdot|Z) = (x_n(\cdot|Z))$, where Z is an infinite-dimensional Banach space. Following Pietsch the nth *Weyl number* of a (bounded and linear) operator T between two Banach spaces X and Y is given by

$$x_n(T) := \sup a_n(TA),$$

where the supremum is taken over all operators A from the Hilbert space ℓ_2 into X of norm ≤ 1, and $a_n(TA)$ stands for the nth approximation number of the composition TA. The definition of Z-Weyl numbers is similar—replace the Hilbert space ℓ_2 by an infinite-dimensional Banach space Z.

Today the notion of Weyl numbers is fundamental within the highly developed theory of eigenvalue distribution of power compact operators in

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Banach spaces, and in particular, within the theory of eigenvalue distribution of integral operators acting in function spaces. The standard references are the monographs of König [13] and Pietsch [19].

The first systematic studies of the more general notion of Z-Weyl numbers were undertaken in [3, 22, 23], and this article continues these works. After some preliminaries in Section 2, we show in Section 3 how to estimate Weyl numbers by Z-Weyl numbers under appropriate geometrical assumptions on Z; the main result is Theorem 3.9. Section 4 deals with converse estimates (i.e., estimates of Z-Weyl numbers by Weyl numbers), and in this context our main contributions are collected in Theorem 4.2. The final Section 5 then shows that Weyl numbers have a certain minimality property. In [4] the first author and Hinrichs proved that every multiplicative s-number sequence (s_n) which is uniformly dominated by the Weyl numbers (x_n) (i.e., $s_n(T) \le C x_n(T)$ for all integers n, all operators T between Banach spaces, and some absolute constant $C > 0$) conversely satisfies in the case $C = 1$ the estimate

$$x_{2n-1}(T) \le e\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n} \tag{1.1}$$

for all n and T. This result emphasizes the unique role of Weyl numbers within the theory of eigenvalue distribution of operators in Banach spaces. The main result (Theorem 5.3) is a variant of (1.1) for Z-Weyl numbers.

2. s-numbers and basic tools. As already noted, the axiomatic approach to so-called *s-number sequences* for operators between Banach spaces is due to Pietsch (cf. [16, 17, 19]). Denote the set of all bounded linear operators $T : X \to Y$ between two Banach spaces X and Y by $\mathscr{L}(X, Y)$, and the class of all bounded linear operators between arbitrary Banach spaces by $\mathscr{L}$. A map $s = (s_n)_{n=1}^{\infty}$ which assigns to every $T \in \mathscr{L}$ a sequence $(s_n(T))_{n=1}^{\infty}$ is called an *s-number sequence* if the following conditions are satisfied:

(1) $\|T\| = s_1(T) \ge s_2(T) \ge \cdots \ge 0$ for all $T \in \mathscr{L}$.
(2) $s_{m+n-1}(S + T) \le s_m(S) + s_n(T)$ for all $S, T \in \mathscr{L}(X, Y)$.
(3) $s_n(BTA) \le \|B\| s_n(T) \|A\|$ for all $A \in \mathscr{L}(X_0, X)$, $T \in \mathscr{L}(X, Y)$, and $B \in \mathscr{L}(Y, Y_0)$.
(4) $s_n(T) = 0$ for all $T \in \mathscr{L}$ with $\operatorname{rank}(T) < n$.
(5) $s_n(I_n) = 1$ for the identity map $I_n : \ell_2^n \to \ell_2^n$ on ℓ_2^n.

We call $s_n(T)$ the nth *s-number* of the operator T. Moreover, an s-number sequence (s_n) is said to be *multiplicative* whenever for all $S \in \mathscr{L}(X, Y)$ and $T \in \mathscr{L}(Y, Z)$ we have

$$s_{m+n-1}(TS) \le s_n(T) s_m(S).$$

Let us recall the most important examples; here, given a (closed) subspace M of a Banach space X, we denote by J_M^X the canonical embedding from M into X, and by Q_M^X the quotient map from X onto X/M. For $T \in \mathscr{L}(X,Y)$ the nth *approximation number* is defined by

$$a_n(T) := \inf\{\|T - A\| : A \in \mathscr{L}(X,Y),\ \operatorname{rank}(A) < n\},$$

the nth *Gelfand number* by

$$c_n(T) := \inf\{\|TJ_M^X\| : M \subset X,\ \operatorname{codim}(M) < n\},$$

the nth *Kolmogorov number* by

$$d_n(T) := \inf\{\|Q_N^Y T\| : N \subset Y,\ \dim(N) < n\},$$

the nth *Weyl number* by

$$x_n(T) := \sup\{a_n(TA) : A \in \mathscr{L}(\ell_2, X),\ \|A\| \le 1\},$$

and finally the nth *Hilbert number* by

$$h_n(T) := \sup\{a_n(BTA) : A \in \mathscr{L}(\ell_2, X),\ B \in \mathscr{L}(Y, \ell_2),\ \|A\|, \|B\| \le 1\}.$$

It is well-known that the approximation numbers (a_n) form the largest s-number sequence, and the Hilbert numbers (h_n) form the smallest one. Moreover, the s-number sequences given by the approximation, Gelfand, Kolmogorov, and Weyl numbers are all multiplicative; but the Hilbert numbers are not. If not credited differently, all needed information on s-number sequences can be found in the monographs [13, 17, 19].

The following lemma is taken from [4, Lemma 1.1] (see also [7, 1, 17]), and it will be crucial in what follows. It relates Gelfand numbers to Hilbert numbers and turns out to be an important tool within the study of optimal inequalities between eigenvalues and s-numbers.

Lemma 2.1. *Let $s = (s_n)$ be an s-number sequence. Then for every T in $\mathscr{L}(X,Y)$ the following inequality holds:*

$$\prod_{k=1}^{n} c_k(T) \le \sup\Big\{\prod_{k=1}^{n} s_k(BTA : \ell_2^n \to \ell_2^n) : \|A : \ell_1^n \to X\|, \|B : Y \to \ell_\infty^n\| \le 1\Big\}.$$

Note that, since the Hilbert numbers form the smallest s-number sequence, this result in fact estimates Gelfand numbers (c_n) by Hilbert numbers (h_n).

Furthermore, to estimate approximation numbers by Gelfand and Kolmogorov numbers we need certain geometrical parameters given in [8, Section 2.4]. For a fixed triple of Banach spaces $(E; X, Y)$ with E a (closed) subspace of X, the *extension constant* $p(E; X, Y)$ is defined by

$$p(E; X, Y) := \inf\big\{\rho \ge 0 : \forall T \in \mathscr{L}(E,Y)\ \exists \widetilde{T} \in \mathscr{L}(X,Y) \\ \text{with } \widetilde{T}_{|X} = T,\ \|\widetilde{T}\| \le \rho\|T\|\big\}.$$

Clearly, $p(E;X,Y)\geq 1$ (provided $E,X,Y\neq\{0\}$). But $p(E;X,Y)$ in general need not be finite. If Y is a Banach space with the metric extension property, then $p(E;X,Y)=1$ for any Banach space X and any subspace $E\subset X$. Moreover, $p(E;H,Y)=1$ for every Hilbert space H, every subspace $E\subset H$, and every Banach space Y.

Given a pair of Banach spaces (X,Y) and $n\in\mathbb{N}$, and letting E vary in the class of all subspaces $E\subset X$ of codimension $n-1$, the nth *extension constant* $p_n(X,Y)$ is defined by

$$p_n(X,Y):=\sup\{p(E;X,Y): E\subset X,\ \operatorname{codim}(E)=n-1\}.$$

We recall that for each n,

$$p_n(X,Y)\leq 1+\sqrt{n-1}\leq\sqrt{2n}; \tag{2.1}$$

see for example [8, (2.4.10)] or [17]. Similarly, we now introduce geometrical parameters which are determined by lifting properties of the underlying pair of Banach spaces (X,Y). An operator T mapping a Banach space X into a quotient space Y/F of a Banach space Y, is said to possess a *lifting to* Y if there exists a (linear and bounded) operator $\widetilde{T}$ such that $T=Q_F^Y\widetilde{T}$, where Q_F^Y denotes the quotient map of Y onto Y/F. For a fixed triple of Banach spaces $(X,Y;F)$ with F a (closed) subspace of Y, the *lifting constant* $q(X,Y;F)$ is given by

$$\begin{aligned}q(X,Y;F):=\inf\bigl\{\rho\geq 0:\forall T\in\mathscr{L}(X,Y/F)\ \exists\widetilde{T}\in\mathscr{L}(X,Y)&\\ \text{with } T=Q_F^Y\widetilde{T},\ \|\widetilde{T}\|\leq\rho\|T\|\bigr\}.&\end{aligned}$$

We always have $q(X,Y;F)\geq 1$, and if X is a Banach space with the metric lifting property, then $q(X,Y;F)=1$ (provided the triple is nontrivial). Moreover, in the case of Hilbert spaces H we have $q(X,H;F)=1$. In general, however, the infimum need not be finite.

Fixing a couple of Banach spaces (X,Y) and $n\in\mathbb{N}$, and letting now F vary within the class of all subspaces $F\subset Y$ of dimension $n-1$, the nth *lifting constant* $q_n(X,Y)$ is defined by

$$q_n(X,Y):=\sup\{q(E;X,Y): F\subset Y,\ \dim(F)=n-1\}.$$

By a result from [8] we know that

$$q_n(X,Y)\leq 1+\sqrt{n-1}\leq\sqrt{2n}.$$

Using extension and lifting constants we deduce as an immediate consequence of [8, Propositions 2.4.1 and 2.4.4] the following estimates for the approximation numbers by Gelfand and Kolmogorov numbers.

LEMMA 2.2. *For each $T\in\mathscr{L}(X,Y)$ the following inequalities hold:*

$$a_n(T)\leq p_n(X,Y)c_n(T)\quad\textit{and}\quad a_n(T)\leq q_n(X,Y)d_n(T).$$

As usual, we denote by $T_p(X)$ and $C_q(X)$, $1 < p \le 2 \le q < \infty$, the type p and cotype q constant of a Banach space X, respectively; if X has no type p or no cotype q, we write $T_p(X) := \infty$ or $C_q(X) := \infty$ (see e.g. [25, Section 1] for more information).

LEMMA 2.3. *Let $T \in \mathscr{L}(X,Y)$.*

(i) *If Y has type p and cotype q with $1 < p \le 2 \le q < \infty$, then with an absolute constant $c \ge 1$, for each n,*
$$a_n(T) \le cT_p(Y)C_q(Y)n^{1/p-1/q}d_n(T).$$

(ii) *If X' has type p and cotype q with $1 < p \le 2 \le q < \infty$, then with an absolute constant $c \ge 1$, for each n,*
$$a_n(T) \le cT_p(X')C_q(X')n^{1/p-1/q}c_n(T).$$

Proof. For a subspace $F \subset Y$ with $\dim(F) < n$, we choose according to a result of [14, Corollary 7] a projection $P \in \mathscr{L}(Y)$ onto F with
$$\|P\| \le cT_p(Y)C_q(Y)n^{1/p-1/q},$$
where $1 \le c \le 2^8\pi^{-1}$ is an absolute constant. This immediately implies
$$q_n(X,Y) \le cT_p(Y)C_q(Y)n^{1/p-1/q},$$
and hence, by Lemma 2.2, the desired estimate (i). The estimate from (ii) now follows by a duality argument: To this end we use a result of Edmunds and Tylli [10, Proposition 2] (see also [8, Proposition 2.5.4]) which states that $a_n(T) \le 5a_n(T')$. Combining this with (i) and using moreover the (obvious) duality relation $d_n(T') = c_n(T)$, we finally arrive at
$$\begin{aligned} a_n(T) &\le 5a_n(T') \\ &\le 5cT_p(X')C_q(X')n^{1/p-1/q}d_n(T') = 5cT_p(X')C_q(X')n^{1/p-1/q}c_n(T). \end{aligned}$$
This completes the proof. ■

3. Estimates of Weyl numbers by Z-Weyl numbers. Given an infinite-dimensional Banach space Z, the definition of Z-Weyl numbers of operators in Banach spaces follows the pattern of classical Weyl numbers (see e.g. [3]). For $T \in \mathscr{L}(X,Y)$ and $n \in \mathbb{N}$ the nth *Z-Weyl number* $x_n(T|Z)$ of T is defined by
$$x_n(T|Z) := \sup\{a_n(TA) : A \in \mathscr{L}(Z,X),\ \|A\| \le 1\}.$$
From [3, Section 2, p. 474] it is known that $(x_n(\cdot|Z))_{n=1}^{\infty}$ forms a multiplicative s-number sequence. Clearly, the classical Weyl numbers are nothing else than the ℓ_2-Weyl numbers. The main purpose of this article is to relate Weyl numbers with Z-Weyl numbers.

We start with the following general result.

Proposition 3.1. *Let Z be an infinite-dimensional Banach spaces. Then for any $T \in \mathscr{L}(X,Y)$ and $n \in \mathbb{N}$ we have*

$$x_n(T) \le \sqrt{n}\Big(\prod_{k=1}^{n} x_n(T|Z)\Big)^{1/n}.$$

Proof. We have $h_n(T) \le x_n(T|Z)$, hence the desired result is an immediate consequence of [4, Corollary 1.2]. ∎

In the case when Z or Z' is a Banach space of type 2, we can estimate Weyl numbers by Z-Weyl numbers up to constants which only depend on Z. The following result is from [3, Proposition 3.1].

Proposition 3.2. *Let Z be an infinite-dimensional Banach space such that Z or Z' is of type 2. Then for all $T \in \mathscr{L}$ we have*

$$x_n(T) \le \min\{T_2(Z), T_2(Z')\} x_n(T|Z).$$

It is well-known that $\ell_{p'}$ for $1 < p \le 2$ is of type 2 (where p' as usual denotes the conjugate exponent of p), and ℓ_p for $2 \le p < \infty$ is of type 2 (see e.g. [25, 4]). Hence the following corollary is immediate.

Corollary 3.3. *Let $1 < p < \infty$. Then for all $T \in \mathscr{L}$ we have*

$$x_n(T) \le \min\{T_2(\ell_p), T_2(\ell_{p'})\} x_n(T|\ell_p).$$

For $p = 1$ this is no longer true.

Remark 3.4. For arbitrary operators T an inequality $x_n(T) \le c x_n(T|\ell_1)$, $n \in \mathbb{N}$, with a constant $c > 0$ independent of n is not true.

For a proof of this remark, first note the following result of independent interest.

Proposition 3.5. *Let X be a separable Banach space, Y a Banach space, and $T \in \mathscr{L}$. Then for all $n \in \mathbb{N}$,*

$$x_n(T|\ell_1) = d_n(T).$$

Proof. Since ℓ_1 has the metric lifting property, for $T \in \mathscr{L}(X,Y)$ and $A \in \mathscr{L}(\ell_1, X)$ we have $a_n(TA) = d_n(TA)$. This implies $x_n(T|\ell_1) \le d_n(T)$. On the other hand we know that the separable Banach space X is isometric to a quotient space of ℓ_1, and hence there exists a metric surjection Q in $\mathscr{L}(\ell_1, X)$. The surjectivity of the Kolmogorov numbers yields

$$d_n(T) = d_n(TQ) \le a_n(TQ) \le x_n(T|\ell_1)\|Q\| \le x_n(T|\ell_1),$$

which completes the proof. ∎

Proof of Remark 3.4. From [17, 11.11.8] or [19, 2.9.11] we know that for all n,

$$\frac{1}{\sqrt{2}} \le a_n(I : \ell_2^{2n} \to \ell_\infty^{2n}) = x_n(I : \ell_2^{2n} \to \ell_\infty^{2n}),$$

and from [12] (see also [6, 11]), with some absolute constant $d \geq 1$,

$$d_n(I : \ell_2^{2n} \to \ell_\infty^{2n}) \leq d\frac{1}{\sqrt{n}}.$$

Clearly, then by Proposition 3.5 a general inequality like $x_n(T) \leq cx_n(T|\ell_1) = cd_n(T)$ is impossible. ∎

In contrast to this, it is unknown whether an estimate as stated in Corollary 3.3 holds for $p = \infty$. In this context we refer to the limit orders of the s-number ideals $\mathscr{L}_r^{(x(\cdot|\ell_\infty))}$, $0 < r < \infty$. In [22] (see also [23]) it was shown that

$$\lambda(\mathscr{L}_r^{(x(\cdot|\ell_\infty))}, p, q) = \lambda(\mathscr{L}_r^{(x)}, p, q), \quad 1 \leq p \leq 2,\ 1 \leq q \leq \infty,$$
$$\lambda(\mathscr{L}_r^{(x(\cdot|\ell_\infty))}, p, q) = \lambda(\mathscr{L}_r^{(a)}, p, q), \quad 2 \leq p \leq \infty,\ 1 \leq q \leq \infty.$$

These limit orders indicate that an inequality as stated in Corollary 3.3 could be true for the case $p = \infty$.

We now turn to the main results of this section—results which generalize Propositions 3.1 and 3.2. We need the *n-dimensional extension constant* $p^{(n)}(X,Y)$ of a pair (X,Y) of Banach spaces given by

$$p^{(n)}(X,Y) := \sup\{p(E;X,Y) : E \subset X,\ \dim(E) \leq n\}.$$

Obviously, the estimate

$$p^{(n)}(X,Y) \leq \sqrt{n}$$

is a simple reformulation of the well-known Kadets–Snobar theorem. We start to estimate approximation numbers by Z-Weyl numbers.

Lemma 3.6. *Let Z be an infinite-dimensional Banach space and let $T \in \mathscr{L}(\ell_2^n, Y)$. Then for all $k \in \mathbb{N}$:*

(i) $a_k(T) \leq p^{(n)}(Z, \ell_2^n) x_k(T|Z)$,
(ii) $a_k(T) \leq p^{(n)}(Z', \ell_2^n) x_k(T|Z)$.

Proof. In order to check (i), let $\varepsilon > 0$. By Dvoretzky's theorem (see e.g. [21] or [25]) there is an n-dimensional subspace $E \subset Z$ and an operator A in $\mathscr{L}(E, \ell_2^n)$ such that $\|A\|\,\|A^{-1}\| \leq 1 + \varepsilon$. The definition of the n-dimensional extension constant gives an operator $\widetilde{A} \in \mathscr{L}(Z, \ell_2^n)$ with $\widetilde{A}J_E^Z = A$ and

$$\|\widetilde{A}\| \leq (1+\varepsilon)p^{(n)}(Z, \ell_2^n)\|A\|.$$

But then for $T \in \mathscr{L}(\ell_2^n, Y)$ we have

$$\begin{aligned} a_k(T) &\leq a_k(TA)\|A^{-1}\| = a_k(T\widetilde{A}J_E^Z)\|A^{-1}\| \\ &\leq a_k(T\widetilde{A})\|A^{-1}\| \leq x_k(T|Z)\|\widetilde{A}\|\,\|A^{-1}\| \\ &\leq (1+\varepsilon)p^{(n)}(Z, \ell_2^n)x_k(T|Z)\|A\|\,\|A^{-1}\| \leq (1+\varepsilon)^2 p^{(n)}(Z, \ell_2^n)x_k(T|Z), \end{aligned}$$

which clearly proves (i).

In order to show (ii) we apply Dvoretzky's theorem to the dual Banach space Z'. For $\varepsilon > 0$ there is an n-dimensional subspace $E \subset Z'$ and an operator $A \in \mathscr{L}(E, \ell_2^n)$ with $\|A\|\,\|A^{-1}\| \le 1+\varepsilon$. Again we may choose $\widetilde{A} \in \mathscr{L}(Z', \ell_2^n)$ with $\widetilde{A} J_E^{Z'} = A$ and

$$\|\widetilde{A}\| \le (1+\varepsilon) p^{(n)}(Z', \ell_2^n)\|A\|,$$

and moreover an operator $B \in \mathscr{L}(Z, \ell_2^n)$ for which $B' = J_E^{Z'} A^{-1}$. Then for $T \in \mathscr{L}(\ell_2^n, Y)$ we get

$$\begin{aligned} a_k(T) = a_k(T') = a_k(AA^{-1}T') &= a_k(\widetilde{A} J_E^{Z'} A^{-1} T') \le \|\widetilde{A}\| a_k(B'T') \\ &\le (1+\varepsilon) p^{(n)}(Z', \ell_2^n)\|A\| a_k(TB) \\ &\le (1+\varepsilon) p^{(n)}(Z', \ell_2^n)\|A\|\,\|B\| x_k(T|Z) \\ &\le (1+\varepsilon) p^{(n)}(Z', \ell_2^n)\|A\|\,\|A^{-1}\| x_k(T|Z) \\ &\le (1+\varepsilon)^2 p^{(n)}(Z', \ell_2^n) x_k(T|Z), \end{aligned}$$

which, since $\varepsilon > 0$ is arbitrary, is the desired inequality (ii). ∎

We now use the preceding lemma to estimate Weyl numbers up to n-dimensional extension constants by Z-Weyl numbers.

LEMMA 3.7. *Let Z be an infinite-dimensional Banach space and $T \in \mathscr{L}$. Then for every $n \in \mathbb{N}$ the following inequalities hold:*

(i) $x_{2n}(T) \le \sqrt{2e}\, p^{(2n)}(Z, \ell_2^{2n}) \Big(\prod_{k=1}^{n} x_k(T|Z) \Big)^{1/n}$,

(ii) $x_{2n}(T) \le \sqrt{2e}\, p^{(2n)}(Z', \ell_2^{2n}) \Big(\prod_{k=1}^{n} x_k(T|Z) \Big)^{1/n}$.

Proof. Fix $T \in \mathscr{L}(X,Y)$, $A \in \mathscr{L}(\ell_2, X)$, and $\varepsilon > 0$. According to Lemma 2.1 we may choose operators

$$\begin{aligned} &A_{2n} \in \mathscr{L}(\ell_1^{2n}, \ell_2) \quad \text{with} \quad \|A_{2n} : \ell_1^{2n} \to \ell_2\| \le 1, \\ &B_{2n} \in \mathscr{L}(Y, \ell_\infty^{2n}) \quad \text{with} \quad \|B_{2n} : Y \to \ell_\infty^{2n}\| \le 1 \end{aligned}$$

such that

$$c_{2n}(TA : \ell_2 \to Y) \le (1+\varepsilon) \Big(\prod_{k=1}^{2n} h_k(B_{2n} T A A_{2n} : \ell_2^{2n} \to \ell_2^{2n}) \Big)^{1/(2n)}.$$

The operator $A_{2n} \in \mathscr{L}(\ell_2^{2n}, \ell_2)$ can be factorized as $A_{2n} = J_M^{\ell_2}(A_{2n})_0$, where M is the range of A_{2n} in ℓ_2, the operator $(A_{2n})_0$ is the astriction of A_{2n} onto M, and $J_M^{\ell_2}$ is the embedding from M into ℓ_2. Using the multiplicativity of the approximation numbers for operators acting between Hilbert spaces (see e.g. [13, 1.b.6]) we obtain

$$
\begin{aligned}
(3.1)\qquad a_{2n}(TA) &= c_{2n}(TA)\\
&\le (1+\varepsilon)\Big(\prod_{k=1}^{2n} a_k(B_{2n}TAJ_M^{\ell_2} : M \to \ell_2^{2n})\Big)^{1/(2n)}\\
&\quad\times \Big(\prod_{k=1}^{2n} a_k((A_{2n})_0 : \ell_2^{2n} \to M)\Big)^{1/(2n)}\\
&= (1+\varepsilon)\Big(\prod_{k=1}^{2n} x_k(B_{2n}TAJ_M^{\ell_2} : M \to \ell_2^{2n})\Big)^{1/(2n)}\\
&\quad\times \Big(\prod_{k=1}^{2n} a_k((A_{2n})_0 : \ell_2^{2n} \to M)\Big)^{1/(2n)}.
\end{aligned}
$$

We now estimate both terms on the right-hand side of the inequality (3.1), and start with the second one. Obviously,

$$a_k((A_{2n})_0) = a_k(A_{2n}),$$

and hence the second term in (3.1) satisfies

$$
\begin{aligned}
\Big(\prod_{k=1}^{2n} a_k((A_{2n})_0)\Big)^{1/(2n)} &= \Big(\prod_{k=1}^{2n} a_k(A_{2n})\Big)^{1/(2n)}\\
&\le (2n)^{-1/2}\Big(\sum_{k=1}^{2n} a_k^2(A_{2n})\Big)^{1/2}.
\end{aligned}
$$

Here on the right-hand side we have the well-known Hilbert–Schmidt norm of $A_{2n} \in \mathscr{L}(\ell_2^{2n}, \ell_2)$. Choosing the unit vector basis $\{e_k : k = 1, \dots, 2n\}$ in ℓ_2^{2n} we obtain

$$
\begin{aligned}
\Big(\sum_{k=1}^{2n} a_k^2(A_{2n} : \ell_2^{2n} \to \ell_2)\Big)^{1/2} &= \Big(\sum_{k=1}^{2n} \|A_{2n}e_k\|^2\Big)^{1/2}\\
&\le \sqrt{2n}\max_{1\le k\le n} \|A_{2n}e_k\| \le \sqrt{2n}\,\|A_{2n} : \ell_1^{2n} \to \ell_2\|,
\end{aligned}
$$

and therefore we control the second factor in (3.1):

$$(3.2)\qquad \Big(\prod_{k=1}^{2n} a_k((A_{2n})_0 : \ell_2^{2n} \to M)\Big)^{1/(2n)} \le \|A_{2n} : \ell_1^{2n} \to \ell_2\| \le 1.$$

It remains to control the first factor on the right-hand side of (3.1): With the multiplicativity of the Weyl numbers (now in Banach spaces) we check

$$(3.3)\qquad \Big(\prod_{k=1}^{2n} x_k(B_{2n}TAJ_M^{\ell_2}: M\to \ell_2^{2n})\Big)^{1/(2n)} \le \Big(\prod_{k=1}^{n} x_k(B_{2n}: Y\to \ell_2^{2n})\Big)^{1/n}\Big(\prod_{k=1}^{n} x_k(TAJ_M^{\ell_2}: M\to Y)\Big)^{1/n} \le \Big(\prod_{k=1}^{n} x_k(B_{2n}: Y\to \ell_2^{2n})\Big)^{1/n}\Big(\prod_{k=1}^{n} a_k(TAJ_M^{\ell_2}: M\to Y)\Big)^{1/n}.$$

Again we consider both factors separately. We use the well-known inequality between Weyl numbers and the 2-summing norm from [18] (see also [13, 2.a.3] or [19, 2.7.3])

$$\sqrt{k}\, x_k(B_{2n}: Y\to \ell_2^{2n}) \le \pi_2(B_{2n}: Y\to \ell_2^{2n}),$$

and moreover the estimate

$$\pi_2(B_{2n}: Y\to \ell_2^{2n}) \le \sqrt{2n},$$

which follows from $\|B_{2n}: Y\to \ell_\infty^{2n}\| \le 1$ and $\pi_2(I: \ell_\infty^{2n}\to \ell_2^{2n}) \le \sqrt{2n}$. Hence we obtain

$$(3.4)\qquad \Big(\prod_{k=1}^{n} x_k(B_{2n}: Y\to \ell_2^{2n})\Big)^{1/n} \le \sqrt{\frac{e}{n}}\,\pi_2(B_{2n}: Y\to \ell_2^{2n}) \le \sqrt{2e}.$$

In order to estimate the second factor in (3.3) we apply the two inequalities proved in Lemma 3.6:

$$(3.5)\qquad \begin{aligned} \Big(\prod_{k=1}^{n} a_k(TAJ_M^{\ell_2}: M\to Y)\Big)^{1/n} &\le p^{(2n)}(Z,\ell_2^n)\Big(\prod_{k=1}^{n} x_k(T|Z)\Big)^{1/n}\|A\|,\\ \Big(\prod_{k=1}^{n} a_k(TAJ_M^{\ell_2}: M\to Y)\Big)^{1/n} &\le p^{(2n)}(Z',\ell_2^n)\Big(\prod_{k=1}^{n} x_k(T|Z)\Big)^{1/n}\|A\|. \end{aligned}$$

Combining (3.1) and (3.3)–(3.5) we finally arrive at

$$a_{2n}(TA) = c_{2n}(TA) \le (1+\varepsilon)\sqrt{2e}\, p^{(2n)}(Z,\ell_2^n)\Big(\prod_{k=1}^{2n} x_k(T|Z)\Big)^{1/n}\|A\|,$$

$$a_{2n}(TA) = c_{2n}(TA) \le (1+\varepsilon)\sqrt{2e}\, p^{(2n)}(Z',\ell_2^n)\Big(\prod_{k=1}^{2n} x_k(T|Z)\Big)^{1/n}\|A\|,$$

which by the definition of Weyl numbers is what we wanted. ∎

In view of the preceding lemma the next aim is to get a better control of the two extension constants $p^{(2n)}(Z,\ell_2^{2n})$ and $p^{(2n)}(Z',\ell_2^{2n})$ whenever we assume additional geometrical assumptions on Z.

Recall the definition of the Banach operator ideal $[\Gamma_2,\gamma_2]$ of all operators factorizing through Hilbert spaces. Given two Banach spaces X and Y, an operator $T \in \mathscr{L}(X,Y)$ belongs to the component $\Gamma_2(X,Y)$ if there is a Hilbert space H and operators $R \in \mathscr{L}(X,H)$ and $S \in \mathscr{L}(H,Y)$ with $T = SR$, and its *hilbertian norm* is defined to be

$$\gamma_2(T) := \inf\{\|S\|\,\|R\| : R \in \mathscr{L}(X,H),\ S \in \mathscr{L}(H,Y),\ T = SR\};$$

we write $\gamma_2(T) := \infty$ whenever T cannot be factorized through a Hilbert space. The γ_2-*factorization constant* of two Banach spaces X and Y is given by

$$\gamma_2(X,Y) := \inf\{\rho \geq 0 : \exists T \in \mathscr{L}(X,Y) \text{ with } \gamma_2(T) \leq \rho\|T\|\}.$$

The following result is from [14, Theorem 3] (see also [24]).

LEMMA 3.8. *Let $1 < p \leq 2 \leq q < \infty$, E a subspace of a Banach space X, and Y an n-dimensional Banach space. Then every $T \in \mathscr{L}(E,Y)$ has an extension $\widetilde{T} \in \mathscr{L}(X,Y)$ for which*

$$\gamma_2(\widetilde{T}) \leq cT_p(X)\min\{C_q(E), C_q(Y)\}n^{1/p-1/q}\|T\|,$$

where $c > 0$ is an absolute constant.

This lemma can be seen as a finite-dimensional variant of Maurey's extension theorem from [15] (see also [25, 13.13]): *Let $E \subset X$ be a subspace of a Banach space X of type 2, and Y a Banach space of cotype 2. Then every $T \in \mathscr{L}(E,Y)$ has an extension $\widetilde{T} \in \mathscr{L}(X,Y)$ satisfying*

$$\gamma_2(\widetilde{T}) \leq cT_2(X)C_2(Y)\|T\|. \tag{3.6}$$

Finally, we are able to state and prove the main result of this section.

THEOREM 3.9. *Let Z be an infinite-dimensional Banach space, $1 < p \leq 2$, and $T \in \mathscr{L}$. Then the following inequalities hold with absolute constants $c_1, c_2 > 0$:*

(i) *If Z has type p, then for every $n \in \mathbb{N}$,*

$$p^{(n)}(Z, \ell_2^n) \leq c_1 T_p(Z)n^{1/p-1/2},$$

$$x_{2n}(T) \leq c_2 T_p(Z)n^{1/p-1/2}\Big(\prod_{k=1}^{n} x_k(T|Z)\Big)^{1/n}.$$

(ii) *If Z' has type p, then for every $n \in \mathbb{N}$,*

$$p^{(n)}(Z', \ell_2^n) \leq c_1 T_p(Z')n^{1/p-1/2},$$

$$x_{2n}(T) \leq c_2 T_p(Z')n^{1/p-1/2}\Big(\prod_{k=1}^{n} x_k(T|Z)\Big)^{1/n}.$$

Proof. We prove statement (i). By Lemma 3.8, given a subspace $E \subset Z$ with $\dim(E) \leq n$ and $S \in \mathscr{L}(E, \ell_2^n)$, there exists an extension $\widetilde{S} \in \mathscr{L}(Z, \ell_2^n)$ such that

$$\|\widetilde{S}\| \leq \gamma_2(\widetilde{S}) \leq cT_p(Z)n^{1/p-1/2}\|S\|,$$

where $c > 0$ is an absolute constant (recall that $C_2(\ell_2^n) = 1$). We conclude that

$$p^{(n)}(Z, \ell_2^n) \leq c T_p(Z) n^{1/p-1/2},$$

and hence (i) is an immediate consequence of Lemma 3.7. Similarly, (ii) follows. ∎

4. Estimates of Z-Weyl numbers by Weyl numbers. In this section we deal with converse inequalities—we estimate Z-Weyl numbers by Weyl numbers.

PROPOSITION 4.1. *Let Z be an infinite-dimensional Banach space and $T \in \mathscr{L}(X, Y)$. Then*

$$x_n(T|Z) \leq \gamma_2(Z, X) x_n(T).$$

Proof. Without loss of generality we may assume that $\gamma_2(Z, X) < \infty$. Given $\varepsilon > 0$, there is some $A \in \mathscr{L}(Z, X)$ with $\|A\| \leq 1$ and $x_n(T|Z) \leq (1 + \varepsilon) a_n(TA)$. Since by definition we have $\gamma_2(A) \leq \gamma_2(Z, X)\|A\|$, there are operators $S \in \mathscr{L}(H, X)$, $R \in \mathscr{L}(Z, H)$ with $A = SR$ and $\|S\|\,\|R\| \leq (1 + \varepsilon)\gamma_2(Z, X)$. But then

$$\begin{aligned} x_n(T|Z) &\leq (1+\varepsilon) a_n(TA) \leq (1+\varepsilon) a_n(TS)\|R\| \\ &\leq (1+\varepsilon) x_n(T)\|S\|\,\|R\| \leq (1+\varepsilon)^2 \gamma_2(Z, X) x_n(T), \end{aligned}$$

which clearly yields the desired estimate. ∎

Similar to the strategy for the proof of Proposition 3.9, estimates of $\gamma_2(Z, X)$ (under additional geometrical assumptions on Z and Y) lead to more concrete inequalities. In order to control the γ_2-factorization constant we mainly use results of König, Retherford, and Tomczak-Jaegermann [14], Maurey [15], and Pisier [20, 21].

THEOREM 4.2. *Let Z be an infinite-dimensional Banach space, X a Banach space, and $T \in \mathscr{L}(X, Y)$. Then, with an absolute constant $c > 0$, the following three inequalities hold:*

(i) *If Z has type 2 and X cotype 2, then for all $n \in \mathbb{N}$,*

$$\gamma_2(Z, X) \leq c T_2(Z) C_2(X) \quad \text{and} \quad x_n(T|Z) \leq c T_2(Z) C_2(X) x_n(T).$$

(ii) *If Z' has cotype 2 and X cotype 2, then for all $n \in \mathbb{N}$,*

$$\gamma_2(Z, X) \leq c\big(C_2(Z')C_2(X)\big)^2 \quad \text{and} \quad x_n(T|Z) \leq c\big(C_2(Z')C_2(X)\big)^2 x_n(T).$$

(iii) *If Z has type p and cotype q, $1<p\leq 2\leq q<\infty$, and X is a finite-dimensional Banach space with $\dim(X)\leq m$, then for all $n\in\mathbb{N}$,*

$$\gamma_2(Z, X) \leq c T_2(Z) C_2(Z) m^{1/p-1/q},$$
$$x_n(T|Z) \leq c T_2(Z) C_2(Z) m^{1/p-1/q} x_n(T).$$

Proof. The first result (i) follows from Maurey's result repeated in (3.6) and Proposition 4.1. The assertion (ii) is based on a result of Pisier—in [20] it is shown that for $A \in \mathscr{L}(Z, X)$,

$$\gamma_2(A) \le \left(\frac{c_0}{\vartheta} C_2(Z')C_2(X)\right)^{1/(1-\vartheta)} \|A\|,$$

where $c_0 > 0$ is an absolute constant and $0 < \vartheta < 1$. Putting $\vartheta = 1/2$ we get

$$\gamma_2(A) \le \big(2c_0 C_2(Z')C_2(X)\big)^2 \|A\|,$$

and therefore

$$\gamma_2(Z, X) \le 2c_0\big(C_2(Z')C_2(X)\big)^2.$$

Again we apply Proposition 4.1 and get (ii). In order to prove (iii) we apply Lemma 3.8 to the operator $A \in \mathscr{L}(Z, X)$ and obtain

$$\gamma_2(A) \le cT_p(Z)C_q(Z)m^{1/p-1/q}\|A\|.$$

This implies

$$\gamma_2(Z, X) \le cT_p(Z)C_q(Z)m^{1/p-1/q},$$

which is the first statement from (iii), and as before we deduce the second assertion from (iii) by Proposition 4.1. ∎

Corollary 4.3. *Under the assumption of Theorem* 4.2(i), *for* $T \in \mathscr{L}$ *and all* $n \in \mathbb{N}$ *we have*

$$c_1 T_2(Z)^{-1} x_n(T) \le x_n(T|Z) \le c_2 T_2(Z) C_2(X) x_n(T),$$

where $c_1, c_2 > 0$ *are absolute constants.*

Proof. The result follows from Proposition 3.2 and Theorem 4.2(i). ∎

5. Minimal multiplicative *s*-numbers. This section is devoted to minimal multiplicative s-numbers which are very useful tools for estimating eigenvalues of operators. Given s-number sequences $s = (s_n)$ and $t = (t_n)$, we write

$$s \lesssim t$$

whenever there exists a constant $\rho > 0$ such that $s_n(T) \le \rho t_n(T)$ for all $T \in \mathscr{L}$ and all $n \in \mathbb{N}$.

The following question is studied: Given a multiplicative s-number sequence $s = (s_n)$ and an infinite-dimensional Banach space Z for which $s \lesssim x(\cdot|Z)$. To what extent does the converse inequality $x(\cdot|Z) \lesssim s$ hold?

For the classical Weyl numbers $x = x(\cdot|\ell_2)$ the following result was proved in [4, Theorem 3.1].

THEOREM 5.1. *Let s be a multiplicative s-number sequence with $s \lesssim x$. Then for all operators $T \in \mathscr{L}$ and $n \in \mathbb{N}$ we have*

$$x_{2n-1}(T) \le c\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n},$$

where $c > 0$ is some absolute constant.

The Weyl numbers are considered to be a minimal s-number sequence in the sense of Theorem 5.1. Replacing the classical Weyl numbers by arbitrary Z-Weyl numbers we have the following

THEOREM 5.2. *Let Z be an infinite-dimensional Banach space, and $s = (s_n)$ a multiplicative s-number sequence with the property that $s \lesssim x(\cdot|Z)$. Then for every $T \in \mathscr{L}$ we have*

$$x_n(T|Z) \le \sqrt{2}\, n^{3/2}\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n}.$$

Proof. Let $T \in \mathscr{L}(X,Y)$, $A \in \mathscr{L}(Z,X)$ with $\|A\| \le 1$ and $\varepsilon > 0$. Then from Lemma 2.2 and (2.1) (see also [16, 2.1.2]) we obtain

$$a_n(TA) \le \sqrt{2n}\, c_n(TA),$$

and as a consequence of Lemma 2.1,

$$a_n(TA : Z \to Y) \le (1+\varepsilon)\sqrt{2n}\,\Big(\prod_{k=1}^{n} s_k(B_nTAA_n : \ell_2^n \to \ell_2^n)\Big)^{1/n} \tag{5.1}$$

with operators

$$\begin{aligned} &A_n \in \mathscr{L}(\ell_1^n, Z) \quad \text{with} \quad \|A_n : \ell_1^n \to Z\| \le 1,\\ &B_n \in \mathscr{L}(Y, \ell_2^n) \quad \text{with} \quad \|B_n : Y \to \ell_\infty^n\| \le 1. \end{aligned}$$

This implies

$$a_n(TA) \le (1+\varepsilon)\sqrt{2}\, n^{3/2}\Big(\prod_{k=1}^{n} s_k(TA)\Big)^{1/n},$$

and consequently

$$x_n(T|Z) \le \sqrt{2}\, n^{3/2}\Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n},$$

which is the desired inequality. ∎

Under geometrical assumptions on Z we obtain a far better control of the constant $\sqrt{2}\, n^{3/2}$.

THEOREM 5.3. *Let Z be an infinite-dimensional Banach space, and $s = (s_n)$ a multiplicative s-number sequence with the property that $s \lesssim x(\cdot|Z)$. If Z has type p and cotype q, $1 < p \le 2 \le q < \infty$, then for every $T \in \mathscr{L}$*

we have

$$x_{3n-2}(T|Z) \le cT_p(Z)^3 C_q(Z)^3 n^{3\min\{1/p-1/q,1/2\}} \Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n},$$

where $c > 0$ is some absolute constant.

The case $p = q = 2$ and $Z = \ell_2$ is of special interest; then the Weyl and Z-Weyl numbers coincide, and $Z = \ell_2$ has type and cotype 2. Hence, we obtain a slightly weaker version of Theorem 5.1 as a corollary of the preceding result; note that in [4] a more direct proof shows that the $(3n-2)$th Weyl number in the previous inequality can even be replaced by the $(2n-1)$th Weyl number.

Proof of Theorem 5.3. In view of Theorem 5.2 we may assume that $1/p - 1/q < 1/2$. Again let $T \in \mathscr{L}(X, Y)$, $A \in \mathscr{L}(Z, X)$ with $\|A\| \le 1$ and $\varepsilon > 0$. Since

$$T_p(Z) = C_{p'}(Z') \quad \text{and} \quad C_q(Z) = T_{q'}(Z')$$

(cf. [25]), we deduce from Lemma 2.3 that, with an absolute constant $c_1 > 0$, we have

$$\begin{aligned} a_{3n-2}(TA) &\le c_1 C_{p'}(Z')T_{q'}(Z')n^{1/q'-1/p'} c_{3n-2}(TA) \\ &= c_1 T_p(Z)C_q(Z)n^{1/p-1/q} c_{3n-2}(TA). \end{aligned} \tag{5.2}$$

By Lemma 2.1 we find operators

$$\begin{aligned} A_{3n-2} &\in \mathscr{L}(\ell_1^{3n-2}, Z) \quad \text{with} \quad \|A_{3n-2} : \ell_1^{3n-2} \to Z\| \le 1, \\ B_{3n-2} &\in \mathscr{L}(Y, \ell_\infty^{3n-2}) \quad \text{with} \quad \|B_{3n-2} : Y \to \ell_\infty^{3n-2}\| \le 1 \end{aligned}$$

such that

$$c_{3n-2}(TA) \le (1+\varepsilon)\Big(\prod_{k=1}^{3n-2} s_k(B_{3n-2}TAA_{3n-2} : \ell_2^{3n-2} \to \ell_2^{3n-2})\Big)^{1/(3n-2)}.$$

The multiplicativity of the s-number sequence (s_n) implies

$$\begin{aligned} c_{3n-2}(TA) \le (1+\varepsilon)&\Big(\prod_{k=1}^{n} s_k(B_{3n-2} : Y \to \ell_2^{3n-2})\Big)^{1/n}\Big(\prod_{k=1}^{n} s_k(T : X \to Y)\Big)^{1/n} \\ &\times \Big(\prod_{k=1}^{n} s_k(A_{3n-2} : \ell_2^{3n-2} \to Z)\Big)^{1/n} \|A : Z \to X\|. \end{aligned} \tag{5.3}$$

By [18] (see again [13, 2.a.3] or [19, 2.7.3]) we know that

$$\begin{aligned} s_k(A_{3n-2} : \ell_2^{3n-2} \to Z) &\le a_k(A_{3n-2} : \ell_2^{3n-2} \to Z) \\ &= x_k(A_{3n-2} : \ell_2^{3n-2} \to Z) \\ &\le k^{-1/q}\pi_{q,2}(A_{3n-2} : \ell_2^{3n-2} \to Z) \end{aligned}$$

(here $\pi_{q,2}$ as usual denotes the $(q,2)$-summing norm), and by [14, Corollary 6], with absolute constants $c_2, c_3 > 0$,

$$\begin{aligned}\pi_{q,2}(A_{3n-2} : \ell_2^{3n-2} \to Z) &\le c_2 T_p(Z) C_q(Z) n^{1/p-1/2} \pi_2(A'_{3n-2} : Z' \to \ell_2^{3n-2})\\ &\le c_3 T_p(Z) C_q(Z) n^{1/p}.\end{aligned}$$

This then implies the following estimate for the third factor in (5.3):

$$(5.4)\qquad \Big(\prod_{k=1}^{n} s_k(A_{3n-2} : \ell_2^{3n-2} \to Z)\Big)^{1/n} \le c_3 T_p(Z) C_q(Z) n^{1/p} (n!)^{-1/(nq)} \le c_4 T_p(Z) C_q(Z) n^{1/p-1/q}.$$

Now we show a similar inequality for the first term on the right-hand side of (5.3). Because of $s \lesssim x(\cdot|Z)$ and the properties of the operators B_{3n-2} we have, with a constant $\rho > 0$, the estimate

$$\begin{aligned}s_k(B_{3n-2} : Y \to \ell_2^{3n-2}) &\le \|B_{3n-2} : Y \to \ell_\infty^{3n-2}\| \, \|s_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2})\|\\ &\le \rho x_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2} | Z).\end{aligned}$$

By Theorem 4.2(iii) we get, with an absolute constant $c_5 > 0$,

$$x_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2} \mid Z) \le c_5 T_p(Z) C_q(Z) n^{1/p-1/q} x_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2}).$$

Moreover, we know that

$$x_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2}) \le \left(\frac{3n}{k}\right)^{1/2}$$

(see again [17, 11.11.8] or [19, 2.9.11]), and hence we deduce, with absolute constants $c_6, c_7 > 0$, the inequality

(5.5)

$$\begin{aligned}\Big(\prod_{k=1}^{n} s_k(B_{3n-2} : Y \to \ell_2^{3n-2})\Big)^{1/n} &\le \rho \Big(\prod_{k=1}^{n} x_k(I : \ell_\infty^{3n-2} \to \ell_2^{3n-2} \mid Z)\Big)^{1/n}\\ &\le c_6 T_p(Z) C_q(Z) n^{1/p-1/q} \left(\frac{n^n}{n!}\right)^{1/(2n)}\\ &\le c_7 T_p(Z) C_q(Z) n^{1/p-1/q}.\end{aligned}$$

Combining the estimates (5.2)–(5.5) we conclude that

$$a_{3n-2}(TA) \le (1+\varepsilon) c T_p(Z)^3 C_q(Z)^3 n^{3(1/p-1/q)} \Big(\prod_{k=1}^{n} s_k(T)\Big)^{1/n},$$

with an absolute constant $c > 0$. This completes the proof. ∎

Remark 5.4. The proof shows that the statement of the above theorem remains valid for a multiplicative s-number sequence satisfying

$$s_k(I : \ell_\infty^n \to \ell_2^n) \le x_k(I : \ell_\infty^n \to \ell_2^n \mid Z), \qquad k, n \in \mathbb{N}.$$

Closing remarks. First, observe that one can prove similarly that the dual Weyl numbers are also minimal multiplicative s-numbers in the sense of Theorem 5.1. In this context the problem arises whether there exist minimal multiplicative s-numbers different from the Weyl or dual Weyl numbers. Second, we remark that a weaker Weyl type inequality for an arbitrary multiplicative s-number sequence (s_n) was proved by the first named author (cf. also [13, 2.d.8]). For inequalities between eigenvalues and s-numbers and their applications we refer to the monographs [8], [9], [13], or [19], as well as to the recent articles [2]–[5]. Here we treated Z-Weyl numbers defined through approximation numbers. Similar procedures can be used to generate new s-number sequences by classical s-numbers; see [3] for details.

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Bernd Carl
Mathematisches Institut
FSU Jena
Ernst-Abbe-Platz 1–3
D-07743 Jena, Germany
E-mail: bernd.carl@uni-jena.de

Andreas Defant
Institut für Mathematik
Carl von Ossietzky Universität
Postfach 2503
D-26111 Oldenburg, Germany
E-mail: andreas.defant@uni-oldenburg.de

Doris Planer
Fachbereich Grundlagenwissenschaften
Ernst-Abbe-Fachhochschule Jena
Postfach 100314
D-07743 Jena, Germany
E-mail: doris.planer@fh-jena.de

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